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Short Communication

On the dynamic stability of a cantilever under tangential follower force according to Timoshenko beam theory

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Abstract

The dynamic stability of a cantilevered Timoshenko beam lying on an elastic foundation of Winkler type and subjected to a tangential follower force is studied. Two models describing this phenomenon are examined and their predictions are compared in several special cases. For the values of the beam parameters considered here, the critical compressive forces obtained using these models differ substantially only for short beams as has already been established in other cases. Both models are found to predict dynamic instability of cantilevers under tension unlike the Bernoulli–Euler beam theory. For a beam of intermediate slenderness the Winkler foundation is found to reduce the critical tensile force.

1. Introduction

The dynamic stability of a cantilever subjected to a tangential follower force within the Bernoulli–Euler beam theory has been studied by Beck [1] who found that there exists a value of the force defined as critical above which this beam loses stability by flutter—vibrations of increasing amplitudes.

To the best of our knowledge, the first studies in which the Timoshenko beam theory has been extended to account for the influence of an axial force are reported by Kolousek [2] and Nemat-Nasser [3]. However, these two authors get to different equations of motion, which in the present paper are referred to as Models I and II, respectively. At a first glance, it seems that this difference arises from the different approaches used in Refs. [2,3] to the derivation of the equations of motion in question. Indeed, Kolousek [2] employs the dynamic equilibrium conditions of a beam element, whereas Nemat-Nasser [3] bases his analysis on variational arguments. Later, however, Kounadis [4] shows that the two models can be derived by balancing the forces and momenta acting on a small element of the beam, and Sato [5] shows that both models can also be derived by Hamilton's principle. For this reason, there are no theoretical arguments for giving preference to Models I or II in the analysis of the dynamic behaviour of axially loaded Timoshenko beams.

In Refs. [3,6], it is found, within the framework of Models II and I, respectively, that a cantilever subjected to a compressive tangential follower force can be unstable, and the critical force in both cases depends on the slenderness of the beam and is less than the critical force of the Bernoulli–Euler beam.

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Kounadis [4] was the first to point out that the critical forces of Timoshenko beams obtained by the two models must be compared to find out whether they predict different results. In a subsequent study Katsikadelis and Kounadis [7] find that for cantilevered beams of shear correction factor $\kappa = 0.186$ both models predict very close values of the critical force for a wide range of beam slenderness ratio. Later, many authors, for instance Lee et al. [8,9], Malekzadeh et al. [10] studied the dynamic stability of Timoshenko beams applying one of the foregoing two models and additionally accounting for different effects—Winkler foundation, partially tangential follower force, concentrated masses, elastically supported ends, intermediate supports, etc. However, we do not know studies, other than [7], where a comparison of the results obtained using the two models is reported. Let us also recall that in all foregoing studies the dynamic stability of Timoshenko beams subjected to a compressive force is considered.

In the present paper, the case of Timoshenko beams subjected to tensile tangential follower forces is investigated. The aim is to compare the results of the two models, to determine whether the beam equilibrium loses stability and to study the influence of Winkler foundation on the beam behaviour.

2. Boundary-value problem

Consider a uniform elastic cantilevered beam of length L, cross-section area A, inertia moment of the cross-section I, resting on a Winkler foundation of modulus c and subjected to an axial force N at the free end, which is always normal to the end cross-section (a follower force[1]). In what follows, the force N is assumed to be positive for tension. Within the framework of the aforementioned two models, the vibration of the beam is governed by the following system of differential equations:

$$EI\frac{\partial^2\Theta}{\partial X^2} + (\kappa GA - \varepsilon_1 N)\left(\frac{\partial W}{\partial X} - \Theta\right) = \rho I\frac{\partial^2\Theta}{\partial t^2},$$

$$\kappa AG\left(\frac{\partial^2 W}{\partial X^2} - \frac{\partial\Theta}{\partial X}\right) + \varepsilon_2 N\frac{\partial^2 W}{\partial X^2} + \varepsilon_1 N\frac{\partial\Theta}{\partial X} - cW = \rho A\frac{\partial^2 W}{\partial t^2}, \tag{1}$$

where the case $\varepsilon_1 = 1$, $\varepsilon_2 = 0$ corresponds to Model I, and the case $\varepsilon_1 = 0$, $\varepsilon_2 = 1$ corresponds to Model II. Here, X is the axial coordinate along the beam axis, t is the time, $W(X,t)$ is the transverse deflection of the beam axis, $\Theta(X,t)$ is the rotation angle of the cross section, E, G and κ are Young's modulus, the shear modulus and the shear coefficient, respectively, and ρ is the mass density of the beam.

