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Composite scheme using localized relaxation with non-standard finite difference method for hyperbolic conservation laws

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Abstract

Non-standard finite difference methods (NSFDM) introduced by Mickens [*Non-standard Finite Difference Models of Differential Equations*, World Scientific, Singapore, 1994] are interesting alternatives to the traditional finite difference and finite volume methods. When applied to linear hyperbolic conservation laws, these methods reproduce exact solutions. In this paper, the NSFDM is first extended to hyperbolic systems of conservation laws, by a novel utilization of the decoupled equations using characteristic variables. In the second part of this paper, the NSFDM is studied for its efficacy in application to nonlinear scalar hyperbolic conservation laws. The original NSFDMs introduced by Mickens (1994) were not in conservation form, which is an important feature in capturing discontinuities at the right locations. Mickens [Construction and analysis of a non-standard finite difference scheme for the Burgers–Fisher equations, *Journal of Sound and Vibration* 257 (4) (2002) 791–797] recently introduced a NSFDM in conservative form. This method captures the shock waves exactly, without any numerical dissipation. In this paper, this algorithm is tested for the case of expansion waves with sonic points and is found to generate unphysical expansion shocks. As a remedy to this defect, we use the strategy of composite schemes [R. Liska, B. Wendroff, Composite schemes for conservation laws, *SIAM Journal of Numerical Analysis* 35 (6) (1998) 2250–2271] in which the accurate NSFDM is used as the basic scheme and localized relaxation NSFDM is used as the supporting scheme which acts like a filter. Relaxation schemes introduced by Jin and Xin [The relaxation schemes for systems of conservation laws in arbitrary space dimensions, *Communications in Pure and Applied Mathematics* 48 (1995) 235–276] are based on relaxation systems which replace the nonlinear hyperbolic conservation laws by a semi-linear system with a stiff relaxation term. The relaxation parameter (λ) is chosen locally on the three point stencil of grid which makes the proposed method more efficient. This composite scheme overcomes the problem of unphysical expansion shocks and captures the shock waves with an accuracy better than the upwind relaxation scheme, as demonstrated by the test cases, together with comparisons with popular numerical methods like Roe scheme and ENO schemes.

1. Introduction

Non-standard finite difference methods (NSFDM) for the numerical integration of differential equations have been reported in the book by Mickens [1]. Most important distinction of the NSFDM is that they are free

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of numerical dissipation and dispersion, which are unavoidable in the traditional finite difference methods. This feature is achieved by enforcing the discrete equations to have the same general solutions as the associated differential equations, apart from utilizing a number of *rules*, which are derived by deep observations of discretization processes [1] in devising numerical methods. Such non-standard finite difference schemes also do not allow numerical instabilities to occur. In NSFDM, the classical denominator h or h^2 of the discrete first or second-order derivative is replaced by a non-negative function ϕ such that $\phi(z) = z + O(z^2)$ or $\phi(z) = z^2 + O(z^3)$ as $0 < z \to 0$. NSFDM techniques have been developed for a large varieties of problems mostly in one dimension. These methods reproduce exact solutions for the linear problems and either exact or nearly exact solutions for the nonlinear problems reported in literature till now. It has also been shown previously that despite their simplicity, appropriate non-standard schemes greatly improve or eliminate numerical instabilities. With these interesting properties, these methods become interesting alternatives to the traditional finite difference methods for solving nonlinear hyperbolic conservation laws which are of importance in several branches of engineering, physics and applied mathematics. To the best of the knowledge of the authors of this paper, the application of the NSFDM to the hyperbolic systems of conservation laws is not reported in literature. In this paper, the NSFDM is extended to hyperbolic vector conservation laws which are linear by the novel use of the decoupling of equations with the use of a characteristic variable. In the second part of this paper, the efficacy of the NSFDM for the application of nonlinear scalar hyperbolic conservation laws is studied in depth. In the numerical simulation of hyperbolic conservation laws, discontinuities often appear in the solutions and using the conservative form in the discretization process has been found to be necessary in capturing the locations of the discontinuities correctly. The earlier versions of the NSFDM did not share this feature. In a recent paper [2], Mickens has introduced NSFDM in conservative form to solve Burger–Fisher equation given as

$$u_t + auu_x = Du_{xx} + \gamma u(1-u) \quad \text{where } a, D, \gamma >= 0 \tag{1}$$

enforcing positivity conditions. Mickens has reduced the Burgers–Fisher equation into six different equations based upon different values of parameters (a, D, γ). We have studied the problem with $D, \lambda = 0$ and $a = 1$, i.e., diffusionless Burgers equation, in this paper. The Burgers equation, together with its generalizations and modifications, is the dominating wave equation for the propagating waves in nonlinear acoustics [5]. Burgers equation also represents a simplification of the more general wave equation, the Kuznetsov's equation, which is derived as a generalization of the d'Alembert's equation with nonlinearity and diffusion terms added. Burgers equation describes plane waves in homogeneous space. The inviscid Burgers equation is a simple example of a nonlinear hyperbolic conservation law in the solution of which discontinuities can develop even if the initial profile is smooth. Apart from discontinuities, this nonlinear hyperbolic partial differential equation also generates expansion (or rarefaction) waves. Apart from the nonlinear acoustics applications of the Burgers equation, nonlinear hyperbolic partial differential equations are also of importance in the aero-acoustics applications. While most of the aero-acoustic problems are linear, the supersonic jet noise problem is nonlinear. Experimental studies have shown that when a supersonic jet is imperfectly expanded, strong screech tones are emitted with intensities being as high as 160 dB, and nonlinear distortions of the acoustic wave forms are expected [6]. Though these nonlinearities do not lead to acoustic shocks, inside the plumes of the imperfectly expanded supersonic jets, shock waves and expansion waves are formed and the shock waves become responsible for the generation of screech tones and broadband shock noise. Therefore, for the direct numerical simulation of noise from shock containing jets, the numerical simulation of nonlinear hyperbolic partial differential equations is required [6].

While the NSFDM has been tested for discontinuities and is found to perform very well, its capability in resolving the expansion waves, especially in the presence of sonic points where the wave speeds change sign, has not been reported in the literature, to the best of the knowledge of the authors of this paper. In this study, we first apply the NSFDM, both in conservative form and in non-conservative form, to the test cases involving both shock and expansion waves. While the NSFDM, especially in conservation form, captures the shock waves well, it encounters difficulties in capturing expansion waves in the presence of sonic points. It is well known in the CFD literature that capturing expansion waves in the presence of sonic points well requires the presence of a finite value of numerical dissipation to avoid entropy condition violating solutions like expansion shocks. The NSFDM seems to have less numerical dissipation than what is required to obtain an entropy condition satisfying solution. In this paper, we utilize the strategy of composite schemes [3] in which two numerical methods of complimentary features are combined to produce a better algorithm. Since the need here is to have a numerical method with sufficient numerical dissipation to avoid expansion shocks and give a stable numerical method, we choose the relaxation schemes [4] to be coupled with NSFDM. The relaxation schemes are simpler alternatives to Riemann solvers and are based on a linearization of a nonlinear PDE using a mathematically introduced relaxation process. The relaxation systems are used for developing efficient numerical methods for hyperbolic conservation laws (see, for example, Refs. [4,7,8]). The composite scheme which results from combining the NSFDM with a localized relaxation NSFDM is found to avoid the problems of the NSFDM in capturing expansion waves and yet captures the shock waves with a reasonably good accuracy. After the next section, the basics of the relaxation schemes are introduced, followed by introduction to localized relaxation NSFDM and its application to some bench-mark test cases, followed by a discussion of the results with the proposed composite scheme and comparison with popular Roe's scheme and ENO schemes and conclusions. In the next section, a novel strategy is presented for extending the idea of NSFDM to the linear hyperbolic systems of conservation laws (vector conservation laws).

