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Short Communication

Solution of the relativistic (an)harmonic oscillator using the harmonic balance method

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Abstract

The harmonic balance method is used to construct approximate frequency–amplitude relations and periodic solutions to the relativistic oscillator. By combining linearization of the governing equation with the harmonic balance method, we construct analytical approximations to the oscillation frequencies and periodic solutions for the oscillator. To solve the nonlinear differential equation, firstly we make a change of variable and secondly the differential equation is rewritten in a form that does not contain the square-root expression. The approximate frequencies obtained are valid for the complete range of oscillation amplitudes, A, while the discrepancy between the second approximate frequency and the exact one never exceeds 0.82% and tends to 0.52% when A tends to infinity. Excellent agreement of the approximate frequencies and periodic solutions with the exact ones are demonstrated and discussed.

The simple harmonic oscillator and its extension to the relativistic case are important in physics because they are usually employed as the basis for analyzing more complicated motion. When the energy of a simple harmonic oscillator is such that the velocities become relativistic, the simple harmonic motion (linear oscillations) at low energy becomes anharmonic (nonlinear oscillations) at high energy [1]; hence the parentheses around the "an" in the title of this paper. Then, the strength of the nonlinearity increases as the total relativistic energy increases, and at the non-relativistic limit the oscillator becomes linear. Synge [2] gave an exact expression for the period in terms of an integral that Goldstein [3] identified as being expressible in terms of elliptical integrals. Mickens [4] showed that all the solutions to the relativistic (an)harmonic oscillator are periodic and determined a method for calculating analytical approximations to its solutions. Mickens considered the first-order harmonic balance method, but we think he did not apply the technique correctly and the analytical approximate frequency he obtained is not the correct one. The purpose of this paper is to determine the high-order periodic solutions to the relativistic oscillator by applying the harmonic balance method [5–14]. To do this, we use the analytical approach developed by Lim and Wu [6,7] in which linearization of the governing equation is combined with the harmonic balance method.

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The governing non-dimensional nonlinear differential equation of motion for the relativistic oscillator is [4]

$$\frac{d^2x}{dt^2} + \left[1 - \left(\frac{dx}{dt}\right)^2\right]^{3/2} x = 0, \tag{1}$$

where x and t are dimensionless variables. The even power term in Eq. (1), $(dx/dt)^2$, acts like the powers of coordinates in that it does not cause a damping of the amplitude of oscillations with time. Therefore, Eq. (1) is an example of a generalized conservative system [5]. At the limit when $(dx/dt)^2 \ll 1$ Eq. (1) becomes $(d^2x/dt^2) + x \approx 0$ the oscillator is linear and the proper time τ becomes equivalent to the coordinate time t to this order.

Introducing the phase space variable (x,y), Eq. (1) can be written in the system form

$$\frac{dx}{dt} = y, \quad \frac{dy}{dt} = -(1-y^2)^{3/2}x \tag{2}$$

and the trajectories in phase space are given by solutions to the first order, ordinary differential equation

$$\frac{dy}{dx} = -\frac{(1-y^2)^{3/2}x}{y}. \tag{3}$$

As Mickens pointed out, since the physical solutions of both Eqs. (1) and (3) are real, the phase space has a "strip" structure [4], i.e.,

$$-\infty < x < +\infty, \quad -1 < y < +1. \tag{4}$$

Then unlike the usual non-relativistic harmonic oscillator, the relativistic oscillator is bounded in the y variable. This is due to the fact that the dimensionless variable y is related with the relativistic parameter $\beta = v/c$, where v is the velocity of the particle and c the velocity of light. In the relativistic case, the condition $-c < v(t) < +c$ must be met, and so we obtain $-1 < y(t) < +1$. Mickens proved that all the trajectories to Eq. (3) are closed in the open region of phase space given by Eq. (4) and then all the physical solutions to Eq. (1) are periodic. However, unlike the usual (non-relativistic) harmonic oscillator, the relativistic (an)harmonic oscillator contains higher-order multiples of the fundamental frequency [4].