Eqs. (1) together with an appropriate set of boundary conditions describe entirely the dynamic behaviour of the considered beam. The boundary conditions for a cantilevered Timoshenko beam subjected to a tangential follower force are

$$W|_{X=0} = 0, \quad \Theta|_{X=0} = 0, \quad \left.\frac{\partial\Theta}{\partial X}\right|_{X=L} = 0, \quad \left.\left(\frac{\partial W}{\partial X} - \Theta\right)\right|_{X=L} = 0. \tag{2}$$

Using the dimensionless variables

$$x = \frac{1}{L}X, \quad \tau = t\sqrt{\frac{EI}{L^4\rho A}}, \quad w = \frac{1}{L}W$$

introducing the parameters

$$\beta = \frac{\kappa}{2(1+\nu)}, \quad \lambda = \frac{I}{L^2 A}, \quad K = \frac{L^2}{AE}c, \quad P = \frac{1}{AE}N,$$

where ν is Poisson's ratio, taking into account the relation $G = E/[2(1+\nu)]$, and separating the variables in the form

$$w = u(x)\exp(\mathrm{i}\omega\tau), \quad \Theta = \theta(x)\exp(\mathrm{i}\omega\tau),$$

[1] For a detail discussion on the notion *follower force* see the exhaustive survey by Elishakoff [11].

Eqs. (1) and conditions (2) transform to the two-point boundary value problem

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}x^2}+\frac{\beta-\varepsilon_1 P}{\lambda}\left(\frac{\mathrm{d}u}{\mathrm{d}x}-\theta\right)+\lambda\omega^2\theta=0,$$

$$\frac{\mathrm{d}^2u}{\mathrm{d}x^2}-\frac{\beta-\varepsilon_1 P}{\beta+\varepsilon_2 P}\frac{\mathrm{d}\theta}{\mathrm{d}x}+\frac{\lambda\omega^2-K}{\beta+\varepsilon_2 P}u=0, \tag{3}$$

$$u|_{x=0}=0,\quad \theta|_{x=0}=0,\quad \left.\frac{\mathrm{d}\theta}{\mathrm{d}x}\right|_{x=1}=0,\quad \left.\left(\frac{\mathrm{d}u}{\mathrm{d}x}-\theta\right)\right|_{x=1}=0. \tag{4}$$

Actually, this constitutes a non-self-adjoint eigenvalue problem, the eigenvalue parameter being the dimensionless frequency ω.

The general solution of Eqs. (3) can be written in the form

$$u=C_1\cosh(a_1x)-C_2\sinh(a_2x)+C_3\sinh(a_1x)+C_4\cosh(a_2x),$$

$$\theta=C_1b_1\sinh(a_1x)+C_2b_2\cosh(a_2x)+C_3b_1\cosh(a_1x)-C_4b_2\sinh(a_2x), \tag{5}$$

where C_i $(i=1,\ldots,4)$ are arbitrary complex numbers,

$$a_1=\frac{1}{2}\sqrt{-2a-2\sqrt{a^2-4b}},\quad a_2=\frac{1}{2}\sqrt{-2a+2\sqrt{a^2-4b}},$$

$$b_1=(a_1^2+\gamma)\frac{\beta+\varepsilon_2 P}{a_1(\beta-\varepsilon_1 P)},\quad b_2=-(a_2^2+\gamma)\frac{\beta+\varepsilon_2 P}{a_2(\beta-\varepsilon_1 P)},$$

$$a=\gamma+\lambda\omega^2-\frac{P(\beta^2-\varepsilon_1 P^2)}{\lambda\beta(\beta+P)},\quad b=\gamma\left(\lambda\omega^2-\frac{\beta-\varepsilon_1 P}{\lambda}\right),\quad \gamma=\frac{\lambda\omega^2-K}{\beta+\varepsilon_2 P}.$$