2. Extension of NSFDM to linear acoustics system

2.1. NSFDM based on decoupling with characteristic variable [9,12]

Consider the equations of linear acoustics as a hyperbolic system of conservation laws.

$$\frac{\partial U}{\partial t} + A\frac{\partial U}{\partial x} = 0, \tag{2}$$

where

$$U = \begin{bmatrix} p \\ u \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 0 & K_0 \\ \dfrac{1}{\rho_0} & 0 \end{bmatrix}. \tag{3}$$

Here, p is the pressure, u is the velocity and K_0 is the *bulk modulus of compressibility of material* [9]. The matrix A has real and distinct eigenvalues $v_1 = -a_0$ and $v_2 = a_0$ where a_0 is the speed of sound given by $a_0 = \sqrt{K_0/\rho_0}$. Thus, the system (2) is a linear hyperbolic one. Since the eigenvalues are real and distinct, we can write the matrix A as

$$A = RDR^{-1}, \tag{4}$$

where R is the right eigenvector matrix of A, R^{-1} is its inverse and D is a diagonal matrix consisting of the eigenvalues of A as its diagonal elements.

$$R = \begin{bmatrix} -\rho_0 a_0 & \rho_0 a_0 \\ 1 & 1 \end{bmatrix}, \quad R^{-1} = \frac{1}{2\rho_0 a_0}\begin{bmatrix} -1 & \rho_0 a_0 \\ 1 & \rho_0 a_0 \end{bmatrix}, \quad D = \begin{bmatrix} v_1 & 0 \\ 0 & v_2 \end{bmatrix}. \tag{5}$$

For applying the NSFDM of Mickens [2] to a system of linear hyperbolic conservation laws, we first decouple the coupled system using a characteristic variable vector. It is well-known that a linear hyperbolic system can be decoupled by introducing a characteristic variable. For the system described above, we introduce the characteristic variable as $W = (w_1, w_2)^{\mathrm{T}} = R^{-1}U$ and *decoupled equations* are given as

$$\frac{\partial w_1}{\partial t} + v_1\frac{\partial w_1}{\partial x} = 0, \tag{6}$$

$$\frac{\partial w_2}{\partial t} + v_2\frac{\partial w_2}{\partial x} = 0. \tag{7}$$

These equations represent the diagonal form of the system (2). We now apply NSFDM of Mickens [2] to each of these equations separately. Our use of the NSFDM for the decoupled equations (decoupled in the sense that we solve each of these equations separately) gives exact solution for each of the equations, unlike the traditional finite difference methods which have significant numerical dissipation.

2.2. NSFDM for linear scalar equations

The NSFDM for linear scalar hyperbolic equations like Eq. (6), given by Mickens [1] has the distinct feature that the discretized equation has a general solution which is the same as the solution of the associated partial differential equation, leading to *exact difference schemes*. Consider a scalar linear hyperbolic equation as

$$\frac{\partial w}{\partial t} + v\frac{\partial w}{\partial x} = 0, \quad v > 0, \tag{8}$$

with the initial condition given by

$$w(x, t = 0) = \phi(x, t). \tag{9}$$

The exact solution of Eq. (8) is given by

$$w(x, t) = \phi(x - vt). \tag{10}$$

Starting from a discrete equation

$$u_j^{k+1} = u_{j-1}^k, \tag{11}$$

the NSFDM for Eq. (8) can be built up as

$$\frac{w_j^{k+1} - w_j^k}{\psi(\Delta t)} + v\frac{w_j^k - w_{j-1}^k}{\psi(\Delta x)} = 0, \tag{12}$$

which has an exact solution

$$u_j^k = \Phi(x_j - vt_k), \tag{13}$$

which is the discretized equivalent of Eq. (10) [1]. The denominator function $\psi(z)$ has the property $\psi(z) = z + O(z^2)$. The simplest choice, $\psi(z) = z$ leads to $v\Delta t = \Delta x$, which makes the NSFDM lead to the exact solution of Eq. (8). We apply such NSFDM for each of the decoupled system of equations derived before.

In the second part of this paper, the efficacy of NSFDM has been studied for scalar nonlinear hyperbolic conservation laws. As a remedy to the defects mentioned in the introduction, a composite scheme is introduced, based on a *Relaxation System*, which is introduced briefly in the next section.

3. Relaxation system for hyperbolic equations

3.1. Relaxation system of Jin and Xin [4]

Consider a scalar conservation equation in 1D,

$$\frac{\partial u}{\partial t} + \frac{\partial g(u)}{\partial x} = 0, \tag{14}$$

with the initial condition given by

$$u(x, t = 0) = u_0(x), \tag{15}$$

where $g(u)$ is a nonlinear function of u. The relaxation system of Jin and Xin [4] for Eq. (14) is given by

$$\frac{\partial u}{\partial t} + \frac{\partial v}{\partial x} = 0,$$

$$\frac{\partial v}{\partial t} + \lambda^2\frac{\partial u}{\partial x} = -\frac{1}{\varepsilon}[v - g(u)], \tag{16}$$

with the initial conditions given by

$$u(x, t = 0) = u_0(x), \quad v(x, t = 0) = g(u_0(x)), \tag{17}$$

where v is a new variable, λ is a positive constant and ε is a small positive constant known as relaxation parameter. Re-arranging Eq. (16), we get

$$\varepsilon \left[\frac{\partial v}{\partial t} + \lambda^2 \frac{\partial u}{\partial x} \right] = -[v - g(u)]. \tag{18}$$

In the limit $\varepsilon \to 0$, the second equation in Eq. (16) leads to

$$v = g(u). \tag{19}$$

By substituting Eq. (19) in the first equation of the relaxation system (16), we get back the original conservation equation (14). The advantage in dealing with the relaxation system (16) instead of the original nonlinear conservation equation (14) is that the convection terms of Eq. (16) are linear. The nonlinear source term (the right-hand side) in Eq. (16) can be separated by using a splitting method [4]. In the initial condition (17), $v(x, t = 0) = g(u_0(x))$ leads to initial local equilibrium and avoids the development of an initial layer [4].