The harmonic balance method can now be applied to obtain analytic approximations to the periodic solutions of Eq. (1). We make a change of variable, $y \to u$, such that

$$-\infty < u < +\infty. \tag{5}$$

The required transformation is [4]

$$y = \frac{u}{\sqrt{1+u^2}} \tag{6}$$

and the corresponding second-order nonlinear differential equation for u is

$$\frac{d^2u}{dt^2} + \frac{u}{\sqrt{1+u^2}} = 0. \tag{7}$$

We consider the following initial conditions in Eq. (7):

$$u(0) = B \quad \text{and} \quad \frac{du}{dt}(0) = 0. \tag{8}$$

Eq. (7) is an example of a conservative nonlinear oscillatory system in which the restoring force has an irrational form. All the motions corresponding to Eq. (7) are periodic and the system will oscillate within symmetric bounds $[-B, B]$.

We can solve Eq. (7) approximately using the harmonic balance method. To do this, we first write this equation in a form that does not contain the square-root expression

$$(1+u^2)\left(\frac{d^2u}{dt^2}\right)^2 = u^2. \tag{9}$$

Since the restoring force is an odd function of u, the periodic solution $u(t)$ has a Fourier series representation which contains only odd multiples of ωt. The first-order harmonic balance solution takes the form [5]

$$u_1(t) = B\cos\omega t. \tag{10}$$

Observe that $u_1(t)$ satisfies the initial conditions, Eq. (8). The angular frequency of the oscillator is ω, which is unknown and to be determined. Both the periodic solution $u(t)$ and frequency ω (thus period $T = 2\pi/\omega$) depend on B. Substituting Eq. (10) into Eq. (9) gives

$$(1 + B^2\cos^2\omega t)\omega^4 B^2\cos^2\omega t = B^2\cos^2\omega t. \tag{11}$$

Expanding and simplifying the above expression gives

$$\tfrac{1}{2}\omega^4 + \tfrac{3}{8}\omega^4 B^2 - \tfrac{1}{2} + \left(\tfrac{1}{2}\omega^4 B^2 + \tfrac{1}{2}\omega^4 - \tfrac{1}{2}\right)\cos 2\omega t + \tfrac{1}{8}\omega^4 B^2\cos 4\omega t = 0 \tag{12}$$

and setting the coefficient of the resulting term $\cos(0\omega t)$ (the lowest harmonic) equal to zero gives the first analytical approximate frequency ω_1 as a function of B:

$$\omega_1(B) = \left(1 + \tfrac{3}{4}B^2\right)^{-1/4}. \tag{13}$$

In this limit, the present method gives exactly the same frequency [15] as the first-order homotopy perturbation method [16,17] applied to Eq. (9). The corresponding first analytical approximate periodic solution is given by

$$u_1(t) = B\cos[\omega_1(B)t]. \tag{14}$$

The approximate frequency in Eq. (13) is the correct one when the harmonic balance method is applied to Eq. (11) and not the following frequency obtained by Mickens [4]:

$$\omega_M(B) = \left(1 + \tfrac{1}{2}B^2\right)^{-1/4}. \tag{15}$$

This is due to the fact that Mickens divided Eq. (11) by $\cos^2\omega t$ and obtained

$$(1 + B^2\cos^2\omega t)\omega^4 B^2 = B^2. \tag{16}$$

It is easy to verify that Eq. (15) can be obtained from Eq. (16). However, ω_M is not the first-order approximate frequency of Eq. (9). We can readily see that ω_M is the first-order approximate frequency of the following differential equation:

$$\left(1 + \frac{2}{3}u^2\right)\left(\frac{d^2u}{dt^2}\right)^2 = u^2, \tag{17}$$

which can be obtained from

$$\frac{d^2u}{dt^2} + \frac{u}{\sqrt{1 + (2/3)u^2}} = 0. \tag{18}$$

As we can see, Eq. (18) does not coincide with Eq. (7).

