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# The component mode transformation method: A fast implementation of fuzzy arithmetic for uncertainty management in structural dynamics

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**Abstract**

The study of mechanical systems with uncertain parameters is of increasing importance in the analysis and in the design of structures, because an adequate description of these uncertainties is necessary to provide a reliable forecasting model. In order to update or evaluate an uncertain model, different methods have been proposed, but many of these, such as the Monte Carlo simulation or the transformation method, require multiple sequential evaluations of the model. Moreover, to cover an adequate portion of the uncertain parameter space, the number of necessary evaluations of the model grows exponentially with the number of uncertain parameters, leading often to a large number of runs.

The proposed methodology deals with uncertain parameters that are modeled by the use of fuzzy numbers, and the transformation method is applied to characterize the dynamic behavior of the structure. In order to reduce the order of magnitude of the computational time, a reduction of the model dimension, obtained through a component mode synthesis, is performed.

Advantages and drawbacks of the proposed method are discussed together with an application to the study of the dynamics of a beam.



## 1. Introduction

In the design process of mechanical systems, the inclusion of uncertainties in the model is of increasing importance. Such uncertainties are inherent to either the model itself or to the inter-variability between several realizations of a single product. The first type of uncertainties arise due to several reasons, which range from modeling errors, via simplification of the model to speed-up calculations, to a lack of knowledge in the design parameters that may be not exactly defined. Those uncertainties are typically reducible by increasing the designer knowledge, and they are usually large with respect to their mean value, even larger than 100%. Furthermore, in general, there is not enough information available to derive a probability density function.

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The second type of uncertainties is generally not reducible, it is rather small and usually the possibility exists of finding reliable data about an appropriate probability distribution. While it is convenient to use stochastic methods for the latter type of uncertainties, it is possible and generally well accepted the use of the fuzzy sets theory for the former.

This paper deals with the solution of a direct, feed-forward uncertain problem, which consists in propagating the uncertainties through a model from an input parameter space to an output space with the purpose of finally obtaining a quantification of the uncertainties in the output variables.

The problem of incorporating model uncertainties into the finite element method has already been addressed by a number of publications, of which the vast majority is based on stochastic description of the uncertainties. In that context, the early paper of Contreras [1], and Handa and Anderson. [2], the monographs of Ghanem and Spanos [3] and of Kleiber and Hien [4], and the papers of Elishakoff et al. (e.g. Refs. [5,6]) are worth to be noted.

Recently, among the others that tackled the problem of the uncertain finite element models from a stochastic point of view, Schueller et al. [7–9] and Soize [10] are worth mentioning.

An alternative concept of describing uncertainties on the basis of fuzzy sets theory [11] emerged more recently; Rao and Sawyer [12] presented an approach for its incorporation into the finite element method. However, since that approach uses fuzzy arithmetic based on interval computation, the results were affected by a non-negligible overestimation. More recently, the transformation method introduced by Hanss [13], which represents the basis of the proposed method, is a practical implementation of the Zadeh's extension principle for fuzzy arithmetical computation within complex finite element models. The studies of Moens and Vandepitte [14,15], who are also the authors of a survey paper on non-statistical approaches to the uncertainty quantification [16], are also worth to be noted. In the context of the application of the fuzzy sets theory to engineering problems it is important to mention the Taylor's expansion with extrema management method (TEEM) approach, proposed by Massa et al. [17] that is an implementation of fuzzy arithmetic not based on the extension principle.

Finally, the work of De Gersem et al. [18] proposes an approach similar to the one discussed in this paper, based on a model condensation through component mode synthesis (CMS) to speed-up the computation of interval frequency response function (FRF) [14,15] of large finite elements models.

It is here important to notice that the fuzzy sets theory was developed to describe linguistic ambiguities, therefore, it can be applied in an engineering context, provided that the uncertain parameters are caused by deficiencies in any part of the model due to *lack of knowledge*.

In other words the uncertainties in the models are related to *incomplete information* that is typically filled by *human subjective opinion* on the unknown quantities.

The transformation method, proposed by Hanss [13] and outlined in Section 2 of this paper, describes uncertain parameters in the input space as fuzzy numbers and provides a simple, but nevertheless very powerful tool to estimate the uncertainties of the outputs as fuzzy-valued quantities. This method represents a very practical and state-of-the-art implementation of fuzzy arithmetic that can be applied to complex engineering problems. However, one drawback of the transformation method can be seen in the rather high computational cost, for it usually requires a large number of evaluations of the eventually large model. Moreover, the number of evaluations increases exponentially with the number of uncertain input parameters.

In order to reduce the required computation time, a reduction of the size of the model through the CMS method is proposed in this paper. The CMS method, introduced by Hurty in 1965 [19], has significant advantages with respect to condensation and is well suited for the modeling and simulation of large and complex systems.

The idea of taking advantage of model reduction, when dealing with parametric analyses of complicated systems, is not original. Among the others, Balmes [20,21] studied improvements of the CMS approach to perform parametric analyses, and Mace and Shorter [22] used CMS techniques to obtain statistics on the natural frequencies of perturbed systems and Guedri et al. [23] deal with stochastic analysis of condensed models.

Moreover, when dealing with the medium frequency range, there is still the need to cover the gap between standard finite element approach and SEA approach. With this aim, several different approaches are proposed in order to reduce the model size and, in the same time, taking into account the variability of the system dynamics that becomes more relevant when the frequency increases [24–26].

This paper proposes a novel method, called component mode transformation method (CMTM) that provides a consistent reduction of the time needed for the evaluation of each model, leading to a significant computational advantage. Unfortunately, the method does not change the scale factor of the computational cost with respect to the number of uncertain input parameters. Since it remains exponential, a large number of parameters (e.g. 100 parameters) still lead to an overwhelming number of models to be evaluated, even if they are reduced.

## 2. Methodology

### 2.1. The transformation method

A practical way to model the uncertainties inherent to the parameters of a model is to represent the model parameter by fuzzy numbers [11]. Forming a special class of fuzzy sets, fuzzy numbers are characterized by a membership function $\mu(x)$, $0 \leqslant \mu(x) \leqslant 1$, that defines the degree to which the parameter can take a certain value (Fig. 1).

For the evaluation of systems with uncertain, fuzzy-valued parameters, the transformation method, a practical implementation of the fuzzy arithmetic, can be used. The method is available in a general, a reduced, and an extended form [13]. Assuming the uncertain system to be characterized by $n$ fuzzy-valued model parameters $\tilde{p}_i$, $i = 1,2,\ldots, n$, the major steps of the method can be briefly described as follows:

In the first step, each fuzzy number $\tilde{p}_i$ is discretized into a number of intervals $X_i^{(j)} = [a_i^{(j)}, b_i^{(j)}]$, assigned to the levels $\mu_j$, $j = 0, 1,\ldots, m$, of membership that result from subdividing the possible range of membership equally spaced by $\Delta\mu = 1/m$ (Fig. 1). In a second step, the input intervals $X_i^{(j)}$, $i = 1, 2,\ldots, n$, $j = 0, 1,\ldots, m$, are transformed into arrays $\widehat{X}_i^{(j)}$ that are obtained from the upper and lower interval bounds after the application of a well-defined combinatorial scheme [27]. Each of these arrays represents a specific sample of possible parameter combinations and can be used as an input parameter set to the problem to be evaluated. As a result of the evaluation of the model for the input arrays $\widehat{X}_i^{(j)}$, $\widehat{Z}_i^{(j)}$ are obtained that are then retransformed into the output intervals $Z^{(j)} = [a^{(j)}, b^{(j)}]$ for each membership level $\mu_j$, and finally recomposed into the fuzzy-valued output $\tilde{q}$ of the system.