Substituting solution (5) in the boundary conditions (4) one obtains a linear homogeneous system for the unknown constants C_i. The condition for existence of a non-trivial solution to this system can be written as

$$\begin{aligned}D\equiv{}&[a_2b_2^2(b_1-a_1)-a_1b_1^2(a_2+b_2)]\cosh a_1\cosh a_2\\&+[a_2b_1^2b_2-a_1b_1b_2(2a_2+b_2)]\sinh a_1\sinh a_2\\&-b_1b_2(a_1^2+a_2^2+b_2a_2-b_1a_1)=0\end{aligned} \tag{6}$$

and the solution is

$$C_2=\frac{a_1b_1\cosh a_1+a_2b_2\cosh a_2}{a_1b_2\sinh a_1-a_2b_2\sinh a_2}C_1,\quad C_3=-\frac{b_2}{b_1}C_2,\quad C_4=-C_1,$$

where C_1 is an arbitrary complex number.

Consequently, for a given set of the beam parameters λ, β, K and P, the eigenfrequencies ω are determined as the solutions of Eq. (6). The critical force P_{cr} is determined as the lowest value of P at which Eq. (6) has a solution with negative imaginary part corresponding to a non-zero solution (u,θ) to the eigenvalue problem (3), (4), the rest of the beam parameters being kept fixed.

In general, given the parameters β, λ and K, the critical force could be obtained computing the evolution of the eigenfrequencies. To obtain the evolution of an eigenfrequency, one can start from the value of this frequency at $P=0$ and increasing the force by a step ΔP to determine a sequence of solutions to the frequency equation (6). We accomplished this algorithm using the routine *FindRoot* in *Mathematica* Version 5 (see Ref. [12], Sec. 1.5.7). Our observation is that a frequency of negative imaginary part appears after a coalescence of two neighbour eigenfrequencies. Another observation is that the step ΔP should be sufficiently small (in some computations $\Delta P{\sim}10^{-5}$) in order to obtain accurate results. This leads to lengthy computations especially when one has to vary the foundation modulus K as well. Therefore, another computational procedure is developed here, based on the following arguments. Given the parameters β, λ and K, Eq. (6) defines ω as an implicit function of P so that Eq. (6) can be written in the form $D(\omega(P),P)=0$. Differentiating with respect to P one obtains

$$\frac{\mathrm{d}}{\mathrm{d}P}D(\omega(P),P)=\frac{\partial D}{\partial P}+\frac{\partial D}{\partial \omega}\frac{\mathrm{d}\omega}{\mathrm{d}P}=0, \tag{7}$$

which is a nonlinear ordinary differential equation for the function $\omega(P)$. Thus, the evolution of an eigenfrequency can be obtained solving this equation with an initial condition

$$\omega(P)|_{P=0}=\omega_0,$$

where ω_0 is any of the eigenfrequencies of the cantilevered beam with $P=0$. We used the routine *NDSolve* in *Mathematica* Version 5 (see Ref. [12], Sec. 1.6.4) to solve numerically Eq. (7). The results, presented in the next section are obtained using both algorithms.

3. Results and discussion

In the case studies presented below, Timoshenko beams of rectangular cross section with shear coefficient $\kappa=5/6$ and Poisson's ratio $v=0.3$ are considered. For convenience, a new slenderness parameter $\mu=\sqrt{12\lambda}$ is introduced and beams with values of μ between 0.01 and 0.30 are regarded.

First, the case of a compressive force is studied. The results of our computations are presented in Table 1. It is seen that for small values of the slenderness parameter μ, the predictions of the two models coincide while for high values of μ Model II predicts lower critical compressive forces.