The corresponding approximation to y is obtained from Eq. (6)

$$y_1(t) = \frac{u_1(t)}{\sqrt{1 + u_1^2(t)}} = \frac{B\cos\omega_1 t}{\sqrt{1 + B^2\cos^2\omega_1 t}}. \tag{19}$$

Likewise, $x_1(t)$ can be calculated by integrating equation $y = dx/dt$ subject to the restrictions

$$x_1(0) = 0, \quad y_1(0) = \frac{B}{\sqrt{1 + B^2}}, \tag{20}$$

which can be easily obtained from Eqs. (6) and (7). This integration gives

$$x_1(t) = \left(1 + \frac{3}{4}B^2\right)^{1/4}\sin^{-1}\left[\frac{B}{\sqrt{1 + B^2}}\sin[\omega_1(B)t]\right]. \tag{21}$$

In order to apply the next level of harmonic balance method, firstly we express the periodic solution to Eq. (7) with the assigned conditions in Eq. (8) in the form of [6,7]

$$u_2(t) = u_1(t) + \Delta u_1(t), \tag{22}$$

where $\Delta u_1(t)$ is the correction part. Linearizing the governing Equations (7) and (8) with respect to the correction $\Delta u_1(t)$ at $u(t) = u_1(t)$ leads to

$$-u_1^2 - 2u_1\Delta u_1 + \left(\frac{d^2u_1}{dt^2}\right)^2 + u_1^2\left(\frac{d^2u_1}{dt^2}\right)^2 + 2u_1\left(\frac{d^2u_1}{dt^2}\right)^2\Delta u_1 + 2\frac{d^2u_1}{dt^2}\frac{d^2\Delta u_1}{dt^2} + 2u_1^2\frac{d^2u_1}{dt^2}\frac{d^2\Delta u_1}{dt^2} = 0 \tag{23}$$

and

$$\Delta u_1(0) = 0, \quad \frac{d\Delta u_1}{dt}(0) = 0. \tag{24}$$

The approximation $\Delta u_1(t)$ in Eq. (22), which satisfies the initial conditions in Eq. (24), takes the form [6,7]

$$\Delta u_1 = c_1(\cos \omega t - \cos 3\omega t), \tag{25}$$

where c_1 is a constant to be determined.

Substituting Eqs. (10), (22) and (25) into Eq. (23), expanding the expression in a trigonometric series, and setting the coefficients of the resulting items $\cos(0\omega t)$ and $\cos(2\omega t)$ equal to zero, respectively, yield

$$-\tfrac{1}{2}B + \tfrac{1}{2}B\omega^4 + \tfrac{3}{8}B^3\omega^4 + \omega^4 c_1 - B^2\omega^4 c_1 - c_1 = 0 \tag{26}$$

and

$$-\tfrac{1}{2}B + \tfrac{1}{2}B\omega^4 + \tfrac{1}{2}B^3\omega^4 - 8\omega^4 c_1 - \tfrac{11}{2}B^2\omega^4 c_1 = 0. \tag{27}$$

From Eq. (26) we can obtain c_1 as follows:

$$c_1 = \frac{-4B + 4B\omega^4 + 3B^3\omega^4}{8(1 - \omega^4 + B^2\omega^4)}. \tag{28}$$

Substituting Eq. (28) into Eq. (27) and solving for the second analytical approximate frequency ω_2, we obtain

$$\omega_2(B) = \left(\frac{40 + 22B^2 + 2\sqrt{256 + 256B^2 + 71B^4}}{72 + 92B^2 + 25B^4}\right)^{1/4}. \tag{29}$$

Furthermore, c_1 can be obtained by substituting Eq. (29) into Eq. (28) and the result is

$$c_1(B) = -\frac{B[64 + 17B^4 - 4\Delta + (80 - 3\Delta)B^2]}{4[32 + 47B^4 - 2\Delta + 2(55 + \Delta)B^2]}, \tag{30}$$

where

$$\Delta(B) = \sqrt{256 + 256B^2 + 71B^4}. \tag{31}$$

The corresponding second analytical approximate periodic solution is given by

$$u_2(t) = [B + c_1(B)]\cos[\omega_2(B)t] - c_1 \cos[3\omega_2(B)t]. \tag{32}$$

The corresponding approximation to y is

$$y_2(t) = \frac{u_2(t)}{\sqrt{1 + u_2^2(t)}} = \frac{B\cos\omega_2 t + c_1(\cos\omega_2 t - \cos 3\omega_2 t)}{\sqrt{1 + [B\cos\omega_2 t + c_1(\cos\omega_2 t - \cos 3\omega_2 t)]^2}}. \tag{33}$$