In addition to the simulation part of the transformation method described above, the analysis part of the method can be used to quantify the influence of each fuzzy-valued input parameter $\tilde{p}_i$ on the overall fuzziness of the model output $\tilde{q}$. For these purposes, so-called gain factors $\eta_i(j)$ or mean gain factors $\bar{\eta}_i$ have been

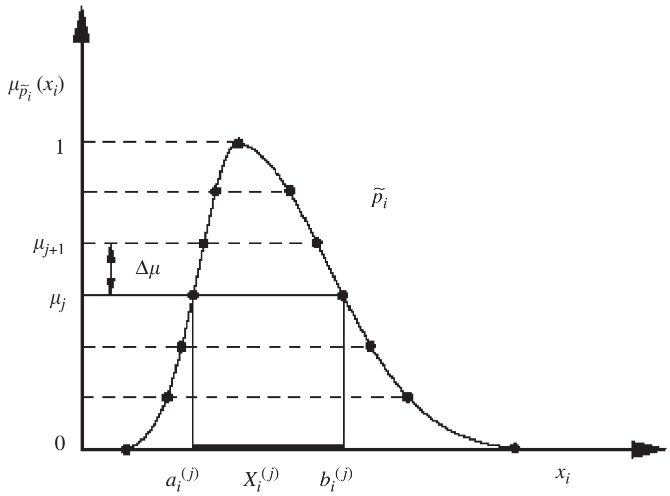


Fig. 1. Decomposition of a fuzzy number $\tilde{p}_i$ into intervals.

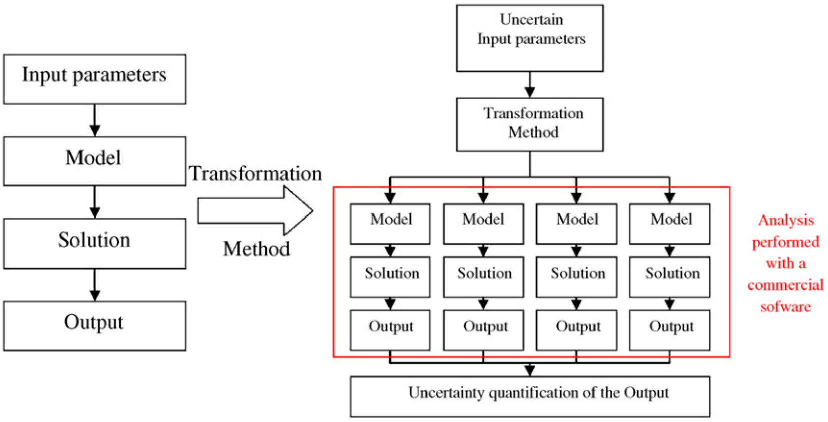


Fig. 2. Flow chart of the transformation method.

introduced that express the effect of the uncertainty of the $i$th model parameter $\tilde{p}_i$ on the overall uncertainty of the model output $\tilde{q}$ at the membership level $\mu_j$, or averaged over all levels, respectively.

Among other advantages of the transformation method, its characteristic property of reducing fuzzy arithmetic to multiple crisp-number operations entails that the transformation method can be implemented without major problems into an existing software environment for system simulation, avoiding expensive rewriting of the software. More explicitly, the steps of decomposition and transformation as well as of retransformation and recomposition can be coupled to the existing software by a separate pre- and post-processing tool. Fig. 2 shows the flow chart of the transformation method.

The transformation method considers the model as a black box that provides only an input–output relation. In other words, the model and the desired analysis perform a mapping of points in the input parameter hyperspace onto points in the output space, reducing fuzzy arithmetic to a set of operations for the intervals at each $\alpha$-cut by avoiding standard interval arithmetic[1] as defined by Moore [28]. Each input interval is sampled independently from the others. Therefore, at each $\alpha$-level the hyper-interval in the input space is defined by the Cartesian product between each vector representing each sampled interval. Fig. 3 shows the combinatorial sampling mechanism for a two-input problem.

If the relationships between a certain input and all the outputs are monotonic, the reduced transformation method can be used to sample the corresponding input parameter, scaling down the sampled interval vector to the pair of lower and upper bounds, and dramatically reducing the number of required evaluations.

### 2.2. *Advantages of the substructuring technique*

A complicated engineering structure is usually characterized by several components that are connected with each other. The substructuring techniques allow considering each subpart independently from the others, providing a condensed model for each substructure and, thus, describing the whole system as the composition of the subparts in a ‘‘compact’’ way.

The following expressions will be adopted in this paper:

- *Overall full model*: The finite element model that will be used for a crisp analysis and provides the reference output for the analysis.
- *Substructure full model*: The finite element model of a substructure.
- *Substructure reduced model*: The condensed model of a substructure, obtained through the CMS method.

[1]Avoiding the use of standard interval arithmetic provides two major advantages: there is no need to implement interval algebra within complex FE programs, and there is no overestimation of the outputs due to a not accounted repetition, i.e. a dependency of the same parameter within the FE matrices. For example, if the thickness of a plate is uncertain, this will affect all the non-zero elements of both the mass and the stiffness matrix, leading to a non-acceptable overestimation of the variability of the system eigenfrequencies.

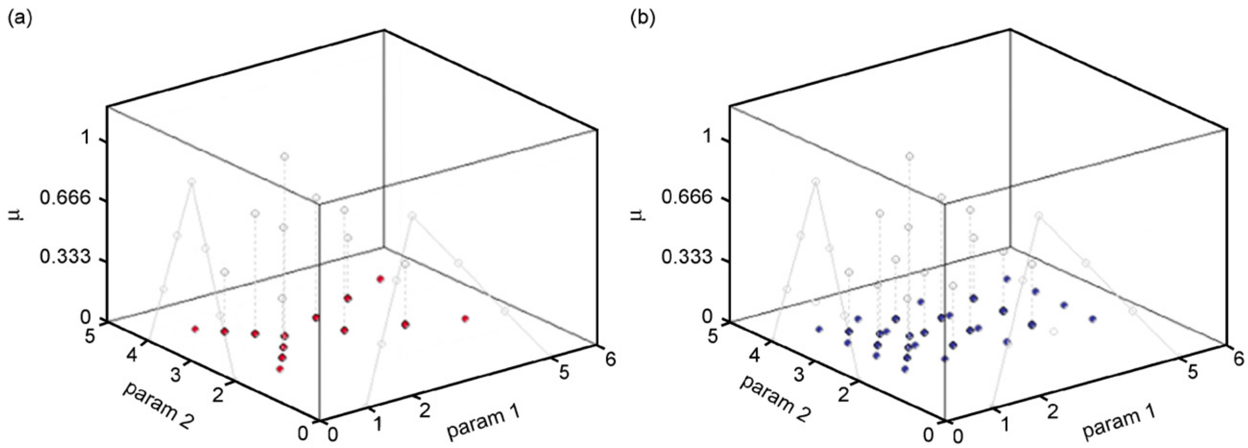


Fig. 3. Combinatorial approach of the transformation method for two input parameters: (a) reduced transformation method; and (b) general transformation method.

- *Overall reduced model*: The reduced model of the whole system obtained by connecting the *substructure reduced models* and adding an eventual residual part of the structure. The evaluation of this model for a given set of parameters is expected to provide a result close to the one obtained from the *overall full model.*

Substructuring techniques are characterized by three steps[2]:

- Definition of the substructures and construction of the *substructure reduced model* for each substructure. This step is the most time consuming.
- Assembling of the *overall reduced model*.
- Evaluation of the *overall reduced model*. This step is usually very inexpensive from a computational point of view, at least compared to the evaluation of the *overall full model*.