As mentioned above, Nemat-Nasser [3] and Lee et al. [8] studied the dynamic stability of cantilevered Timoshenko beams but they gave only graphs of the critical forces. In order to compare our results to theirs, the numeric values of the critical forces are recovered from the graphs in Refs. [3,8] for several values of the slenderness parameter μ. Lee et al. [9] also investigated the dynamic stability of cantilevered Timoshenko beams using Model II and presented numeric data for the critical forces. The comparison of our results to the results in these three papers is shown in Table 2. It is seen, that our results for the critical forces achieved through Model II agree within 7% with the results of Nemat-Nasser [3] and are in excellent agreement with the results of Lee et al. [9]. As far as the critical forces achieved through Model I are concerned, our results agree within 3% with those, given in Ref. [8].

Next, the case of a tensile force is considered. It turned out that in this case both models also predict dynamic instability of cantilevered beams for the whole range of the slenderness parameter μ under consideration unlike the Bernoulli–Euler beam theory.

Table 1
Dimensionless critical compressive forces $P_{cr}\times 10^4$ for cantilevered Timoshenko beams without foundation—comparison of the two models

μ	0.01	0.02	0.03	0.04	0.05	0.10	0.15	0.20	0.25	0.30
$P_{cr}\times 10^4$ (Model I)	1.67	6.66	14.9	26.4	40.9	154	319	513	722	936
$P_{cr}\times 10^4$ (Model II)	1.67	6.66	14.9	26.4	40.9	154	314	497	681	855

Table 2
Dimensionless critical compressive forces $P_{cr}\times 10^4$ for cantilevered Timoshenko beams without foundation—comparison to results by other authors

μ	$\frac{2\sqrt{3}}{50}$	$\frac{2\sqrt{3}}{50}$	$\frac{2\sqrt{3}}{25}$	$\frac{2\sqrt{3}}{15}$	$\frac{2\sqrt{3}}{10}$
β	0.25	0.3269	0.3269	0.3269	0.3269
Present work	76.2	77.0	278.1	643.9	1010.3
Ref. [3]	71.0	–	–	–	–
Ref. [8]	–	76.0	272.0	622.0	–
Ref. [9]	–	77.0	–	–	1010.0

For Model I, it is found that the critical tensile force P_{cr} is slightly greater than the value of the parameter β, see Table 3. For each value of the slenderness parameter μ, the instability occurs after coalescence of the fourth and fifth eigenfrequencies.

As for Model II, the evolution of the eigenfrequencies of the beam of slenderness $\mu = 0.10$ is presented in Fig. 1. It is found that the first eigenfrequency decreases while the eigenfrequencies from second to tenth increase with the increasing of the tensile force but, as seen in Fig. 1(a), they remain real for values of P up to 0.382. The eleventh and twelfth eigenfrequencies also increase, but they coincide at the critical force $P_{cr} = 0.3817$ as it is seen in Fig. 1(b). Beyond P_{cr} a complex eigenfrequency of real part 721 and a negative imaginary part appears indicating the dynamic instability of the beam. Timoshenko beams with other values of μ possess stability features that are similar to the case $\mu = 0.10$. The critical forces, real parts and serial numbers n and m of the merging eigenfrequencies are presented in Table 4.

Finally, the dynamic stability of a Timoshenko beam of slenderness $\mu = 0.10$ resting on a Winkler foundation is studied for values of the dimensionless foundation modulus K up to 120. For both models, it is found that there exist regions of parameters (K, P) where the beam is unstable. These regions are shown in Fig. 2(a) and (b) for Models I and II, respectively. The lowest critical forces can be observed in the regions RI and RII. They are found to be $P_{cr} = 0.0134$ achieved at $K = 79$ and $P_{cr} = 0.0133$ achieved at $K = 78$, for Models I and II, respectively. These critical forces are smaller than the compressive critical force for the same beam without foundation, which is 0.0154 for both models (see Table 1). It is noteworthy that the regions RI and RII almost coincide.