A Chapman–Enskog-type expansion for the relaxation system will show the condition under which the relaxation system is a dissipative approximation to the given conservation laws. Chapman–Enskog-type expansion (see Ref. [10] for the derivation) for the relaxation system (16) is given as

$$\frac{\partial u}{\partial t} + \frac{\partial g(u)}{\partial x} = \varepsilon \frac{\partial}{\partial x} \left[\left\{ \lambda^2 - \left(\frac{\partial g(u)}{\partial x} \right)^2 \right\} \frac{\partial u}{\partial x} \right] + O(\varepsilon^2). \tag{20}$$

The right-hand side of Eq. (20) contains a second derivative of u, and hence represents a viscous (or dissipation) term. The coefficient represents the coefficient of viscosity. Therefore, the relaxation system provides a vanishing viscosity model for the original conservation laws (14). For Eq. (20) to be parabolic λ should be chosen according to

$$\lambda^2 \geqslant |g'(u)|^2. \tag{21}$$

3.2. Diagonal form of relaxation system

Consider the scalar conservation law (14) and the relaxation system (16). The relaxation system (16) can be re-written in the vector form as

$$\frac{\partial \mathcal{Q}}{\partial t} + \mathbb{A} \frac{\partial \mathcal{Q}}{\partial x} = \mathbb{H}, \tag{22}$$

where

$$\mathcal{Q} = \begin{bmatrix} u \\ v \end{bmatrix}, \quad \mathbb{A} = \begin{bmatrix} 0 & 1 \\ \lambda^2 & 0 \end{bmatrix} \quad \text{and} \quad \mathbb{H} = \begin{bmatrix} 0 \\ -\frac{1}{\varepsilon}(v - g(u)) \end{bmatrix}. \tag{23}$$

Since system (22) is hyperbolic, matrix $\mathbb{A}$ can be diagonalized as $\mathbb{A} = B \Lambda B^{-1}$. By introducing characteristic variables as $\mathbf{f} = B^{-1}\mathcal{Q}$, the relaxation system (22) can be decoupled as given below.

$$\frac{\partial \mathbf{f}}{\partial t} + \Lambda \frac{\partial \mathbf{f}}{\partial x} = -\frac{1}{\varepsilon}[\mathbf{f} - \mathbf{F}], \tag{24}$$

where

$$\mathbf{f} = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} = B^{-1}\mathcal{Q} = \begin{bmatrix} \frac{u}{2} - \frac{v}{2\lambda} \\ \frac{u}{2} + \frac{v}{2\lambda} \end{bmatrix} \quad \text{and} \quad \mathbf{F} = \begin{bmatrix} \frac{u}{2} - \frac{g(u)}{2\lambda} \\ \frac{u}{2} + \frac{g(u)}{2\lambda} \end{bmatrix}. \tag{25}$$

In the limit $\varepsilon \to 0$ the de-coupled system (24) is equivalent to the conservation law (14). It is obvious that above system would be much simpler to solve using upwind schemes as compared to the original equation (14), where $g(u)$ is a nonlinear term in u. The variables u and v can be recovered as

$$u = f_1 + f_2 \quad \text{and} \quad v = \lambda(f_2 - f_1). \tag{26}$$

Here the initial conditions (17) can be written as

$$u(x, t=0) = u_0(x) \quad \text{and} \quad \mathbf{f}(t=0) = \mathbf{F}(u_0(x)). \tag{27}$$

For $\varepsilon \ll 1$, Eq. (24) is similar to the classical Boltzmann equation with B-G-K collision model, with ε taking the role of the relaxation time. The difference between the two equations is that Eq. (24) contains two discrete velocities $(-\lambda, \lambda)$ whereas the molecular velocity in the classical Boltzmann equation is continuous. f_1 and f_2 represent the corresponding components of the distribution function as discussed by Aregba-Driollet and Natalini [7] and Eq. (24) is called discrete (velocity) Boltzmann equation.

A splitting method for Eq. (24) is applied by first splitting Eq. (24) into two steps, which can be solved independently. First is a convective step and second is a relaxation step. Eq. (24) can be written as

$$\frac{\partial f}{\partial t} = S^{(t)}(f) + C^{(t)}, \tag{28}$$

where S and C represent the source and convection terms, respectively. We can solve Eq. (28) by a two-step method as

$$\begin{aligned} \frac{\partial f}{\partial t} &= S^{(t)} \quad \text{(relaxation step)}, \\ \frac{\partial f}{\partial t} &= C^{(t)} \quad \text{(convection step)}. \end{aligned} \tag{29}$$

The convection part can be solved by a NSFDM based on upwind method while the relaxation part can be solved by an implicit method. Therefore, the time step is independent of the relaxation parameter. Using a splitting method, we can split the above equation (24) into two steps, as

$$\begin{aligned} \text{Relaxation step:} \quad & \frac{\mathrm{d}f}{\mathrm{d}t} = \frac{1}{\varepsilon}(F - f), \\ \text{Convection step:} \quad & \frac{\partial f}{\partial t} + \Lambda \frac{\partial f}{\partial x} = 0. \end{aligned} \tag{30}$$

The solution of the relaxation step is given by

$$f = (f(t=0) - F)e^{-t/\varepsilon} + F. \tag{31}$$

Convection equation is solved here using upwind NSFDMs. Note that the Mexwellian F does not change in the relaxation step as it is a function of u and $g(u)$ therefore changes only in the convection step.

4. Composite non-standard finite difference methods with localized relaxation

The convection step in the above equation (30) can be solved using NSFDM as discussed by Mickens [1] for linear advection equation $(f_i)_t + \lambda_i (f_i)_x = 0$ where the discrete velocities $\lambda_i, i = 1, 2$ are the wave speeds of the relaxation system. The NSFDM based on upwind method for this equation is

$$f_j^{k+1} = f_j^k - (\psi(\Delta t))\lambda_i^+ \frac{f_j^k - f_{j-1}^k}{\psi(\Delta x)} - (\psi(\Delta t))\lambda_i^- \frac{f_{j+1}^k - f_j^k}{\psi(\Delta x)}, \quad i = 1, 2, \tag{32}$$

where $\psi(z)$ has the property

$$\psi(z) = z + O(z^2) \quad \text{as } 0 < z \to 0 \tag{33}$$

on a three points stencil of grid points $(j, j+1, j-1)$ with a grid spacing Δx. Any $\psi(z)$ that satisfies the condition given in Eq. (33) will work. The simplest choice is $\psi(z) = z$. Here, Courant splitting is used for the diagonal elements of Λ, i.e., λ_i ($\lambda_1 = -\lambda$ and $\lambda_2 = \lambda$) as $\lambda_i = \lambda_i^+ + \lambda_i^-$ where $\lambda_i^{\pm} = \lambda_i \pm |\lambda_i|/2$. Since $\lambda^2 \geqslant |g'(u)|^2$