The CMS, in particular, provides a good and reliable tool, capable of reducing the dimension of the model. In fact, in modal coupling techniques, a reduced number of principal coordinates is used to describe each substructure. According to the choice of those principal coordinates, it is possible to distinguish two groups of methods: fixed interface methods and free interface methods.

In fixed interface methods [29], the total displacement of each substructure is obtained as a superposition of a set of fixed interface normal modes plus constraint modes.[3] In free interface methods [30–32], the total displacement of each substructure is obtained as a superposition of a set of normal modes with free interface plus (depending on the particular method) rigid body modes, attachment modes, inertia relief modes,[4] etc.

The proposed CMTM can use any of these substructuring techniques, so that a particular one can be freely chosen with respect to other aspects of the analysis to be performed.

A first advantage resulting from the application of substructuring techniques to uncertain problems arises from the possibility to build reduced models of those parts of the system that are not uncertain, reducing in this way the computational cost of each evaluation by reducing the numerical dimension of the model.

More relevant computational improvements result from the fact that the number of required evaluations of the model is defined by the Cartesian product between the sampling vectors of the input intervals when the transformation method is applied for each $\alpha$-cut. Thus, if the uncertain parameters are geometrically inherent to different substructures or if the overall structure can be subdivided into parts, each of them characterized by

[2]Appendix A presents the mathematical description of the CMS technique in the case of fixed interface and constraint modes.

[3]I.e. static deformation shape caused by a unit displacement of each connection dof, with the other interface dof fixed.

[4]E.g. An attachment mode is defined as the static deformation shape that results by imposing a unit force at one of the connection dof, while the remaining interface dof are force free.

a small/smaller number of uncertainties, few/fewer reduced model of each part can be computed. In this context, it is important to notice that the number of models within the overall structure that need to be evaluated remains unchanged. However, these models are reduced and their evaluation can be up to 100 times faster.

A final consideration concerns the observation that in mechanical systems the uncertainties are often inherent to joints and constraints. Such uncertain parameters can be neglected while creating the reduced models of the substructures, and must be considered only when evaluating the reduced model of the overall system.

### 2.3. The component mode transformation method

The CMTM can be applied in two different ways depending on the characteristics of the problem and the topological distribution of the uncertainties within the overall structure. We distinguish between the ‘‘*simple case*’’ and the ‘‘*general case*’’.

It is important to notice that, since the transformation method requires several evaluations of the model, it is always convenient to reduce the computational cost of each evaluation, thus, it is convenient to check whether the model has any possible part that can be expressed as a substructure and condensed through the CMS (or other condensation) technique.

The simple case of the CMTM can be used when all the uncertainties of the model are inherent to the boundary conditions or the joints between the substructures. In this case, a straightforward application of the CMTM method is possible. The model is decomposed into one or more substructures so that the models representing the substructures are crisp. Then, the uncertainties are introduced into the boundary conditions, or into the joints between the substructures, only in the reduced-order model. Fig. 4 shows the flow chart of the proposed procedure. Starting from the full model, the model is first subdivided into substructures. For each substructure, a *substructure reduced model* is obtained through the CMS, and the *overall reduced model* of the system is then assembled. Since the parameters characterizing the overall reduced model are uncertain (joints, boundary conditions, etc.), the transformation method is used to propagate the uncertainties and to calculate the uncertain output. In this way, a relevant number of evaluations of the model is still necessary but the actual model exhibits a number of degrees of freedom (dof) that can be drastically reduced.

In the general case, uncertainties are inherent also to the parameters that define the geometry or the material properties of the substructures. For this reason, it is not possible to consider just a crisp model for these

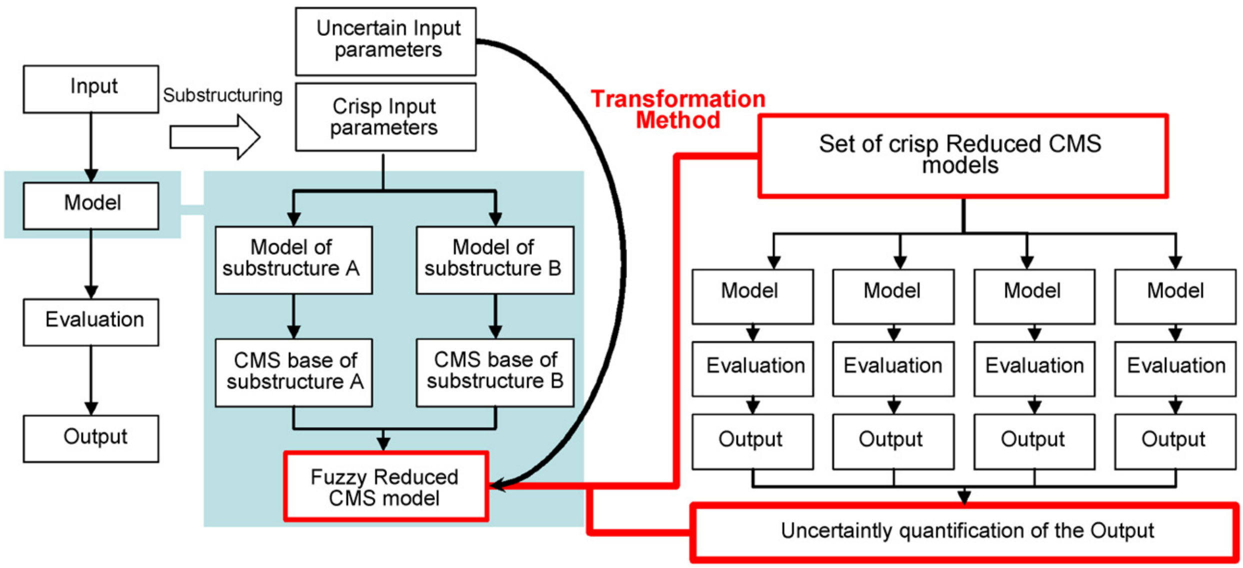


Fig. 4. Flow chart of the CMTM procedure: simple case.