However, the analytical integration of Eq. (33) to obtain $x_2(t)$ is not possible. To obtain an analytical expression for $x_2(t)$, Eq. (25) is written as follows:

$$y_2(t) = \frac{B\cos\omega_2 t + 4c_1(\cos\omega_2 t - \cos^3\omega_2 t)}{\sqrt{1 + [B\cos\omega_2 t + 4c_1(\cos\omega_2 t - \cos^3\omega_2 t)]^2}} = \frac{B(\cos\omega_2 t + (4c_1/B)\cos\omega_2 t\sin^2\omega_2 t)}{\sqrt{1 + B^2(\cos\omega_2 t + (4c_1/B)\cos\omega_2 t\sin^2\omega_2 t)^2}}. \tag{34}$$

The above equation would be easily integrable if $(4c_1/B)\cos\omega_2 t\sin^2\omega_2 t \ll \cos\omega_2 t$. Then Eq. (34) would have the same functional form as Eq. (19). From Eqs. (30) and (31) we can obtain the following limits:

$$\lim_{B\to 0}\frac{4c_1(B)}{B} = 0, \tag{35}$$

$$\lim_{B\to\infty}\frac{4c_1(B)}{B} = \frac{-2337+4283\sqrt{71}}{143867+13822\sqrt{71}} = 0.12965. \tag{36}$$

Then, $4c_1(B)/B$ takes values between 0 and 0.12965 when B varies between 0 and ∞. For example, for $B=1$ and 10, $4c_1(B)/B$ takes the values 0.0424 and 0.1271, respectively. We can write

$$\left|\frac{4c_1}{B}\cos\omega_2 t\sin^2\omega_2 t\right| \leqslant \left|0.12965\cos\omega_2 t\sin^2\omega_2 t\right| \leqslant 0.05, \tag{37}$$

where we have taken into account that the maximum value of $|\cos\omega_2 t\sin^2\omega_2 t|$ is 0.3849. In Fig. 1 we have plotted $\cos(2\pi h)$ and $\cos(2\pi h)+(4c_1/B)\cos(2\pi h)\sin^2(2\pi h)$ as a function of $h=\omega_2 t/2\pi = t/T_2$ for $B\to\infty$ $(4c_1/B=0.12965)$. From Eqs. (34)–(37) and Fig. 1 we can conclude that Eq. (33) can be approximately written as follows:

$$y_2(t) \approx \frac{B\cos\omega_2 t}{\sqrt{1+B^2\cos^2\omega_2 t}}. \tag{38}$$

Later we will verify that such a simple approximation gives very good results for $x(t)$.

Likewise, $x_2(t)$ can be calculated by integrating equation $y=\mathrm{d}x/\mathrm{d}t$ subject to the restrictions

$$x_2(0)=0, \quad y_2(0)=\frac{B}{\sqrt{1+B^2}} \tag{39}$$

and this integration gives

$$x_2(t) = \left(1+\frac{3}{4}B^2\right)^{1/4}\sin^{-1}\left[\frac{B}{\sqrt{1+B^2}}\sin[\omega_2(B)t]\right]. \tag{40}$$

We will show Eq. (40) gives good results for $x_2(t)$.

However, we should not forget what we are really looking for is an approximate analytical solution to Eq. (1), that is, $x(t)$. Moreover, it is convenient to express the approximate angular frequency and the solution in terms of oscillation amplitude A rather than as a function of B. It is now necessary to find a relation between oscillation amplitude A and parameter B used to solve Eq. (7) approximately.

