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Scattering of flexural waves and boundary-value problem in Mindlin's plates of soft ferromagnetic material with a cutout

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Abstract

In this paper, based on the equation of wave motion in Mindlin's plate of magneto-elastic interaction, the problem of scattering of flexural waves and dynamic stress concentrations in Mindlin's plates of ferromagnetic material with a cutout is analyzed using the wave function expansion method. And an analytical solution and numerical examples of the problems are given. It can be seen from the results that the magnetic induction intensity has great influences on the dynamic bending moment concentration factors.

1. Introduction

The problem of elastic waveguide and dynamic stress concentration in plates with a cutout is the important subject in the research of solid structure dynamics. The cutout in structures directly influences the carrying capacity and the life-span of structures. Therefore, many researchers have devoted to the theoretical and experimental researches on this problem [1–10].

As analyzing and computing dynamic stress concentration factors or dynamic stress intensity factors, the theory of classical thin plate has been restricted in theory. Mindlin's thick plate theory is made up for the shortage of classical thin plate theory by considering the influences of plate's moment of inertia and shearing strain on the problem. The satisfactory results have been gained in engineering analysis and calculation [4]. With the wave function expansion method, Pao and Mao [5] first studied the problem of the flexural wave scattering and dynamic stress concentrations in Mindlin's thick plates with cutouts and gave an analytical solution and numerical examples.

With the developments of modern engineering, the ferromagnetic material has been considered for structural applications in superconduct nuclear power station and magnetically levitated trains. It has better physical and mechanical prosperities. The margin stress of crack or cavity in ferromagnetic material structures may be increased in a uniform magnetic field. It has influences on the carrying capacity and the life-span of

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structures. According to the corresponding documents, the dynamic behaviors of ferromagnetic-elastic structures can be significantly affected by the presence of a uniform magnetic field [7].

In this paper, based on the equation of wave motion in Mindlin's plate of magneto-elastic interaction, the problem of scattering of flexural waves and dynamic stress concentration in a plate of ferromagnetic material with a cutout is analyzed with wave function expansion method.

2. The equation of wave motion in Mindlin's plate of soft ferromagnetic materials

Suppose that the thickness of the soft ferromagnetic–elastic plate is $2h$. The Cartesian coordinates x–y are in the middle plane of the plate. The z direction is along the thickness direction. The plate is placed in a static uniform magnetic field of vertical incidence, of which magnetic induction density is B_0.

The whole physical quantity of magnetic field is assumed to be divided into two parts. One is the basic physical quantity condition, which is state of rigidity. The other is the slightly disturbed physical quantity condition. Then, the total magnetic field may be described as

$$B = B_0 + b, \tag{1a}$$

$$M = M_0 + m, \tag{1b}$$

$$H = H_0 + d, \tag{1c}$$

where B, M and H are magnetic induction intensity, intensity of magnetization and intensity of magnetic, respectively, subscript 0 physical quantity in the permanent magnetic field, and the minuscule slightly disturbed quantities. The physical quantities of the magnetic field under rigidity condition are given below:

For $|z| > h$

$$B_{0z}^{e} = B_0, \tag{2a}$$

$$\left(H_{0z}^{e} = \frac{B_0}{\mu_0}\right), \tag{2b}$$

$$M_{0z}^{e} = 0. \tag{2c}$$

For $|z| \leqslant h$

$$B_{0z} = B_0, \tag{2d}$$

$$H_{0z} = \frac{B_0}{\mu_0 \mu_r}, \tag{2e}$$

$$M_{0z} = \frac{\chi B_0}{\mu_0 \mu_r}, \tag{2f}$$

where B_{0z}, H_{0z} and M_{0z} are the partial quantities of B_0, H_0, M_0 along the z-axis, respectively, superscript e denotes the value in outer plate, $\mu_0 = 4\pi \times 10^{-7}\,\mathrm{N/A^2}$ the permeability of vacuum and $\mu_r = 1 + \chi$ the relative magnetic permeability.

For convenience, the magnetic field in plate is written as

$$\nabla \cdot \mathbf{B} = \nabla \cdot (\mu_0 \mu_r \mathbf{d}) = \frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z} = 0, \tag{3a}$$

$$\nabla \times \mathbf{d} = \left(\frac{\partial h_z}{\partial y} - \frac{\partial h_y}{\partial z}\right) i + \left(\frac{\partial h_x}{\partial z} - \frac{\partial h_z}{\partial x}\right) j + \left(\frac{\partial h_y}{\partial x} - \frac{\partial h_x}{\partial y}\right) k = 0. \tag{3b}$$