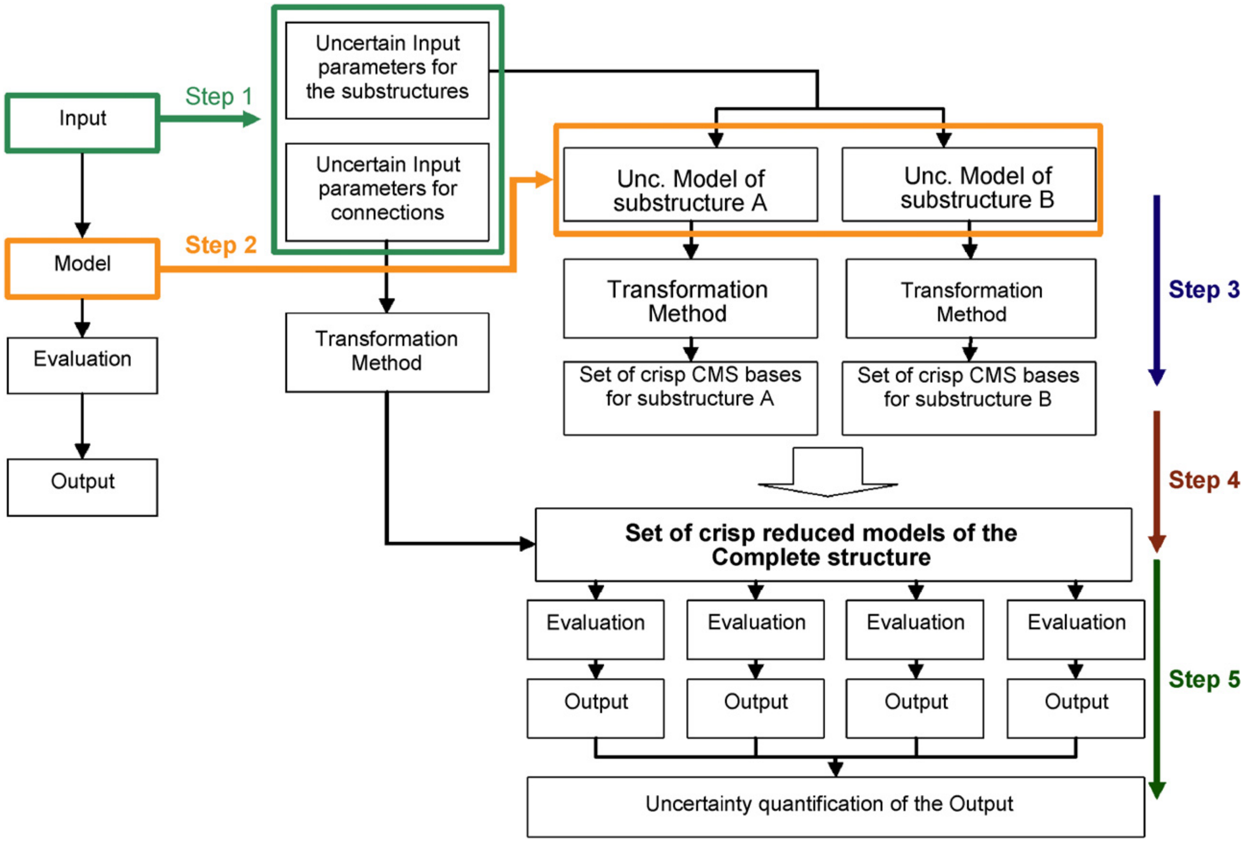


Fig. 5. Flow chart of the CMTM procedure: general case.

substructures. The general formulation of the CMTM, which must be applied in this case, can be described in five sequential steps, as follows. A flow chart of the procedure is presented in Fig. 5:

- *Step* 1: Definition and sampling of the parameter hyperspace: The input parameter hyperspace is sampled according to the transformation method or, when possible, to the reduced transformation method.
- *Step* 2: Substructuring of the model: The uncertain model is substructured so that the minimum possible number of uncertain parameters characterizes a single substructure and the maximum number of uncertain parameters occurs in joints or boundary conditions.
- *Step* 3: Application of the transformation method to the substructures: Since the model of each substructure is uncertain, the transformation method is applied. Instead of just a single crisp reduced model of the substructure, a set of models is required to cover the portion of the parameter hyperspace related to that substructure uncertainty.
- *Step* 4: Assembly of the crisp reduced model of the overall structure: Each crisp reduced model of the overall structure is characterized by a well-defined set of crisp parameters (this is obtained at step 1 when the parameter hyperspace is sampled). Therefore, the models of the substructures to be used in this step must be chosen accordingly. At this step, the uncertainties related to joints and boundary conditions are also considered.
- *Step* 5: Solution and post-processing: Each crisp reduced model is solved, and the fuzzy outputs are reconstructed from the data obtained from the analysis of each crisp model, according to the formulation of the transformation method.

Basically, in the first step all those models that are necessary for the analysis are defined. This information is later used to obtain all the models needed for each substructure in step 3. Step 2 defines the substructures.

In order to obtain a fast solution, it is important to separate as well as possible the set of uncertain parameter between the substructures. In fact, the number of models needed at this step depends on the number of fuzzy parameters inherent to each substructure. Moreover, the number of models grows exponentially with the number of uncertain parameters that characterize a single substructure. In step 3, the master nodes of the substructure must be selected. Finally, steps 4 and 5 represent a straightforward application of the transformation method to the reduced model and do not need further explanations.

### 2.4. Discussion

The proposed methodology couples a practical implementation of the fuzzy arithmetic granted by the transformation method, with the computational advantages of the CMS technique.

Generally, when CMS is applied to solve a mechanical problem, the generation of the reduced models of the substructures requires the major part of the computational effort (generally comparable with computational cost of the whole analysis). In fact, the solution of the problem is then obtained by dealing with a few dof system.

The CMTM requires in its general formulation several extractions of reduced models of the substructures but the number of evaluations is usually much lower than the number of solutions needed by the standard transformation method, if the substructuring is selected carefully. In any case, the advantage arising from considering the uncertainties in the boundary conditions or in the joints only in the reduced model provides a significant reduction of the number of full models to be evaluated.

It is important to note that, if the boundary conditions of a substructure are uncertain, it is generally convenient to exclude them from the model of the substructure itself. So that they can be considered in the reduced model of the system only, leading to an improvement of the overall performance even if this requires an increase of the attachment modes necessary to describe the substructure.

Final considerations deal with the error related to the reduced base of the model. Since CMS techniques are well-developed methodologies, this error can be usually bound within 2% of the eigenvalues, and it is of negligible effect on the eigenvectors.[5] It is also important to notice that, in general, when dealing with an uncertain problem, the variation of the eigenvalues due to variations of the input parameters is much larger than their variation induced by reducing the modal base. In other words, the use of the CMTM adds to the studied case another source of uncertainty that must have a negligible effect on the results with respect to the other sources of uncertainty.

To conclude it is important to notice that, like the transformation method, the CMTM basically consists of a pre-processor and a post-processor of the data, and can be easily integrated in any commercial software for finite element analysis.

### 2.5. The computational gain: preliminary analysis of the model

The CMTM can improve consistently the computational performance with respect to the standard transformation method. However, both the computational advantage and the accuracy of the results are very "*case dependent*". For this reason, it is not possible to derive a general expression for the computational gain because it depends on several characteristics of both the model and the analysis to be performed.

An important advantage of the proposed method is that it allows for an a-priory accurate estimation of the time needed by the whole analysis and, moreover, of the computational advantage that can be reached. Also, the overall accuracy of the method can be forecasted with this preliminary analysis that requires a rather low computational time.

[5]The accuracy of CMS technique are extremely "*case dependent*". However, the fact that CMS techniques are well established, allows an expert user of these methods to choose wisely which part of the system can be modeled with a superelement, the number of dof necessary to describe correctly the global modes of the system, and which is the specific CMS technique to be adopted in order to maximize the efficiency and minimize the errors. In this context it is possible to state that 2% is a reasonable bound for the error for the proposed method. In order to have a case-dependent forecast of both the accuracy and the performance of the CMTM it is possible to perform a preliminary analysis that is described in Section 2.5.

Once the substructuring strategy is defined, the preliminary analysis consists of the following computations performed for the nominal value for all uncertain parameters:

(1) *Solution of the overall full model*: The computational time needed by this operation is $C_f$.
(2) *Extraction of the substructures reduced models*: The computational time for this operation performed on the $i$th substructure is $C_r^i$.
(3) *Solution of the overall reduced model*: This process requires the computation time $C_0$.

Performing these three operations requires the computational time of about two evaluations of the overall full model, and it is then generally negligible with respect to the time necessary to solve the whole uncertain analysis.

Comparing the results of solution 1 and 3, it is possible to estimate the error introduced by the reduction process.

Furthermore, by comparing the time of solution 1 and 3 it is possible to evaluate the asymptotic computational gain $G_a = C_f/C_0$. This is the maximum gain possible, given the specific problem and the chosen substructuring strategy.