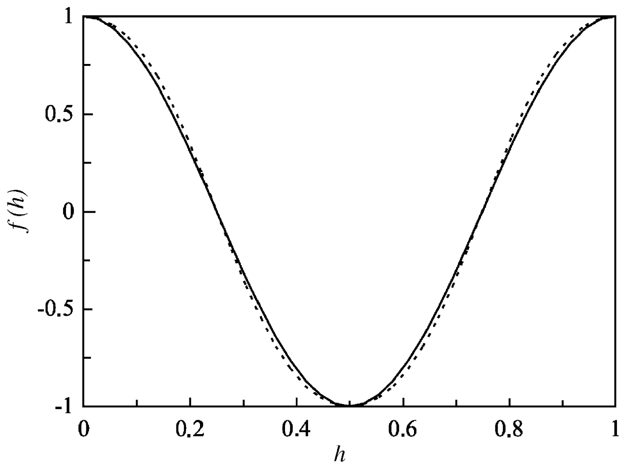

Fig. 1. $\cos(2\pi h)$ (continuous line) and $\cos(2\pi h)+0.12965\cos(2\pi h)\sin^2(2\pi h)$ (dashed line) as a function of $h=\omega_2 t/2\pi = t/T_2$.

From Eq. (3) we get

$$\frac{1}{(1-y^2)^{1/2}}+\frac{1}{2}x^2=C, \tag{41}$$

where C is a constant to be determined as a function of initial conditions. From Eq. (20) we can easily obtain $C=(1+B^2)^{1/2}$ and Eq. (41) can be written as follows:

$$\frac{1}{(1-y^2)^{1/2}}+\frac{1}{2}x^2=(1+B^2)^{1/2}. \tag{42}$$

In addition, when $x=A$, the velocity $y=\mathrm{d}x/\mathrm{d}t$ is zero. Taking this into account in Eq. (49), we obtain the following relation between amplitude A and parameter B:

$$1+\tfrac{1}{2}A^2=(1+B^2)^{1/2}. \tag{43}$$

From the above equation we can easily find that the solution for B is

$$B=A\left(1+\tfrac{1}{4}A^2\right)^{1/2}. \tag{44}$$

Substituting Eq. (44) into Eqs. (13) and (21), the first analytical approximate periodic solution for the relativistic oscillator as a function of the oscillation amplitude A is given by

$$\omega_1(A)=\left(1+\tfrac{3}{4}A^2+\tfrac{3}{16}A^2\right)^{-1/4}, \tag{45}$$

$$x_1(t)=\frac{1}{\omega_1(A)}\sin^{-1}\left[\sqrt{\frac{4A^2+A^4}{4+4A^2+A^4}}\sin[\omega_1(A)t]\right]. \tag{46}$$

Substituting Eq. (44) into Eqs. (29) and (40), the second analytical approximate periodic solution for the relativistic oscillator as a function of the oscillation amplitude A is given by

$$\omega_2(A)=\left(\frac{640+352A^2+88A^4+8\sqrt{4096+4096A^2+2160A^4+568A^6+71A^8}}{1152+1472A^2+768A^4+200A^6+25A^8}\right)^{1/4}, \tag{47}$$

$$x_2(t)=\frac{1}{\omega_2(A)}\sin^{-1}\left[\sqrt{\frac{4A^2+A^4}{4+4A^2+A^4}}\sin[\omega_2(A)t]\right]. \tag{48}$$

Now we illustrate the applicability, accuracy and effectiveness of the proposed approach by comparing the approximate analytical periodic solutions obtained in this paper with the exact ones. Calculation of the exact period frequency, $T_{\mathrm{ex}}(A)$, proceeds as follows. Substituting Eq. (44) into Eq. (42) and integrating, we obtain

$$T_{\mathrm{ex}}(A)=4\int_0^A\frac{1+\frac{1}{2}(A^2-x^2)}{\sqrt{A^2-x^2+\frac{1}{4}(A^2-x^2)^2}}\mathrm{d}x, \tag{49}$$

which allows us to obtain the exact angular frequency, $\omega_{\mathrm{ex}}(A)$, in terms of elliptical integrals as follows:

$$\omega_{\mathrm{ex}}(A)=\frac{2\pi}{T_{\mathrm{ex}}(A)}=2\pi\left[4\sqrt{4+A^2}E\left(\frac{A^2}{4+A^2}\right)-\frac{8}{\sqrt{4+A^2}}K\left(\frac{A^2}{4+A^2}\right)\right]^{-1}, \tag{50}$$

where $K(q)$ and $E(q)$ are the complete elliptic integrals of the first and second kind, respectively, defined as follows [18]:

$$K(q)=\int_0^1\frac{\mathrm{d}z}{\sqrt{(1-z^2)(1-qz^2)}},\quad E(q)=\int_0^1\sqrt{\frac{1-qz^2}{1-z^2}}\mathrm{d}z. \tag{51}$$