as given in Eq. (21), we introduce the parameter $\mu_j = \max\{|g'(u_{j-1})|, |g'(u_j)|, |g'(u_{j+1})|\}$ locally over the three point stencil of grid and then we take wave speed λ_i as $\lambda_1 = -\mu$ and $\lambda_2 = \mu$ (notice that λ_1, λ_2 are vectors now). The CFL condition is chosen as $(\Delta t/\Delta x)\lambda = 1$ (unless until mentioned) where $\lambda = \max_j(\mu_j)$ to obtain the NSFDM. Note that with this condition the NSFDM (32) yields $f_{1,j}^{k+1} = f_{1,j+1}^{k}$ for negative wave speed $f_{2,j}^{k+1} = f_{2,j-1}^{k}$ for positive wave speed, which is the discretization of the exact solution, $f_i(x, t+\Delta t) = f_i(x - \lambda_i \Delta t, t)$ for the equation $(f_i)_t + \lambda_i (f_i)_x = 0$. Though the above scheme is exact for the discrete velocity Boltzmann equation, when we use $u_j^{k+1} = f_{1,j}^{k+1} + f_{2,j}^{k+1}$, together with the relaxation step $f_{1,j}^{k} = F_{1,j}^{k}$, we obtain *Lax-Friedrichs method with low dissipation* (*because of localization*) for the inviscid Burgers equation, which does not yield exact solution (see Refs. [10,4]). This is because of inherent dissipative nature of the relaxation system. Thus, the application of NSFDM to the localized relaxation system does not yield a very low dissipative algorithm. However, the dissipative nature of this non-standard finite difference relaxation scheme can be useful as a complimentary scheme to a low dissipative scheme like pure NSFDM for developing a composite scheme, as will be seen in the following discussion. Let us first consider the NSFDM applied directly to the inviscid Burgers equation.

The non-conservative explicit NSFDM for inviscid Burgers equation, as given by Mickens in Ref. [1], is

$$\frac{u_j^{k+1} - u_j^k}{\Delta t} + u_j^{k+1}\left(\frac{u_j^k - u_{j-1}^k}{\Delta x}\right) = 0 \tag{34}$$

and the conservative non-standard finite difference scheme as recently given by Mickens [2] is

$$\frac{u_j^{k+1} - u_j^k}{\Delta t} + \frac{1}{2}\left[\frac{(u_j^k)^2 - (u_{j-1}^k)^2}{\Delta x}\right] = 0. \tag{35}$$

Here, the nonlinear convection term is modeled by a backward-Euler representation as discussed in Ref. [11]

$$\left(\frac{u^2}{2}\right)_x = uu_x = \left(\frac{u_j + u_{j-1}}{2}\right)\left(\frac{u_j - u_{j-1}}{\Delta x}\right).$$

The non-conservative NSFDM does not capture the shock waves at right locations, as will be demonstrated in the next section. The conservative NSFDM captures shock waves not only at the right locations, but also very accurately. However, as will be shown in the next section, even the conservative NSFDM does not capture expansion waves well, if the sonic points (where the wave speeds change sign) are present in the expansion waves. As is well known in the literature on numerical methods for hyperbolic conservation laws, a finite (non-zero) numerical diffusion is required to capture the expansion waves, especially in the presence of sonic points. The NSFDM seems to be having too little numerical diffusion and is unable to generate fully entropy condition satisfying solution. As a remedy to this defect, we propose to utilize the idea of composite schemes (see Ref. [3]) in which a base-line numerical method is coupled with another numerical method having complimentary features as a filter. We propose to use the conservative NSFDM as the basic numerical method with the localized relaxation NSFDM as the filter. We run the code initially for few iterations using NSFDM and then localized relaxation. Since the NSFDM are first-order accurate and having very little dissipation, we need first few iterations to get the initial solution profile with localized relaxation scheme which is very dissipative. We have found that the first four iterations are good enough for our test problems. However, for comparison, we will also present the results with non-conservative NSFDM, conservative NSFDM, hybrid of conservative NSFDM with localized relaxation NSFDM and localized relaxation NSFDM for some typical test cases.

We also explore in this study the possibility of applying the NSFDM to hyperbolic vector conservation laws. To the best of the knowledge of the authors, extension of NSFDM to hyperbolic vector conservation laws is not available in the literature. In the next section, the NSFDM is first extended to the linearized Euler equations representing the propagation of acoustics waves.

5. Numerical results and discussion

In this section, the two ideas introduced in the previous sections are tested with standard bench-mark test problems. First, the results with the NSFDM for the linear hyperbolic vector conservation laws are presented. Next, the results are presented for two test cases for a scalar hyperbolic conservation law with a convex flux function (inviscid Burgers equation) and also some typical test cases for hyperbolic conservation laws with non-convex flux functions.

5.1. NSFDM for linear hyperbolic vector conservation laws

Test case 5.1.1 (Toro [12]): In this test problem we deal with the Riemann problem for linearized equations of gas dynamics (acoustics)

$$U_t + AU_x = 0, \tag{36}$$

with

$$U = \begin{bmatrix} \rho \\ u \end{bmatrix}, \quad A = \begin{bmatrix} 0 & \rho_0 \\ \dfrac{a^2}{\rho_0} & 0 \end{bmatrix}, \tag{37}$$

where ρ_0 is a constant reference density and a is the sound speed. We define the characteristic variables as

$$W = (w_1, w_2)^{\mathrm{T}} = R^{-1}U,$$

where R is the matrix of right eigenvectors for the corresponding eigenvalues $\lambda_1 = -a, \lambda_2 = a$ and R^{-1} is its inverse, both given as

$$R = \begin{bmatrix} \rho_0 & \rho_0 \\ -a & a \end{bmatrix}, \quad R^{-1} = \frac{1}{2a\rho_0}\begin{bmatrix} a & -\rho_0 \\ a & \rho_0 \end{bmatrix}. \tag{38}$$

Using the diagonalization, we get the system of decoupled equations as

$$\begin{aligned} \frac{\partial w_1}{\partial t} - a\frac{\partial w_1}{\partial x} &= 0, \\ \frac{\partial w_2}{\partial t} + a\frac{\partial w_2}{\partial x} &= 0, \end{aligned} \tag{39}$$

where $w_1 = (\rho a - \rho_0 u)/2\rho_0 a$ and $w_2 = (\rho a + \rho_0 u)/2\rho_0 a$. We solve the problem for $\rho(x,t)$ and $u(x,t)$ at time $t = 1$ for the parameter values $\rho_0 = 1, a = 1$ and initial data $\rho_L = 1, \rho_R = 1/2, u_L = 0$ and $u_R = 0$. The two symmetric waves that emerge from the initial position of the discontinuity carry a discontinues jump in both density ρ and velocity u. The results obtained with the NSFDM are shown in Fig. 1. The NSFDM presented in this paper captures the solution with a very high accuracy. The numerical results are exact for this case.

Test case 5.1.2 (Leveque [9]): In this test problem we study the propagation of sound waves in a one-dimensional (1D) tube of gas. An acoustic wave is a very small pressure disturbance that propagates through the compressible gas, causing infinitesimal changes in density and pressure of the gas via small motions of the gas with infinitesimal values of the velocity u. Linearized acoustic equations with $u_0 = 0$ is given as

$$U_t + AU_x = 0, \tag{40}$$

with

$$U = \begin{bmatrix} p \\ u \end{bmatrix}, \quad A = \begin{bmatrix} 0 & K_0 \\ 1/\rho_0 & 0 \end{bmatrix}, \tag{41}$$

which are the same equations presented before.