Moreover, called $N_i$ the number of reduced models needed for the $i$th substructure and $N$ the number of needed overall reduced models,[6] one has

$$T_c = \sum_i N_i C_r^i + NC_0 = T_{rt} + T_{r0} \text{ being } T_{rt} = \sum_i N_i C_r^i, \tag{1}$$

i.e. the computational cost of the CMTM $T_c$ depends on two factors: the time $T_{rt}$ required to generate the models of the substructure, and the time $T_{r0}$ required to solve the reduced model.

The computational gain is then

$$G = \frac{T_c}{NC_0} = \frac{T_c}{T_f}. \tag{2}$$

While the asymptotic computational gain is:

$$G_a = \lim_{N_i/N \to 0} G = \frac{C_0}{C_f}. \tag{3}$$

Eq. (1) shows that the computational cost of the CMTM is composed by two terms $T_{rt}$, and $T_{r0}$; it is important to notice that for a small number of fuzzy parameters, the first is prevalent, while the second prevails for a high number of fuzzy parameters (Fig. 6 shows an example of the time required for a solution of a problem both with the TM and the CMTM).

For a given analysis and a given number of $\alpha$-cuts, it is possible to define the critical number $N_c$ of fuzzy parameters as:

- the number of parameters for which the computational time needed to generate all the required *substructure reduced models* equals the computational time needed to evaluate all the *overall reduced models* (i.e. when $T_{rt} = T_{r0}$).

The critical number $N_c$ of fuzzy parameters depends on the analysis, the topological distribution of the uncertainties, the substructuring possibilities, the sampling method, i.e. the general transformation, the reduced transformation, etc., as well as the number of $\alpha$-cuts.

If the number of fuzzy parameters is greater than the critical number, the performance of the CMTM depends mainly on the computational performance of the reduced model, the computational gain approaches the asymptotic computational gain and the method is well suited and appropriate to solve this problem.

[6] Both $N_i$ and $N$ are known a priori when the number of $\alpha$-cuts is chosen. Their value depends also on the chosen sampling of the fuzzy number used (i.e. if the reduced or general transformation method is used).

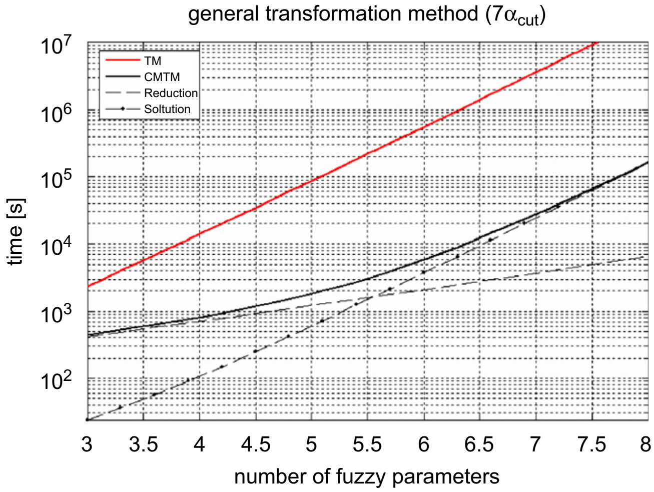


Fig. 6. Time needed to obtain an eigenvalue extraction of a 10,000 dof system characterized by two substructures (bold: full solution; black: reduced solution; dashed: time spent generating the reduced model of the substructures; dash-dot: time spent solving the reduced model).

If, on the other hand, the number of fuzzy parameters is smaller than the critical number, the preliminary analysis provides a forecast of the computational gain, based on which one can decide whether or not it is convenient to use the CMTM.

The aim of this section is to provide information on how the computational gain scales for varying number of fuzzy parameters, number of substructures, or number of alpha-cut used to discretize the fuzzy parameter.

For this purpose and given the high case dependency of the performance of the CMTM technique, let us make the following assumptions:

- If the model is characterized by $n$ fuzzy parameters, it is assumed that these parameters are equally distributed among the substructures and the joints (e.g. if there are six fuzzy parameters and two substructures, each substructure is characterized by two fuzzy parameters and the joints by the remaining two).
- The substructuring of the model does not provide any computational advantage for a crisp analysis. The computational cost for the analysis performed using the full model equals the one performed by substructuring (which includes the generation of the reduced models of the substructures, and the actual analysis performed on the reduced model).
- The computational cost of the solution performed on a reduced model is 1/100 of the cost of the same analysis performed on the full model.

The results are expressed in terms of the gain factor $G$.

Fig. 7 shows the gain factor as a function of the number of uncertain parameters and the number of $\alpha$-cuts for two substructures.

The results show that, if the reduced transformation method is used, the gain factor is mostly independent on the number of $\alpha$-cuts. On the contrary, if the general transformation method is used, there is always a dependence on the numbers of used cuts, and the gain factor increases if the number of cuts increases. However, for a high number of uncertain parameters or cuts, the gain factor stabilizes at 100 (that is the assumed ratio between the time needed for the standard analysis and the reduced one).

Fig. 8 shows the gain factor as a function of the number of uncertain parameters and the number of substructures, for 10 $\alpha$-cuts. Under the assumption made in this paragraph, a decomposition of the model into an increasing number of substructures always increases the performance of the CMTM; however, especially if the general transformation method is used, the advantage arising from an additional substructure becomes negligible as the number of parameters increases.

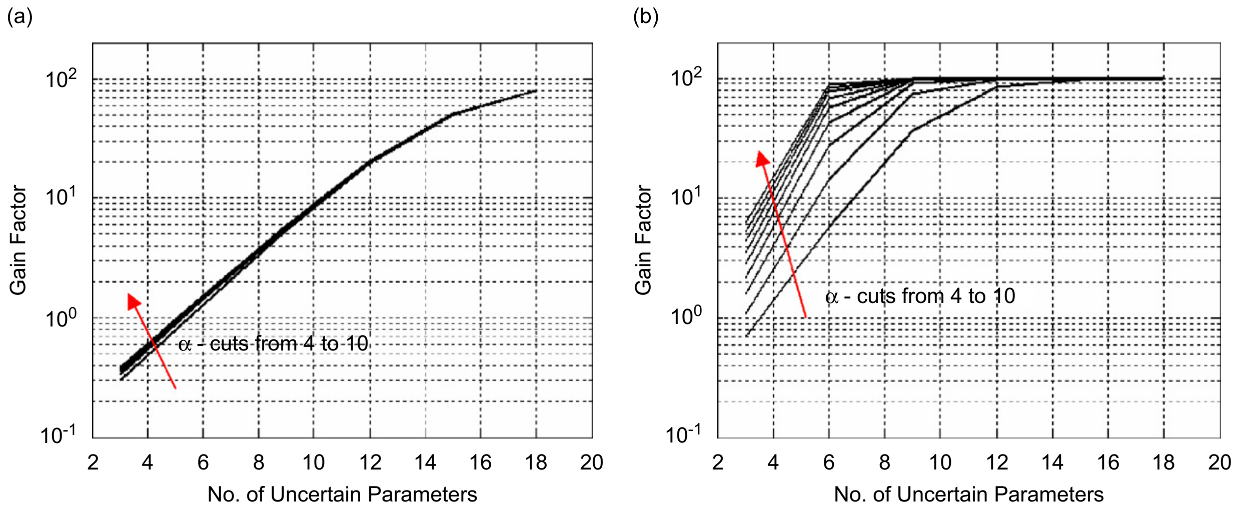


Fig. 7. Gain factor as a function of the number of uncertain parameters and the number of α-cuts: reduced transformation method (a) and general transformation method (b).