By introducing magnetic potential function φ, the slight disturbed condition in Eq. (3) may satisfy

$$\mathbf{d} = \nabla \varphi, \tag{4a}$$

$$\nabla^2\varphi = 0. \tag{4b}$$

Neglecting magnetic striction effect, for $\left|M_{0z}(\partial u/\partial z)\right| < |m|$, the equation of motion of Mindlin's plate of the soft ferromagnetic material may be described as

$$\nabla \cdot \mathbf{t} + \mu_0 \mathbf{M} \cdot \nabla \mathbf{H} = \rho \frac{\partial^2 \mathbf{u}}{\partial t^2} \tag{5}$$

and the physical equation

$$\mathbf{t} = \boldsymbol{\sigma} + \mu_0 \mathbf{M}\mathbf{H} = \boldsymbol{\sigma} + \chi\mu_0 \mathbf{H}\mathbf{H}, \tag{6a}$$

where ρ is the material density, $\mathbf{u} = (u_i)$ $(i = 1, 2, 3)$ the displacement vector and $\mathbf{t} = (t_{ij})$ $(i, j = 1, 2, 3)$ the magneto-elastic stress tensor in which

$$\begin{aligned} &t_{xx} = \sigma_{xx}, \quad t_{yy} = \sigma_{yy}, \quad t_{zz} = \sigma_{zz} + \frac{\chi B_0^2}{\mu_0 \mu_r^2} + 2\frac{\chi B_0}{\mu_r}\frac{\partial \varphi}{\partial z}, \\ &t_{xy} = t_{yx} = \sigma_{xy}, \quad t_{zy} = t_{yz} = \sigma_{yz} + \frac{\chi B_0}{\mu_r}\frac{\partial \varphi}{\partial y}, \quad t_{xz} = t_{zx} = \sigma_{zx} + \frac{\chi B_0}{\mu_r}\frac{\partial \varphi}{\partial x}, \end{aligned} \tag{6b}$$

where $\sigma_{xx}, \sigma_{yy}, \sigma_{zz}, \sigma_{xy} = \sigma_{yx}, \sigma_{zy} = \sigma_{yz}, \sigma_{zx} = \sigma_{xz}$ are the elastic stress components. The mechanical constitutive equation is

$$\boldsymbol{\sigma} = \lambda \mathbf{I}\theta + 2\mu\boldsymbol{\varepsilon} = \lambda \mathbf{I}\nabla \cdot \mathbf{u} + 2\mu(\mathbf{u}\nabla + \nabla\mathbf{u}), \tag{7}$$

where $\lambda = (vE)/((1+v)(1-2v))$ and $\mu = E/(2(1+v))$ are the Lamé constants, E, v the modulus of elasticity of the material and Poisson ratio and $\nabla = (\partial/\partial x)i + (\partial/\partial y)j + (\partial/\partial z)k$ the gradient operator. According to Eqs. (6) and (7), we can get the equation of wave motion in Mindlin's plate of soft ferromagnetic material as follows:

$$\nabla \cdot \mathbf{t} + \frac{2\chi B_0}{\mu_r}\frac{\partial}{\partial z}\nabla\varphi = \rho \frac{\partial^2 \mathbf{u}}{\partial t^2}. \tag{8}$$

For $|z| = h$, the boundary-value condition of the magnetic stress is written as

$$\mathbf{n} \cdot \mathbf{t} = \frac{\mu_0}{2} M^2 \mathbf{n} \quad \text{or} \quad \begin{bmatrix} t_{xx} & t_{xy} & t_{xz} \\ t_{yx} & t_{yy} & t_{yz} \\ t_{zx} & t_{zy} & t_{zz} \end{bmatrix} \begin{Bmatrix} -\dfrac{\partial u_z}{\partial x} \\ -\dfrac{\partial u_z}{\partial y} \\ 1 \end{Bmatrix} = \frac{\mu_0}{2} M^2 \begin{Bmatrix} -\dfrac{\partial u_z}{\partial x} \\ -\dfrac{\partial u_z}{\partial y} \\ 1 \end{Bmatrix}. \tag{9a}$$

Comparing the two sides of the above equation, it can be obtained that

$$t_{xz} = 0, \quad t_{yz} = 0, \quad t_{zz} = \frac{\mu_0}{2} M^2.$$

The linearized boundary-value conditions are

$$\sigma_{zx} = -\frac{\chi B_0}{\mu_r}\frac{\partial \varphi}{\partial x}, \quad \sigma_{zy} = -\frac{\chi B_0}{\mu_r}\frac{\partial \varphi}{\partial y}, \quad \sigma_{zz} = \frac{\chi(\chi - 2)}{\mu_r}\left[\frac{B_0^2}{2\mu_0\mu_r} + B_0\frac{\partial \varphi}{\partial z}\right]. \tag{9b}$$