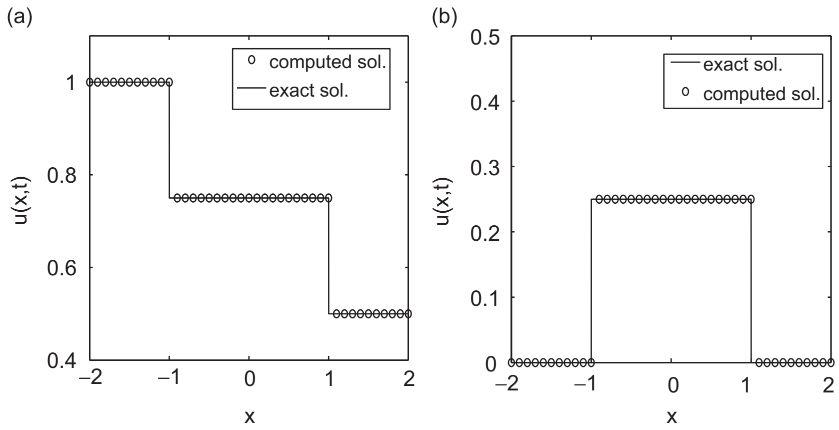

Fig. 1. Test case 5.1.1 at time $t = 1$ using NSFDM for $\Delta x = 0.1$: (a) density profile; (b) velocity profile.

After diagonalization, we get the system of decoupled equations which are then solved in the same way as in test case 5.1.1 using NSFDM. The initial conditions for this test case are taken as

$$p(x,0) = \tfrac{1}{2}\exp(-80x^2) + s(x), \quad u(x,0) = 0, \tag{42}$$

with

$$s(x) = \begin{cases} 0.5 & \text{if } -0.3 < x < -0.1, \\ 0 & \text{otherwise.} \end{cases} \tag{43}$$

In this test case we take $\rho_0 = 1, K_0 = 0.25$, so that $a_0 = 1/2$. The results are shown in Fig. 2. For this test case too, the NSFDM gives a solution with very high accuracy and numerical results are very close to the exact solution but because of discretization error of the initial data and for $\Delta x = 0.02$ results are not exact.

5.2. Composite scheme for scalar nonlinear hyperbolic conservation laws

5.2.1. Inviscid Burgers' equation in 1D

Test case 5.2.1: The first test case is the inviscid Burgers' equation

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}\left(\frac{u^2}{2}\right) = 0, \tag{44}$$

with the initial conditions as

$$u = \begin{cases} 1 & \text{for } |x| < \tfrac{1}{3}, \\ -1 & \text{for } \tfrac{1}{3} < |x| \leqslant 1. \end{cases} \tag{45}$$

The exact solution for this test case is given by

$$u_{ex}(x,t) = \begin{cases} -1 & \text{for } -\infty < x < b_1, \\ -1 + 2\dfrac{x - b_1}{b_2 - b_1} & \text{for } b_1 < x < b_2, \\ 1 & \text{for } b_2 < x < b_{\text{shock}}, \\ -1 & \text{for } b_{\text{shock}} < x < \infty, \end{cases} \tag{46}$$

where

$$b_1 = -\tfrac{1}{3} - t, \quad b_2 = \tfrac{1}{3} + t, \quad b_{\text{shock}} = \tfrac{1}{3}.$$

The initial conditions prescribed above describe a square wave. Periodic boundary conditions are applied at the boundaries. The problem consists of a jump from zero to one at $x = -1/3$ which creates an expansion fan

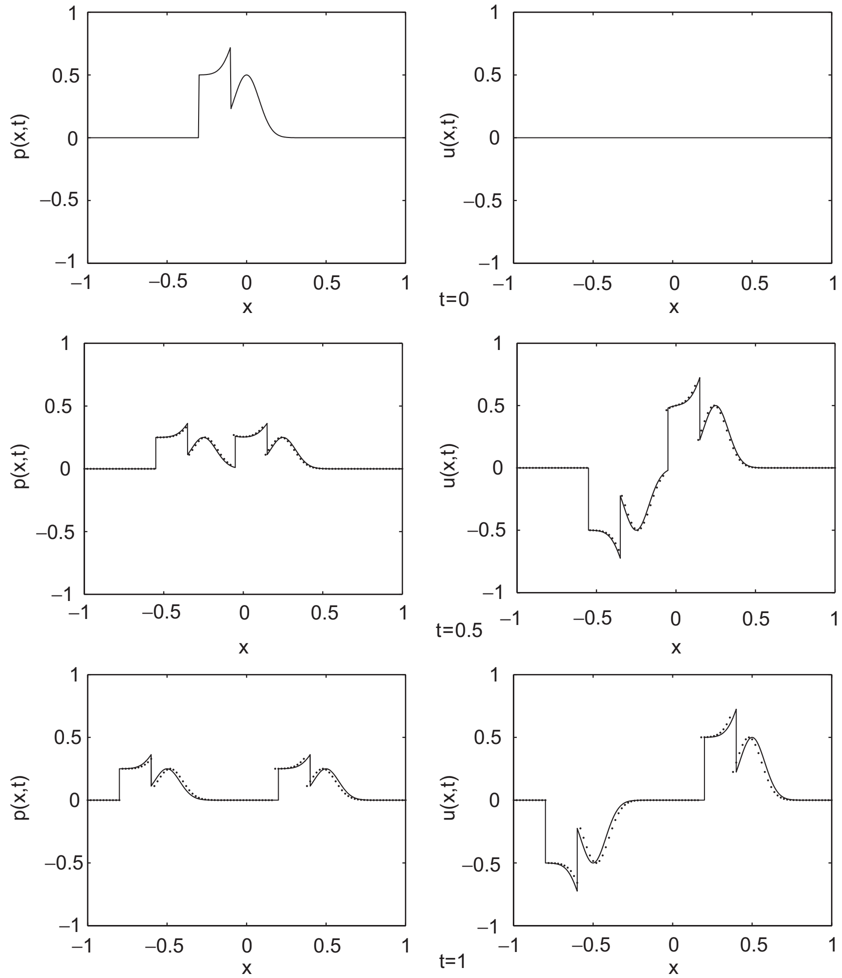

Fig. 2. Test case 5.1.2: the left column shows the pressure p and the right column shows the velocity u at different times. The symbols ('.') represent the computed solution using NSFDM for $\Delta x = 0.02$ and the solid lines ('-') represent the exact solution.

while the jump from one to zero at $x = 1/3$ creates a shock wave [13]. Note that there is a sonic point at $u = 0$ (where the wave speed, $g'(u)$ changes sign, as u varies from -1 to 1). The exact solution, together with the numerical solution obtained with four different numerical methods (non-conservative NSFDM, conservative NSFDM, hybrid of conservative NSFDM and localized relaxation NSFDM), are presented for $u(x, t = 0.3)$, with only 40 points in the domain.