Table 3
Dimensionless critical tensile forces $P_{cr} \times 10^4$ and real parts $\mathrm{Re}(\omega)$ of the coalescing eigenfrequencies for cantilevered Timoshenko beams of slenderness μ without foundation according to Model I

μ	0.01	0.02	0.03	0.04	0.05	0.10	0.15	0.20	0.25	0.30
$P_{cr} \times 10^4$	3205	3205	3205	3206	3206	3208	3212	3217	3224	3233
$\mathrm{Re}(\omega)$	1580	790	527	395	316	158	105	79	63	53

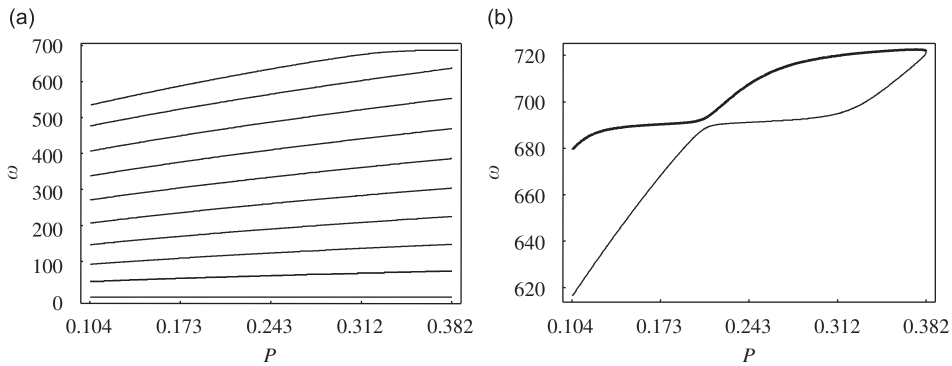

Fig. 1. Evolution of the dimensionless eigenfrequencies with the dimensionless force P of a Timoshenko beam with $K = 0$ and $\mu = 0.10$ according to Model II: (a) first ten eigenfrequencies; (b) eleventh (thin line) and twelfth (thick line) eigenfrequencies.

Table 4
Dimensionless critical tensile forces $P_{cr} \times 10^4$, real parts $\mathrm{Re}(\omega)$ and serial numbers n, m of the coalescing eigenfrequencies for cantilevered Timoshenko beams of slenderness μ without foundation according to Model II

μ	0.01	0.02	0.03	0.04	0.05	0.10	0.15	0.20	0.25	0.30
$P_{cr} \times 10^4$	5706	5709	4102	3472	4096	3817	3541	4131	7711	6210
$\mathrm{Re}(\omega)$	67946	16992	7558	4257	2759	721	312	208	213	168
n, m	77, 78	39, 40	29, 30	23, 24	19, 20	11, 12	7, 8	7, 8	9, 10	9, 10

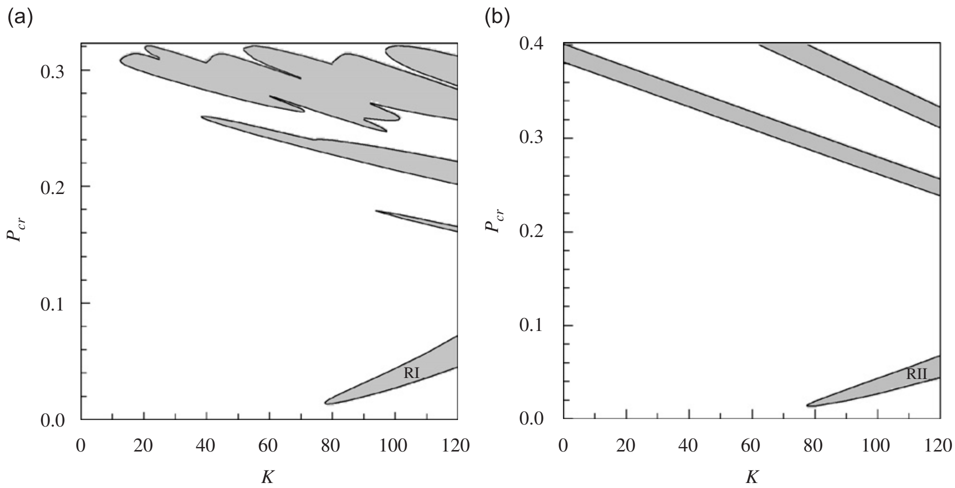

Fig. 2. Instability regions (shaded) of a cantilevered Timoshenko beam with $\mu = 0.10$: (a) Model I; (b) Model II.