For small values of the amplitude A it is possible to take into account the following power series expansion of the angular frequencies

$$\omega_{ex}(A) \approx 1 - \frac{3}{16}A^2 + \frac{51}{1024}A^4 - \cdots, \tag{52}$$

$$\omega_1(A) \approx 1 - \frac{3}{16}A^2 + \frac{42}{1024}A^4 - \cdots, \tag{53}$$

$$\omega_2(A) \approx 1 - \frac{3}{16}A^2 + \frac{56}{1024}A^4 - \cdots. \tag{54}$$

These series expansions were carried out using MATHEMATICA. As can be seen, in the expansions of the angular frequencies $\omega_1(A)$ and $\omega_2(A)$, the first two terms are the same as the first two terms of the equation obtained in the power-series expansion of the exact angular frequency, $\omega_{ex}(A)$. If we compare the third term in Eqs. (53) and (54) with the third term in the series expansion of the exact frequency $\omega_{ex}(A)$ (Eq. (52)), we can see that the third term in the series expansions of $\omega_2(A)$ is more accurate than the third term in the expansion of $\omega_1(A)$.

For very large values of the amplitude A it is possible to take into account the following power series expansion of the exact angular frequency:

$$\omega_{ex}(A) \approx \frac{\pi}{2A} + \cdots = \frac{1.5708}{A} + \cdots, \tag{55}$$

$$\omega_1(A) \approx \frac{2}{3^{1/4}A} + \cdots = \frac{1.5197}{A} + \cdots, \tag{56}$$

$$\omega_2(A) \approx \left(\frac{88 + 8\sqrt{71}}{25}\right)^{1/4} \frac{1}{A} + \cdots = \frac{1.5790}{A} + \cdots. \tag{57}$$

Once again we can see than $\omega_2(A)$ provides excellent approximations to the exact frequency $\omega_{ex}(A)$ for very large values of oscillation amplitude. Furthermore, we have the following equations:

$$\lim_{A\to 0} \omega_{ex}(A) = \lim_{A\to 0} \omega_1(A) = \lim_{A\to 0} \omega_2(A) = 1, \tag{58}$$

$$\lim_{A\to \infty} \omega_{ex}(A) = \lim_{A\to \infty} \omega_1(A) = \lim_{A\to \infty} \omega_2(A) = 0, \tag{59}$$

$$\lim_{A\to\infty} \frac{\omega_1(A)}{\omega_{ex}(A)} = 0.96745, \tag{60}$$

$$\lim_{A\to\infty} \frac{\omega_2(A)}{\omega_{ex}(A)} = 1.00523. \tag{61}$$

Eqs. (58)–(61) illustrate the very good agreement of the approximate frequency $\omega_2(A)$ with the exact frequency $\omega_{ex}(A)$ for small as well as large values of oscillation amplitude. In Fig. 2 we plotted the percentage error of approximate frequencies ω_1 and ω_2, as a function of A. In this figure the percentage errors were calculated using the following equation:

$$\text{Relative error of } \omega_j\,(\%) = 100\left|\frac{\omega_j - \omega_{ex}}{\omega_{ex}}\right|, \quad j = 1, 2. \tag{62}$$

As we can see from Fig. 2, the relative errors for $\omega_1(A)$ are lower than 0.82% for all the range of values of amplitude of oscillation A (see Eq. (60)), and these relative errors tend to 0.52% when A tends to infinity (see Eq. (61)).