In Fig. 3(a), the results of test problem 5.2.1 using non-conservative NSFDM as given by Eq. (34) are shown. This method not only captures the shock wave at the wrong location, but also encounters problems in capturing the expansion fan with a sonic point. Conservative NSFDM is able to capture the shock wave exactly but fails to capture the expansion fan with a sonic point and generates an unphysical expansion shock, as shown in Fig. 3(b). The localized relaxation NSFDM does not encounter any problems in capturing both the waves, but is overly diffusive. The hybrid of conservative NSFDM and localized relaxation NSFDM gives the best results, with well captured shock and expansion waves and the numerical diffusion falling in between the conservative NSFDM and localized relaxation NSFDM (see Figs. 4(a) and (b)). Hybrid method is able to capture the expansion fan nearly exactly whereas it is little dissipative at the shock wave. Fig. 5 shows the solution generated by first-order accurate upwind method, Roe's scheme, with Harten's entropy fix and third-order accurate essentially non-oscillatory (ENO) scheme of Shu and Osher [13]. It is clear from Fig. 5(a) that Roe's scheme (with Harten's entropy fix) gives dissipation around the corners at the head and tail of the

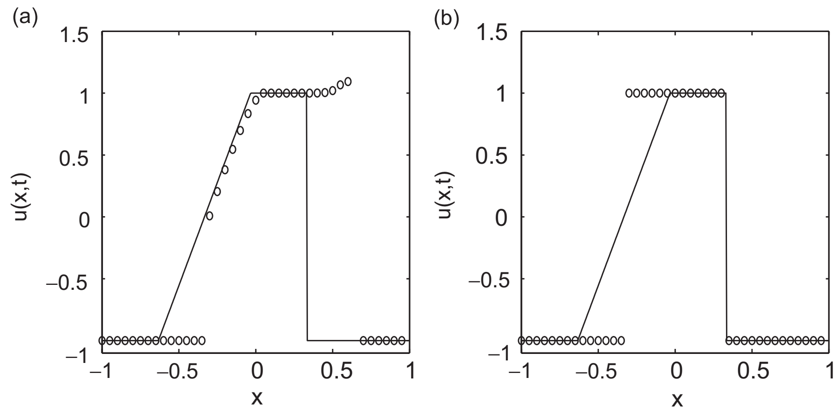

Fig. 3. Non-conservative and conservative NSFDM for test case 5.2.1: inviscid Burgers equation with sonic points; the symbols ('o') represent the computed solution and the solid lines ('-') represent the exact solution: (a) non-conservative NSFDM; (b) conservative NSFDM.

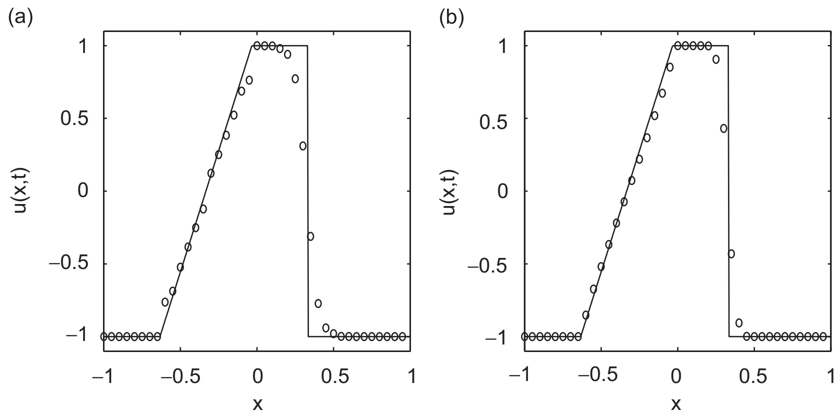

Fig. 4. Localized relaxation NSFDM and composite localized relaxation NSFDM for test case 5.2.1: (a) localized relaxation NSFDM; (b) hybrid of conservative and localized relaxation NSFDM with first iteration using conservative NSFDM.

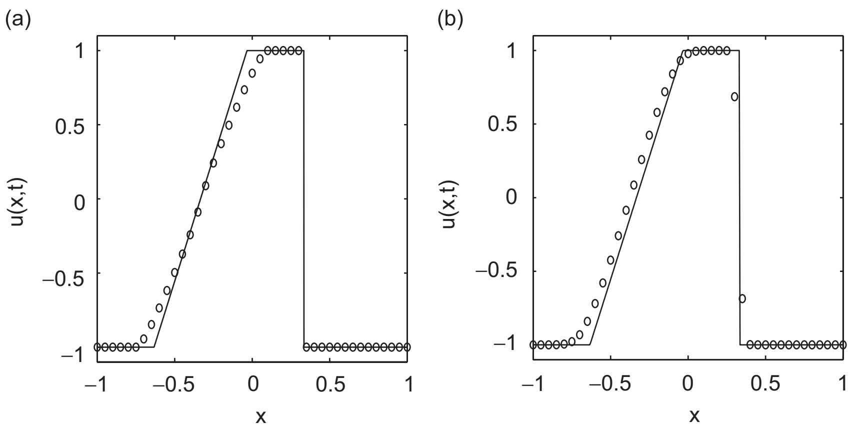

Fig. 5. Test case 5.2.1: (a) Roe's first-order upwind scheme with Harten's entropy fix [13]; (b) Shu-Osher ENO third-order scheme [13].

expansion wave but produces exact solution at the shock. The proposed method is better at the expansion fan but gives not as good results as produced by Roe's scheme with entropy fix. It is to be noted here that the entropy fix with Roe's scheme is used only at the expansive sonic points, and while this is easy for a 1D scalar conservation law, the entropy fix is typically applied everywhere for vector conservation laws, without separating shocks and expansions. The ENO scheme 5(b) also gives better results for this problem at the shock but the proposed composite scheme is better at expansion fan. It is worth noting that the composite scheme used is only first-order accurate and still gives comparable results even with third-order ENO scheme.

Test case 5.2.2: This test case is similar to the test case 5.2.1, except that there are no sonic points in the solution. The inviscid Burgers' equation is same as Eq. (44) with initial conditions given by

$$u = \begin{cases} 1 & \text{for } |x| < \frac{1}{3}, \\ 0 & \text{for } \frac{1}{3} < |x| \leqslant 1 \end{cases} \tag{47}$$

and exact solution is

$$u_{ex}(x,t) = \begin{cases} 0 & \text{for } -\infty < x < b_1, \\ \dfrac{x-b_1}{b_2-b_1} & \text{for } b_1 < x < b_2, \\ 1 & \text{for } b_1 < x < b_{\text{shock}}, \\ 0 & \text{for } b_{\text{shock}} < x < \infty, \end{cases} \tag{48}$$

where

$$b_1 = -\tfrac{1}{3}, \quad b_2 = -\tfrac{1}{3} + t, \quad b_{\text{shock}} = \tfrac{1}{3} + \tfrac{1}{2}t.$$

This problem is also for inviscid Burgers' equation with initial conditions as a square wave. The jump from zero to one at $x = -1/3$ creates an expansion fan, while the jump from one to zero at $x = 1/3$ creates a shock. However, there are no sonic points here, as u varies from 0 to 1 [13].