For $|z| = h$, at the boundary of a plate, the boundary conditions of the magnetic field can be expressed in vectors as the following form:

$$\begin{cases} (\mathbf{B} - \mathbf{B}^e) \cdot \mathbf{n} = 0, \\ (\mathbf{H}^e - \mathbf{H}) \cdot \mathbf{s}_x = 0, \\ (\mathbf{H}^e - \mathbf{H}) \cdot \mathbf{s}_y = 0. \end{cases} \tag{10a}$$

Considering the plate in a uniform transverse magnetic field $\mathbf{B}_0 = B_0\mathbf{k}$ (B_0 = constant) and neglecting magnetic field's effects in the edges of the plate and high-order quantities, one can obtain

$$\begin{cases} B^e_{1z} - B_{1z} = 0, \\ (H^e_{1x} - H_{1x}) + (H^e_{0z} - H_{0z})\dfrac{\partial u_z}{\partial x} = 0, \\ (\mathrm{H}^e_{1y} - H_{1y}) + (H^e_{0z} - H_{0z})\dfrac{\partial u_z}{\partial y} = 0. \end{cases} \tag{10b}$$

Furthermore, one may get

$$\begin{cases} \dfrac{\partial \varphi^e}{\partial z} - \mu_r \dfrac{\partial \varphi}{\partial z} = 0, \\ \dfrac{\partial \varphi^e}{\partial x} - \dfrac{\partial \varphi}{\partial x} = -\dfrac{\chi B_0}{\mu_0 \mu_r}\dfrac{\partial u_z}{\partial x}, \\ \dfrac{\partial \varphi^e}{\partial y} - \dfrac{\partial \varphi}{\partial y} = -\dfrac{\chi B_0}{\mu_0 \mu_r}\dfrac{\partial u_z}{\partial y}, \end{cases} \tag{10c}$$

where $\mathbf{n} = (-\partial u_z/\partial x, -\partial u_z/\partial y, 1)$, $\mathbf{s}_x = (1, 0, \partial u_z/\partial x)$, $\mathbf{s}_y = (0, 1, \partial u_z/\partial y)$ are the unit normal vector, unit shearing vectors along the x and y axes in the upper and lower surfaces of the plate, respectively, subscript 0 a quantity before the deformation of the plate and subscript 1 a slight disturbed quantity after the deformation.

According to the Mindlin's plate theory, the components of displacement u_x, u_y, u_z in a rectangular coordinate system are defined as

$$u_x = z\Psi_x(x, y, t), \quad u_y = z\Psi_y(x, y, t), \quad u_z = W(x, y, t), \tag{11}$$

where W is the normal displacement of the plate and Ψ_x, Ψ_y are the normal rotations with respect to x and y axes, respectively. The bending and torsional moments M_{xx}, M_{yy} and $M_{xy} = M_{yx}$ and shearing forces Q_x and Q_y in unit length can be described by Ψ_x and Ψ_y as follows:

$$M_{xx} = \int_{-h}^{h} z\sigma_{xx}\,\mathrm{d}z = D\left[\frac{\partial \Psi_x}{\partial x} + v\frac{\partial \Psi_y}{\partial y}\right], \tag{12a}$$

$$M_{yy} = \int_{-h}^{h} z\sigma_{yy}\,\mathrm{d}z = D\left[\frac{\partial \Psi_y}{\partial y} + v\frac{\partial \Psi_x}{\partial x}\right], \tag{12b}$$

$$M_{xy} = M_{xx} = \int_{-h}^{h} z\sigma_{xy}\,\mathrm{d}z = \frac{(1-v)}{2}D\left[\frac{\partial \Psi_y}{\partial x} + \frac{\partial \Psi_x}{\partial y}\right], \tag{12c}$$

$$Q_x = \int_{-h}^{h} \sigma_{xz}\,\mathrm{d}z = 2\kappa^2\mu h\left[\frac{\partial W}{\partial x} + \Psi_x\right], \tag{13a}$$

$$Q_y = \int_{-h}^{h} \sigma_{yz}\,\mathrm{d}z = 2\kappa^2\mu h\left[\frac{\partial W}{\partial y} + \Psi_y\right], \tag{13b}$$

where $\kappa^2 = \pi^2/12$ is a numerical factor which is used to consider the effects of uneven shearing forces in the thick plate and $D = (2Eh^3)/3(1 - v^2)$ is the bending stiffness of the plate.