The exact periodic solutions $x(t)$ achieved by numerically integrating Eq. (1), and the proposed first-order approximate periodic solutions $x_1(t)$ in Eq. (46) and $x_2(t)$ in Eq. (48) are plotted in Figs. 3–6 for $A = 0.1$, 1, 2 and 10 ($B = 0.1001$, 1.12, 2.83 and 51), respectively. In these figures parameter h is

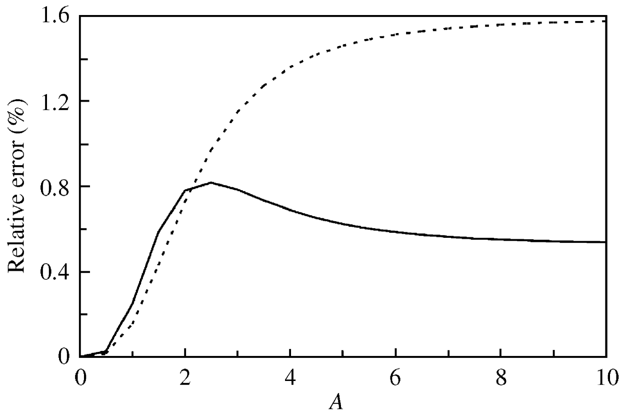

Fig. 2. Relative error for approximate frequencies ω_1, (Eq. (45), dashed line) and ω_2 (Eq. (47), continuous line).

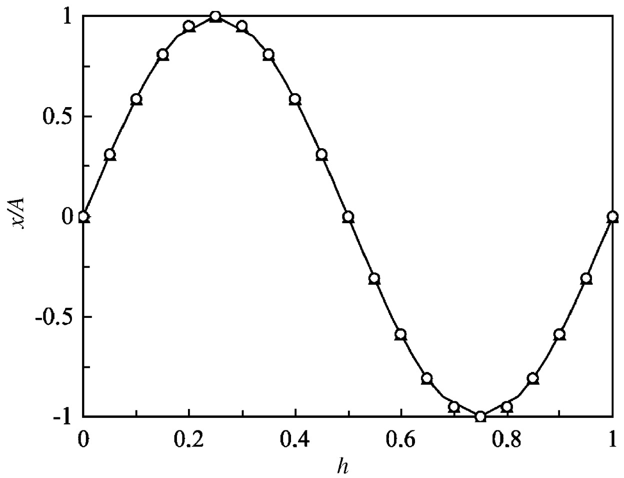

Fig. 3. Comparison of the normalized approximate analytical solutions x_1/A (▲) and x_2/A (○) with the exact solution (continuous line) for $A = 0.1$ ($\beta_0 = v_0/c = 0.09963$).

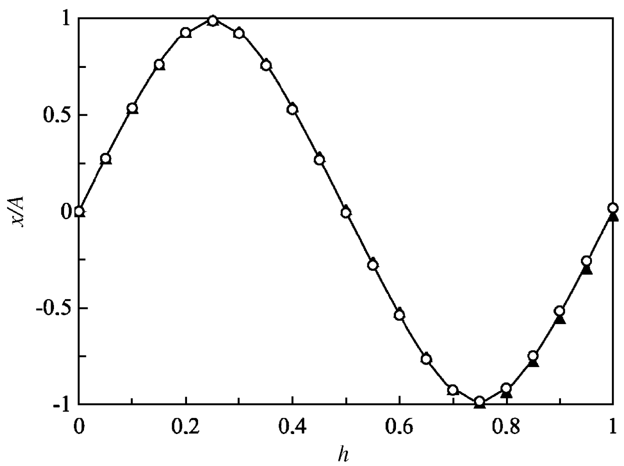

Fig. 4. Comparison of the normalized approximate analytical solutions x_1/A(▲) and x_2/A (○) with the exact solution (continuous line) for $A = 1$ ($\beta_0 = v_0/c = 0.74536$).