In Fig. 6 the results with the non-conservative method are presented with CFL = 1 at different times $t = 0.3$, 0.6 and we can see that the expansion fan is captured quite well but the method fails in capturing the shock wave completely. Notice that this method produces the unbounded solution around the shock and it can be theoretically proved easily from Eq. (34). In Fig. 7(a) the result with non-conservative method for CFL = 0.8 has been shown. The conservative NSFDM, shown in Fig. 7(b), captures both the shock wave and expansion fan reasonably well but with some numerical dissipation. The result with the hybrid of non-conservative NSFDM and localized relaxation for CFL = 0.8 is shown in Fig. 8(a), where the method is seen to be able to capture shock wave and is dissipative in capturing the expansion. The results with the hybrid of conservative and localized relaxation NSFDM, as shown in Fig. 8(b), are similar to the results obtained with the conservative NSFDM. All the results have been generated with 40 points in the domain and for time

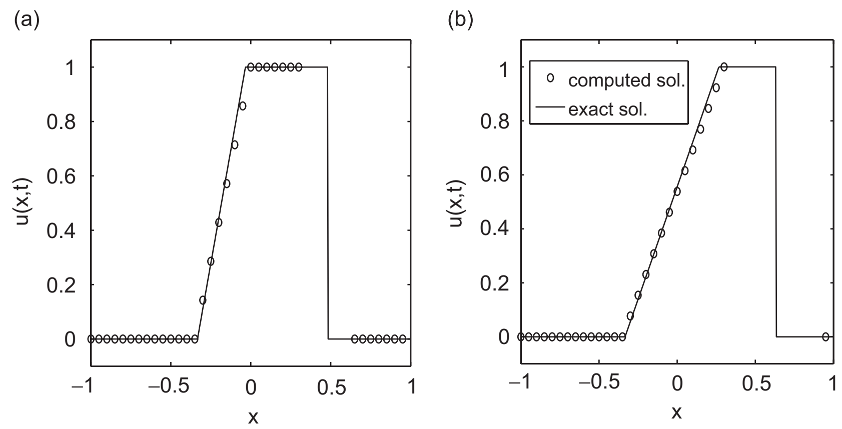

Fig. 6. Test case 5.2.2 with non-conservative NSFDM for CFL = 1: (a) at $t = 0.3$; (b) at $t = 0.6$.

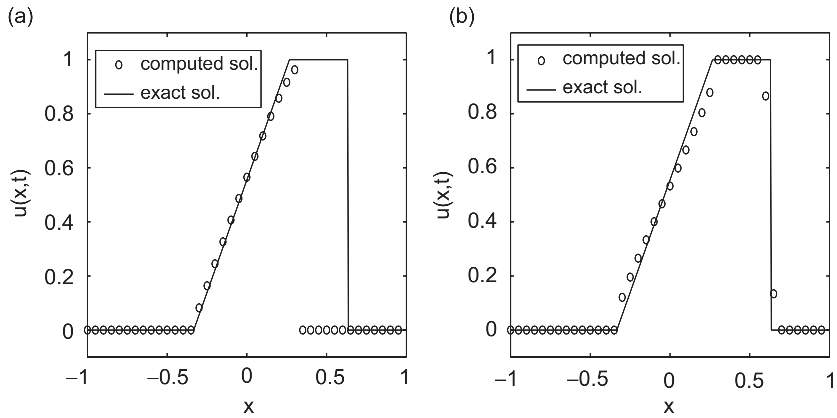

Fig. 7. Test case 5.2.2: (a) non-conservative NSFDM with CFL = 0.8; (b) conservative NSFDM.

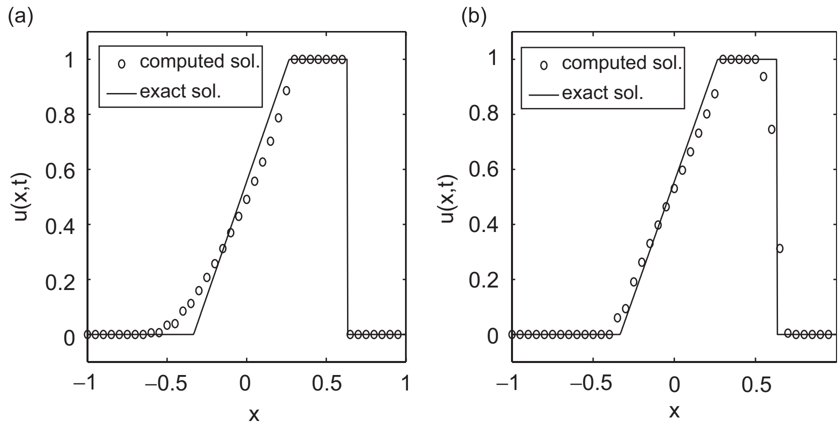

Fig. 8. Test case 5.2.2: (a) hybrid of non-conservative and localized relaxation for CFL = 0.8; (b) hybrid of conservative and localized relaxation NSFDM.

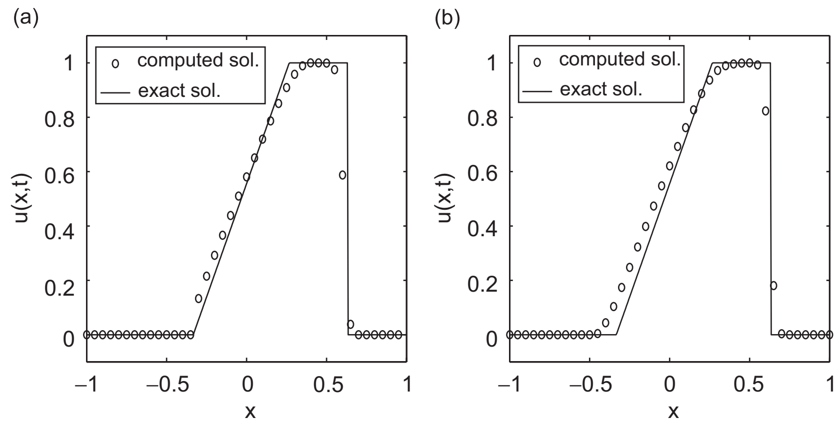

Fig. 9. Test case 5.2.2: (a) Roe's first-order upwind scheme [13]; (b) Shu-Osher ENO third-order scheme [13].

$t = 0.6$. For this problem too, the comparisons are given with the Roe's and ENO schemes mentioned before (see Fig. 9). It is clear from Fig. 9 that the proposed composite method gives comparable results with Roe's and ENO schemes.

Test case 5.2.3 (Seaid [14]): In this test problem, the flux is defined in a slightly different way in the hyperbolic conservation law as

$$\frac{\partial u}{\partial t}+\frac{\partial g(k(x),u)}{\partial x}=0, \tag{49}$$

where

$$g(k(x),u)=k(x)u(1-u), k(x)=\begin{cases}2 & \text{for } 0.0\leqslant x\leqslant 2.5,\\ \dfrac{25-2x}{10} & \text{for } 2.5<x<7.5,\\ 1 & \text{for } 7.5\leqslant x\leqslant 10\end{cases} \tag{50}$$

and the initial condition is given by

$$u_0(x)=\begin{cases}0.9 & \text{for } 0\leqslant x\leqslant 2.5,\\ \dfrac{1+\sqrt{0.28}}{2} & \text{for } 2.5<x\leqslant 10.\end{cases} \tag{51}$$

This problem has a steady-state exact solution defined by

$$u_\infty(x)=\begin{cases}0.9 & \text{for } 0\leqslant x\leqslant 2.5,\\ \dfrac{1}{2}+\dfrac{\sqrt{k(x)^2-0.72k(x)}}{2k(x)} & \text{for } 2.5<x<7.5,\\ \dfrac{1+\sqrt{0.28}}{2} & \text{for } 7.5\leqslant x\leqslant 10.\end{cases} \tag{52}$$

We have computed the approximate solution at $t=10$. At this time the approximated solutions are almost stationary. Fig. 10 shows the results by NSFDM as given by Eq. (34) which represents the unphysical behavior. Conservative NSFDM has not been discussed for such flux function in the literature. Fig. 11(a) shows that the localized relaxation NSFDM method itself is able to capture the expansion fan reasonably well even with $\Delta x=0.25$ (40 points). With 100 points, the solution is very close to the exact solution (Fig. 11(b)).