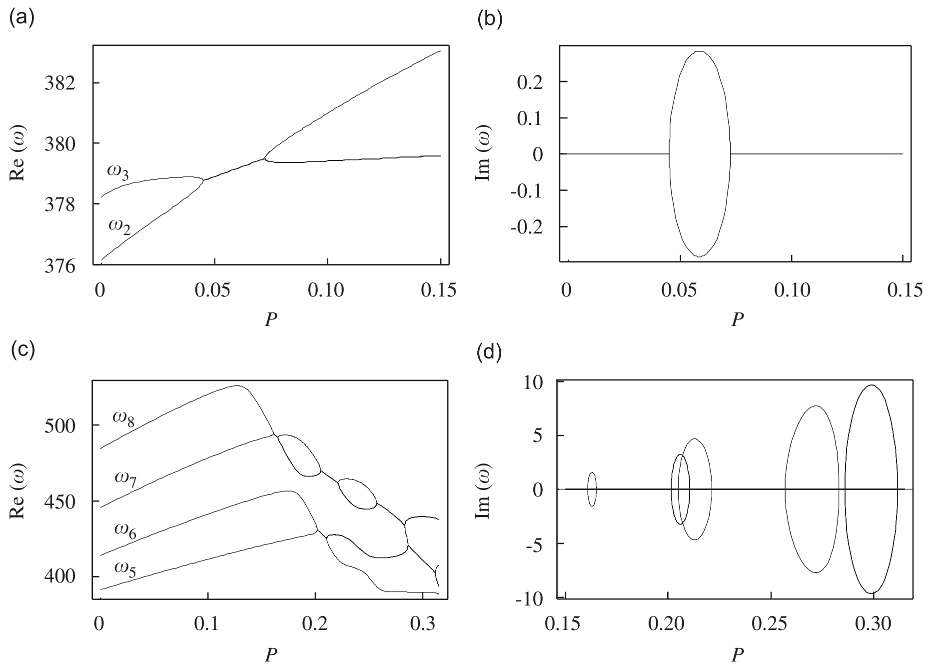

Fig. 3. Evolution of the real and imaginary parts of the dimensionless eigenfrequencies of a cantilevered Timoshenko beam with $\mu = 0.10$ and $K = 120$ responsible for the occurrence of instability according to Model I.

This complicated picture of instability regions is due to the fact that multiple instability–restabilization–instability transitions occur with the increasing of the force P. For instance, the evolution of the eigenfrequencies responsible for the occurrence of the instability within the framework of Model I for $K = 120$, $P < 0.32$ is shown in Fig. 3. The instability region RI depicted in Fig. 2(a) is related to the evolution of eigenfrequencies ω_2 and ω_3. It is seen in Fig. 3(a) and (b) that these eigenfrequencies are real for values of the tensile force up to $P_{cr} = 0.045$ where they coalesce giving rise to a pair of complex conjugate eigenfrequencies. At $P = 0.0725$ these two complex eigenfrequencies coalesce again and evolve into a pair of eigenfrequencies that are real for values of P up to 0.32. The instability regions above RI depicted in Fig. 2(a) are related to the evolution of eigenfrequencies ω_5, ω_6, ω_7 and ω_8 shown in Fig. 3(c) and (d). Their evolution is similar to the evolution of ω_2 and ω_3. Several coalescences are seen in Fig. 3(c) and (d) indicating that the beam undergoes a sequence of instability–restabilization–instability transitions. Observing Fig. 3(b) and (d) one can easily identify the instability regions. They correspond to the intervals of P in which the imaginary parts of the eigenfrequencies are non-zero.

Thus, it is found that the Timoshenko beam theory based on Eqs. (1) predicts dynamic instability of cantilevers under tension unlike the Bernoulli–Euler theory. The Winkler foundation is found to destabilize a Timoshenko beam of slenderness $\mu = 0.10$ for values of K up to 120 in the sense that the critical forces for $0 < K < 120$ are less than the critical force at $K = 0$.

Finally, we would like to stress that although the two models considered are found to give different predictions for the critical forces and corresponding eigenfrequencies especially for short beams and beams of intermediate slenderness, this purely theoretical result is not sufficient to judge which of the two models describes more adequately the dynamic stability of Timoshenko cantilevers. This matter could be clarified through further experimental work.

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