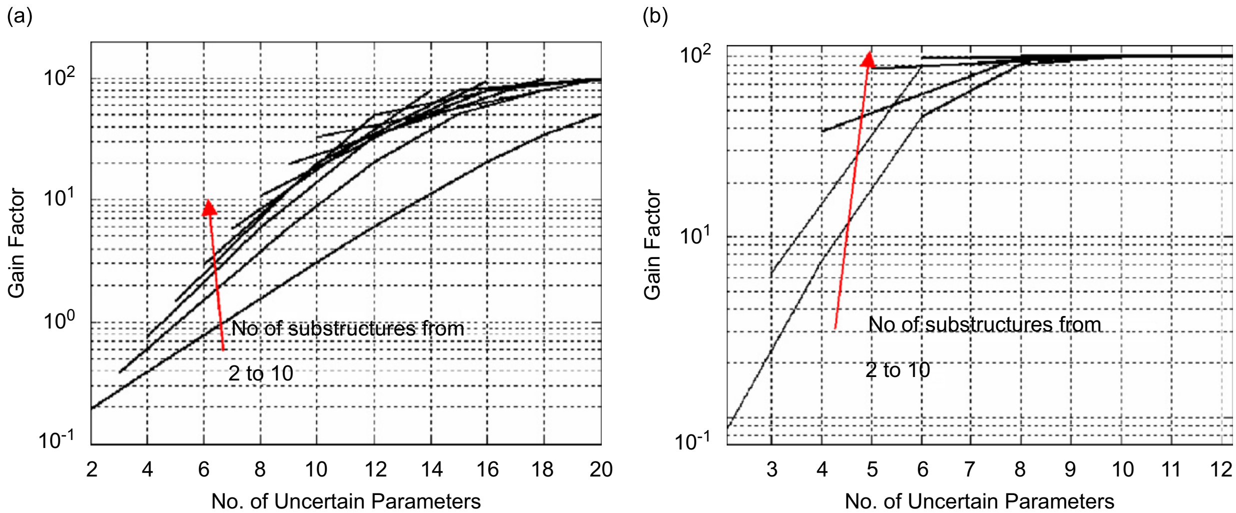


Fig. 8. Gain factor as a function of the number of uncertain parameters and the number of substructures: reduced transformation method (a) and general transformation method (b).

## 3. Test case: hollow cantilever beam

### 3.1. Description of the model

In order to test the effectiveness of the proposed method, a simple test case is proposed. A hollow cantilever beam, with a rectangular cross section area of height $h$, width $s$, and of length $2l$, is considered (Fig. 9). The cantilever beam consists of two parts welded together, characterized by different geometry and material properties, each part of length $l$ and labeled as beam $a$ and beam $b$.

The constraint on one edge of beam $a$ is assumed to be not perfect and characterized by a stiffness $k_n$ in the normal direction and $k_t$ in the tangential direction (Fig. 9).

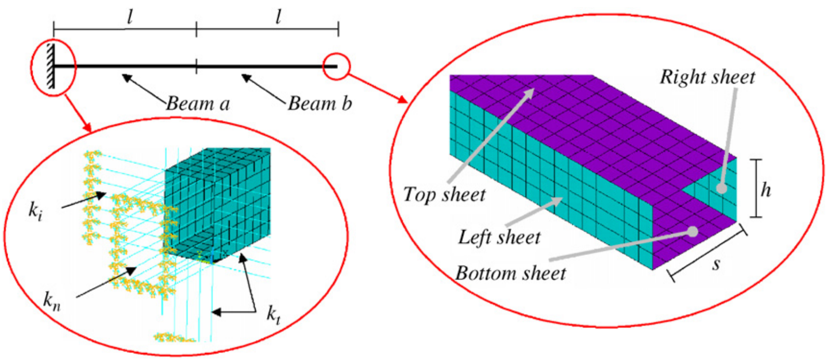


Fig. 9. Schematic of the test case FE model. Detail of the hollow structure and of the constraints.

Two cases are considered in the following:

*Case* 1: The system exhibits three independent uncertain parameters: the sheet thickness of beam $a$, the sheet thickness of beam $b$, and the stiffness $k_n = k_t$ of the constraint.
*Case* 2: The system exhibits six independent uncertain parameters: the thickness of the top and bottom sheets of beam $a$, the thickness of the side sheets of beam $a$, the thickness of the top and bottom sheets of beam $b$, the thickness of the side sheets of beam $b$, the stiffness $k_n$, and finally, the stiffness $k_t$.

For both cases, the first 60 eigenvalues of the system are considered as the output quantities.
The FEM model is characterized by plate elements with six dof per node for the sheets,[7] and linear springs are used to model the stiffness of the constraint on the left edge of the beam. The model can be subdivided into two substructures: the beams $a$ and $b$, each of length $l$. The substructuring method applied is given by the *CMS with fixed interface and constraint modes*, where 80 modes are used for each reduced model. Substructuring and calculation of the eigensolution are performed by the commercial software Ansys®, while the preprocessing of the models as well as the post-processing of the results is performed on the basis of MatLab®.

### 3.2. Case 1

For the numerical simulation of case 1, the following numerical values for the parameters are considered: each beam has length $l = 2$ m, while the cross section is qualified by $h = 30$ mm and $s = 20$ mm. The thickness parameters of the four sheets of beam $a$ and beam $b$ are quantified by triangular fuzzy numbers with a nominal value $t_1 = 2$ mm and a support $t_0$ ranging from 1.9 to 2.1 mm. The general transformation method is applied to sample these fuzzy numbers.
The constraint stiffnesses $k_p = k_t$ are modeled by triangular fuzzy numbers with a nominal value $k_1 = 5 \times 10^8$ N/m and a support ranging from $5 \times 10^7$ and $5 \times 10^9$ N/m. The general transformation method is used to sample these fuzzy numbers, and the sample points are exponentially spaced (Fig. 10).
The number of $\alpha$-cut used for the simulation is five.
Given these fuzzy parameters, 15 reduced models are needed for each substructure,[8] and 225 reduced model of the overall system[9] have to be evaluated to perform the uncertain analysis.
The number of modes used to generate the superelement of beam $a$ is 80, plus 240 dofs due to the attachment modes. For beam $b$, 80 modes plus 120 attachment dofs are considered.

[7]This problem can be solved with the same accuracy using a model with few beam elements. However, solving efficiently this problem is not the aim of this section. The test case was chosen to have an extremely simple dynamics, to have wavelength of the global modes quite larger than the mesh size, so that CMS technique are suitable to improve the computation of the solution, and having a number of dofs reasonably high in order to represent a possible real problem.
[8]One model for the nominal value and two to five for the other four $\alpha$-cuts, respectively.
[9]One model for the nominal values and 8, 27, 64, 125 for the other four $\alpha$-cuts, respectively.

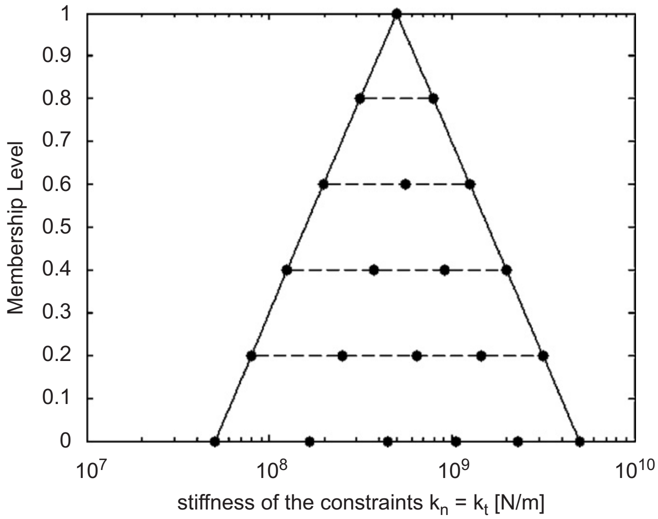


Fig. 10.