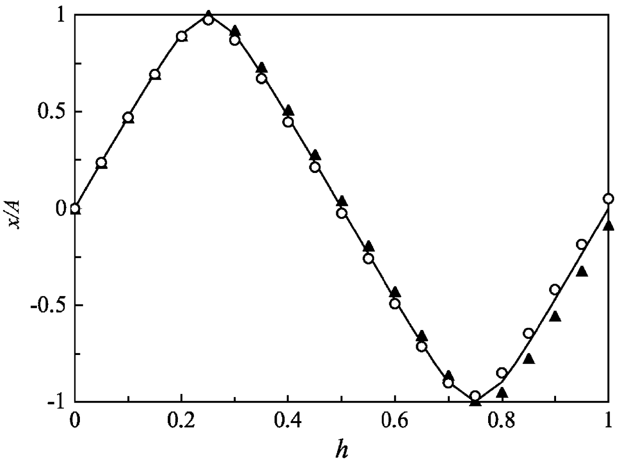

Fig. 5. Comparison of the normalized approximate analytical solutions x_1/A (▲) and x_2/A (○) with the exact solution (continuous line) $A = 2$ ($\beta_0 = v_0/c = 0.94281$).

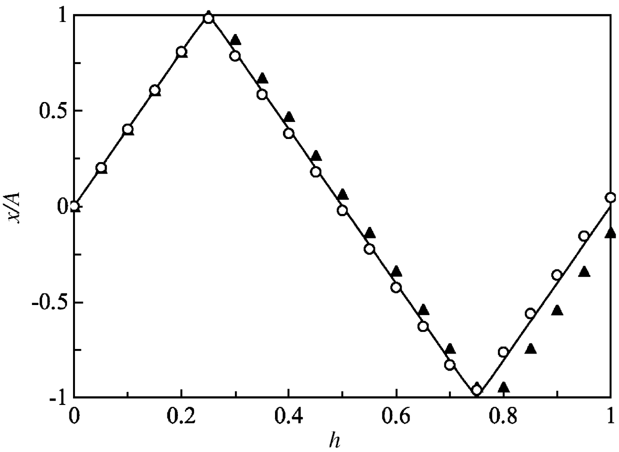

Fig. 6. Comparison of the normalized approximate analytical solutions x_1/A (▲) and x_2/A(○) with the exact solution (continuous line) $A = 10$ ($\beta_0 = v_0/c = 0.99981$).

defined as follows:

$$h = \frac{t}{T_{\mathrm{ex}}(A)} = 2\pi\omega_{\mathrm{ex}}(A)t. \tag{63}$$

From Eqs. (20) and (44) we have

$$\beta_0 = \frac{v_0}{c} = y(0) = \frac{B}{\sqrt{1+B^2}} = \sqrt{\frac{4A^2 + A^4}{4 + 4A^2 + A^4}}, \tag{64}$$

where v_0 is the maximum velocity of the particle and c is the velocity of light.

Figs. 3–6 show that Eq. (48) provides a good approximation to the exact periodic solutions and that the approximation considered in Eq. (38) is adequate to obtain the approximate analytical expression of $x_2(t)$ given in Eq. (48). As we can see, for small values of A (Figs. 3 and 4) $x(t)$ is very close to the sine function form of non-relativistic simple harmonic motion. For higher values of A the curvature becomes more concentrated at the turning points ($x = \pm A$). For these values of A, $x(t)$ becomes markedly anharmonic and is almost straight between the turning points. Only in the vicinity of the turning points, where the magnitude of the

Hooke's law force is maximum and the velocity becomes relativistic, is the force effective in changing the velocity [1]. Fig. 6 are a typical example of the motion in the ultra-relativistic region where $\beta_0 \to 1$.

In summary, the harmonic balance method was used to obtain two approximate frequencies for the relativistic oscillator. To do this we rewrite the nonlinear differential equation in a form that does not contain an irrational expression. We can conclude that the approximate frequencies obtained are valid for the complete range of oscillation amplitude, including the limiting cases of amplitude approaching zero and infinity. Excellent agreement of the approximate frequencies with the exact one was demonstrated and discussed and the discrepancy between the second approximate frequency, ω_2, and the exact one never exceeds 0.82%. Some examples have been presented to illustrate the excellent accuracy of the approximate analytical solutions. Finally, we can see that the method considered here is very simple in its principle, and is very easy to apply.

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