Test case 5.2.4: Let us now consider a non-convex flux in the hyperbolic conservation law, the Buckley–Leverett equation. This equation has served as one of the simplest models of two-phase flow in a porous medium [9]. The flux function is not convex and is given in the hyperbolic equation as

$$\frac{\partial u}{\partial t}+\frac{\partial g(u)}{\partial x}=0, \tag{53}$$

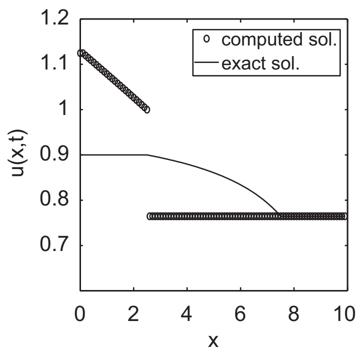

Fig. 10. Non-conservative explicit NSFDM with 100 points in the domain.

where

$$g(u) = \frac{u^2}{u^2 + a(1-u)^2}, \quad a = \frac{1}{2} \tag{54}$$

and the initial condition is given by

$$u_0(x) = \begin{cases} 1 & \text{for } -0.5 \leqslant x \leqslant 0, \\ 0 & \text{for } 0 < x \leqslant 1.5. \end{cases} \tag{55}$$

Here the wave speed is given as

$$g'(u) = \frac{2au(1-u)}{(u^2 + a(1-u)^2)^2}.$$

If we solve this test problem using non-conservative explicit NSFDM given by Eq. (34) then we end up with very complicated mathematical expression which seems difficult to compute. Non-conservative implicit NSFDM as given by Mickens [1] is

$$\frac{u_j^{k+1} - u_j^k}{\Delta t} + u_j^k \left(\frac{u_j^{k+1} - u_{j-1}^{k+1}}{\Delta x} \right) = 0. \tag{56}$$

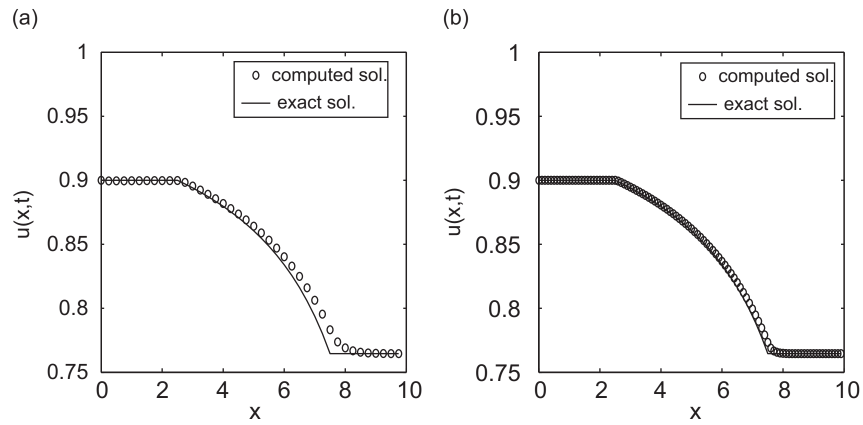

Fig. 11. Test case 5.2.3 using localized relaxation NSFDM: (a) with 40 points; (b) with 100 points.

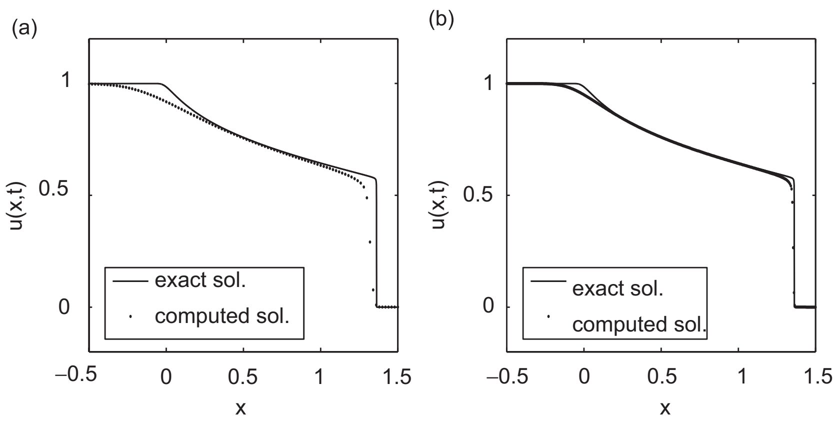

Fig. 12. Test case 5.2.4 using relaxation NSFDM: (a) with 100 points; (b) with 400 points.

This gives $u_j^{k+1} = 0$ as $0 < x_j \leqslant 1.5$, for all time t because of the specific initial data used which shows unphysical behavior. Again conservative NSFDM with non-convex flux function has not been discussed in the literature. Fig. 12 shows the computed solution using relaxation NSFDM (with $\lambda_1 = -\max(g'(u))$ and $\lambda_2 = \max(g'(u))$ as $g'(u)$ is zero at initial data u_0) for different numbers of points in the domain at $t = 1$.

6. Conclusions

The NSFDMs of Mickens, which have been shown to be quite successful for the linear scalar equations, are extended to solve the hyperbolic vector conservation laws using the strategy of decoupling the system of equations with a characteristic variable vector in a novel way. The results are presented for the linear acoustics equations and are of high accuracy. In the second part of this paper, the NSFDM is studied in depth for obtaining the solution of scalar nonlinear hyperbolic conservation laws. The NSFDM, both in the non-conservative form presented earlier and in the conservative form presented recently, has been tested for the case of both shock waves and expansion waves with and without sonic points for hyperbolic conservation laws with convex and non-convex flux functions. While the non-conservative form of the NSFDM captures the shock locations incorrectly, the conservative form of NSFDM captures the shock waves well but fails in capturing the expansions waves when sonic points are present. As an alternative, a composite scheme is presented, which combines the robust localized relaxation NSFDM scheme which is used a filter to the basic low diffusive NSFDM. This composite scheme is also tested on other hyperbolic conservation laws with different flux functions and also non-convex flux functions. The composite NSFDM scheme consistently produced better results compared to the usual NSFDM. Though this composite method produces an alternative, this method still cannot produce exact (numerical dissipation free) solutions in all cases and it is desirable to introduce improvements in the basic NSFDMs to overcome the drawbacks. Till such methods are available, the composite schemes can be considered as attractive alternatives.

Acknowledgments

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