As a first step it is convenient to analyze the computational performance of the method in order to obtain an estimation of the expected accuracy and of the computational gain through the preliminary analysis.

The following operations are then computed for the nominal values of the uncertain parameters:

(1) Extraction of the first 60 eigenvalues of the full system model. This operation requires a computation time $C_f = 62\,\mathrm{s}$.
(2) Creation of the reduced models of the two substructures, which are computationally equal. This requires the computation time $C_r^i = 42\,\mathrm{s}$.
(3) Extraction of the first 60 eigenvalues of the reduced model. This process requires the computation time $C_0 = 2.5\,\mathrm{s}$.

Comparing the results of solution 1 and 3, it is possible to estimate the error introduced by the reduction process. For the present example, the maximum percentage error is negligible being less 0.01%.

Furthermore, by comparing the time of solution 1 and 3 it is possible to evaluate the asymptotic computational gain $G_a = C_f/C_0 = 24.8$. This is the maximum gain possible, given the specific problem and the chosen substructuring strategy.

Following the definitions provided in Section 2.5, the case 1 is characterized by few uncertain parameters. In fact, the time needed to obtain all the *substructure reduced models* is $T_{rt} = 1260\,\mathrm{s}$, which is greater than the time $T_{r0} = 562\,\mathrm{s}$, required for the solution of all the *overall reduced models*. Therefore, if it is necessary to further improve the performance, one should focus on the improvement of the computation time of the substructuring part.

Finally the expected computational gain is $G = 7.6$.

It is interesting to notice that, for a single eigensolution, the substructuring technique does not provide any computational advantage, while the computational time of the overall uncertain analysis can be reduced of a factor of 7.6.

### 3.3. Case 2

For the numerical simulation of case 2, the same values as in the previous case are assigned to the crisp parameters of the model, i.e. to $l$, $s$, $h$. The thicknesses of the sheets of the hollow beam are quantified by triangular fuzzy numbers with the nominal value $t_1 = 2\,\mathrm{mm}$ and the support $t_0$ ranging from 1.9 to 2.1 mm. The general transformation method is applied to sample the fuzzy numbers.

Table 1
Analysis of the asymptotic gain prediction

| | $N_1 = N_2$ | $N$ | $C_r^1 = C_r^2$ (s) | $C_0$ (s) | $G_a$ | $G$ |
|---|---|---|---|---|---|---|
| Test case 1 | 15 | 225 | 42 | 2.5 | 24.8 | 7.6 |
| Test case 2 (80 modes) | 55 | 20,515 | 42 | 2.5 | 24.8 | 22.7 |
| Test case 2 (50 modes) | 55 | 20,515 | 42 | 1.4 | 41.3 | 35.9 |

Table 2
Analysis of the asymptotic gain prediction

| CMS modes | Red. model dof | $C_0$ (s) | $G_a$ | Maximum error (%) |
|---|---|---|---|---|
| 80 | 400 | 2.5 | 24.8 | 0.01 |
| 50 | 340 | 1.5 | 41.3 | 0.02 |
| 30 | 300 | 1.4 | 44.2 | 0.8 |

Making up the difference to the previous case, the top and bottom sheets constituting the beam have now a thickness that is different from the thickness of the side sheets, leading to four independent fuzzy numbers.

The constraint stiffnesses $k_p$ and $k_t$ are modeled, as before, using triangular fuzzy numbers, with a modal value $k_1 = 5 \times 10^8$ N/m and a support ranging from $5 \times 10^7$ and $5 \times 10^9$ N/m. The general transformation method is used to sample these fuzzy numbers and the sample points are exponentially spaced. In this case, however, the two stiffness are considered independent.

The numbers of $\alpha$-cut used for the simulation is five.

Given these fuzzy parameters, 55 reduced models are needed[10] for each substructure, and 20,515 reduced models of the overall system have to be evaluated to perform the uncertain analysis.[11] For this case, the preliminary analysis provides the same computational performance, leading to the following conclusions:

- The asymptotic computational gain remains the same: $G_a = C_f/C_0 = 24.8$.
- In this case, the problem is characterized by a high number of uncertain parameters, with $T_{rt} = 4620\,\mathrm{s} > T_{r0} = 5.1 \times 10^4\,\mathrm{s}$. For this reason, it is highly recommended to improve the performance of the reduced models.
- The expected computational gain is $G = 22.7$, which is very close to the asymptotic gain of $G_a = 24.8$.

In this case it is recommended to perform an additional preliminary analysis, reducing the size of the modal base to improve the overall performance of the method. Tables 1 and 2 provide the results of this analysis.

Since the number of dof of the overall reduced model mainly depends on the number of attachment modes, that cannot be reduced by reducing the number of modes used in the CMS, it is not recommended to use less than 50 modes for each substructure. In this way, the expected computational gain is $G = 35.9$.

### 3.4. Results and discussion

After a rather fast post-processing phase in which the fuzzy outputs are calculated on the basis of the crisp outputs provided by the FEM analysis, the proposed method provides a fuzzy-valued representation of the output quantities. For the test case 2, a fuzzy FRF (Fig. 11) and the fuzzy eigenvalues (Fig. 12 shows the lowest seven eigenvalues) are given and compared with those obtained using the standard TM. The results of test case 1 are qualitatively similar and numerically very close.

[10] One model for the nominal value and 4, 9, 16, 25 for the other four $\alpha$-cuts, respectively for a total number of reduced models $N_i = 55$.

[11] One model for the nominal value and 64, 729, 4096, 15625 for the other four $\alpha$-cuts, respectively, for a total number of reduced models $N = 20{,}515$.

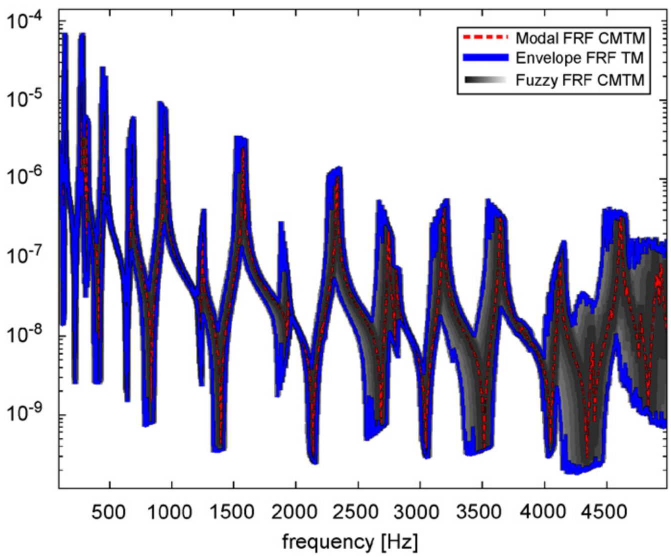


Fig. 11. Comparison between the fuzzy FRF computed with the component mode transformation method and envelope FRF computed with the transformation method of the beam (case 2) between 0 and 5 kHz.

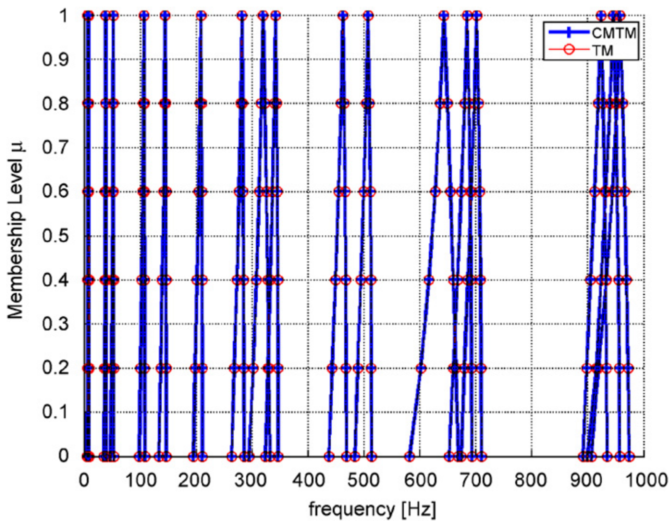


Fig. 12. Comparison between the component mode transformation method and the transformation method: fuzzy eigenvalues (case 2).

For both cases, the effective computation time complies with the expected one, given by the preliminary analysis. However, it is important to note that, in general, this is not true, because the size of both the substructure models and the overall reduced models may change depending on the $\alpha$-cuts. For this reason, the preliminary analysis generally provides only a tendency of the computational trend. Furthermore, it is important to note that the effective time needed for the simulation may exceed the computational time significantly, due to several disc write/read requests necessary to the CMS approach. Finally, the time needed for the pre- and post-processing phase is rather negligible with respect to the FE computation time.

## 4. Conclusions

For the analysis of the propagation of the uncertainties inherent to the model parameters of mechanical structures, a new method, the component mode transformation method, is presented, and its computational merits are analyzed and discussed. The method couples the capabilities of the standard transformation method, which provides a practical and reliable implementation of fuzzy arithmetic for the solution of *large* uncertain mechanical problems, with the computational advantages of the CMS approach. Since the transformation method requires a rather large number of model evaluations for problems with a high number of uncertain parameters, its combination with the CMS approach proves to be very promising, leading to a major reduction of the overall computational effort.

The analysis of the computational costs shows that the gain factor may be relevant in the case of a high number of uncertain parameters. For a low number of fuzzy parameters, instead, the gain factor is lower, but it increases with the number of parameters or the number of substructures. Furthermore, the method allows also for a preliminary analysis that provides reliable expectations on the computational gain as well as the errors related to the proposed procedures; the preliminary analysis requires the costs of only about two evaluations of the model.

The original contribution of this study relies on two major capabilities of the proposed approach that are: the possibility of easily integrating an uncertain analysis in standard FEM programs, because it consist in a appropriate pre-processing of the input parameters and post-processing of the outputs; and the possibility of directly propagating the uncertainties in the physical and geometrical parameters of the model to the desired outputs.

## Acknowledgment

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## Appendix A. The component mode synthesis technique

The well-assessed technique of the component mode synthesis (CMS) method reaches two major goals: First of all, it can reduce the computational time necessary to evaluate a large dynamic model by reducing its size; second, it allows to study and optimize each component of the system independently form the others.

For these reasons CMS is nowadays integrated in most of the commercial FE programs and it is largely used in the design process of industrial cases.

CMS technique consists of three steps that are mathematically described in the following:

(1) The structure is divided into a residual structure and a number of substructures. For each substructure a set of modes is computed, these modes reproduce its static and dynamic behavior.
(2) The modes of each substructures are used to reduce the size of each substructure.
(3) Finally, the reduced models of each substructure are connected together into the overall reduced model of the system.

### *A.1. Calculation of component modes*

After a substructuring strategy is defined, the dofs of each substructure are divided among two sets: one set includes all the dofs that are not connected to other substructures, these are the internal dof (subscript $i$). The second set includes the dofs that are connected either to the residual structure or to another substructure (subscript $c$).

As a result, the mass and stiffness matrices are partitioned as follows:

$$\mathbf{K} = \begin{bmatrix} \mathbf{K}_{cc} & \mathbf{K}_{ci} \\ \mathbf{K}_{ic} & \mathbf{K}_{ii} \end{bmatrix}, \quad \mathbf{M} = \begin{bmatrix} \mathbf{M}_{cc} & \mathbf{M}_{ci} \\ \mathbf{M}_{ic} & \mathbf{M}_{ii} \end{bmatrix}, \quad \mathbf{u} = \begin{bmatrix} \mathbf{u}_c \\ \mathbf{u}_i \end{bmatrix}. \tag{4}$$

For each component, a set of component modes is calculated.

In literature, several approaches using different types of component modes for static and dynamic reduction are described. In the following, the Craig–Bampton method [33] that uses constraint modes and fixed-interface normal modes to describe the static and the dynamic behavior of a component is presented.

The constraint mode is the static deformation of the substructure when a unit displacement (or rotation) is applied to one of the connected dof, while the other connection dof are constrained. The static transformation matrix $[\mathbf{G}_{ic}]$ with all constraint modes is

$$\mathbf{G}_{ic} = [\{\boldsymbol{\phi}_1^C\}\{\boldsymbol{\phi}_2^C\} \quad \ldots \quad \{\boldsymbol{\phi}_{N_c}^C\}] = \mathbf{K}_{ii}^{-1}\mathbf{K}_{ic}, \tag{5}$$

where $N_c$ is the number of connected dof, and $\boldsymbol{\Phi}_i{}^C$ is the constraint mode when the $i$th dof has unitary displacement.

The fixed interface normal mode matrix has the form

$$\mathbf{G}_{iq} = [\{\boldsymbol{\phi}_1\}\{\boldsymbol{\phi}_2\} \quad \ldots \quad \{\boldsymbol{\phi}_{N_q}\}], \tag{6}$$

where the normal modes $\boldsymbol{\Phi}_i$ are the eigenvectors obtained by the following equation:

$$\mathbf{K}_{ii}\boldsymbol{\phi}_q = \lambda_q \mathbf{M}_{ii}\boldsymbol{\phi}_q. \tag{7}$$

The reduction of the size of the model is obtained by truncating the number of considered normal modes $N_{q.}$ Also the accuracy of the method depends on the number of considered normal modes $N_q$ that is usually much smaller than the internal dof $u_i$.

### A.2. Reduction of the substructure full model

The reduction of the substructure model is obtained through the following coordinates transformation:

$$\begin{aligned} \mathbf{K}_{\text{red}} &= \mathbf{G}^t\mathbf{K}\mathbf{G} = \begin{bmatrix} \mathbf{K}_{cc} + \mathbf{K}_{co}\mathbf{G}_{iq} & 0 \\ 0 & \mathbf{G}_{iq}^t\mathbf{K}_{ii}\mathbf{G}_{iq} \end{bmatrix}, \\ \mathbf{M}_{\text{red}} &= \mathbf{G}^t\mathbf{M}\mathbf{G}, \end{aligned} \tag{8}$$

where for the case of constraint modes and fixed-interface normal modes,

$$\mathbf{G} = \begin{bmatrix} \mathbf{I} & 0 \\ \mathbf{G}_{ic} & \mathbf{G}_{iq} \end{bmatrix}. \tag{9}$$

The matrices $\mathbf{K}_{\text{red}}$ and $\mathbf{M}_{\text{red}}$ express the dynamic behavior of the substructure as seen from its connection to the rest of the system.

### A.3. Assembly of the overall reduced model

Once the reduction of each component model is carried out, the reduced stiffness and mass matrices of all components are assembled with the non-reduced part of the structure, to form the reduced stiffness and reduced mass matrix of the overall reduced model.

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