The first two equations in Eq. (12) are the results of stresses σ_{xx}, σ_{yy} multiplying by z, respectively, and integrated with respect to the thickness of the plate. Using Eqs. (9) and (11)–(13), one obtains

$$\frac{\partial M_x}{\partial x} + \frac{\partial M_{yx}}{\partial y} - Q_x + m_{xx} = \frac{2}{3}\rho h^3\frac{\partial^2 \Psi_x}{\partial t^2}, \tag{14a}$$

$$\frac{\partial M_{xy}}{\partial x} + \frac{\partial M_y}{\partial y} - Q_y + m_{yy} = \frac{2}{3}\rho h^3\frac{\partial^2 \Psi_y}{\partial t^2}. \tag{14b}$$

Considering the resultant external moments M_x and M_y, one gets

$$m_{xx} = -\frac{\chi B_0 h}{\mu_r}\left[\frac{\partial \varphi(h)}{\partial x} + \frac{\partial \varphi(-h)}{\partial x}\right] + \frac{2\chi B_0}{\mu_r}\int_{-h}^{h} z\frac{\partial^2 \varphi}{\partial x \partial z}\,\mathrm{d}z, \tag{15a}$$

$$m_{yy} = -\frac{\chi B_0 h}{\mu_r}\left[\frac{\partial \varphi(h)}{\partial y} + \frac{\partial \varphi(-h)}{\partial y}\right] + \frac{2\chi B_0}{\mu_r}\int_{-h}^{h} z\frac{\partial^2 \varphi}{\partial y \partial z}\,\mathrm{d}z. \tag{15b}$$

Eq. (12c) is integrated with respect to the thickness of the plate. Using Eqs. (9), (11) and (13), one obtains

$$\frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} + q = 2\rho h\frac{\partial^2 W}{\partial t^2}. \tag{16}$$

Hence, the external load q acting on the surface of the plate may be described as

$$q = \frac{\chi(\chi-2)B_0}{\mu_r}\left[\frac{\partial \varphi(h)}{\partial z} - \frac{\partial \varphi(-h)}{\partial z}\right] + \frac{2\chi B_0}{\mu_r}\int_{-h}^{h} \frac{\partial^2 \varphi}{\partial z^2}\,\mathrm{d}z. \tag{17}$$

3. Analytical solution of scattering of flexural waves

Substituting Eqs. (13) and (14) into Eqs. (15) and (17), we can obtain the dynamic equations in Mindlin's plate of magnetic action

$$D\left[\frac{\partial^2 \Psi_x}{\partial x^2} + \frac{(1-v)}{2}\frac{\partial^2 \Psi_x}{\partial y^2} + \frac{(1+v)}{2}\frac{\partial^2 \Psi_y}{\partial x \partial y}\right] - C\left(\frac{\partial W}{\partial x} + \Psi_x\right) = \rho J\frac{\partial^2 \Psi_x}{\partial t^2} - m_{xx}, \tag{18a}$$

$$D\left[\frac{\partial^2 \Psi_y}{\partial y^2} + \frac{(1-v)}{2}\frac{\partial^2 \Psi_y}{\partial x^2} + \frac{(1+v)}{2}\frac{\partial^2 \Psi_x}{\partial x \partial y}\right] - C\left(\frac{\partial W}{\partial y} + \Psi_y\right) = \rho J\frac{\partial^2 \Psi_y}{\partial t^2} - m_{yy}, \tag{18b}$$

$$C\left[\frac{\partial^2 W}{\partial x^2} + \frac{\partial^2 W}{\partial y^2} + \frac{\partial \Psi_x}{\partial x} + \frac{\partial \Psi_y}{\partial y}\right] = 2h\rho\frac{\partial^2 W}{\partial t^2} - q, \tag{18c}$$

where $C = 2\kappa^2\mu h$, $\kappa^2 = \pi^2/12$ the shearing coefficient of reduction, $J = \frac{2}{3}h^3$ the rotary inertia of the plate and $C/D = \kappa^2 3(1-v)/2h^2 = (\pi^2/8)(1-v)/h^2$.

The potential function of disturbed electromagnetic wave along z direction varies directly with the transverse displacement function. The magnetic potential function along z direction can be written as

$$\varphi = a_1 \cosh(k_1 z)W \quad (|z| \leqslant h). \tag{19}$$

For convenience, we assume that the center of the cutout is displaced at $x = y = 0$, and the radius, a. It is subjected to a vertical uniform magnetic field whose magnetic induction density is B_0. Without loss of generality, we assume that the incident wave transmits along x direction. Then, its mathematic expression can be defined as follows:

$$F_1^{(i)} = F_0\,\mathrm{e}^{\mathrm{i}(k_1 x - \omega t)}, \tag{20a}$$

$$F_2^{(i)} = 0, \tag{20b}$$

$$f^{(i)} = 0, \tag{20c}$$

where F_0 is the amplitude of the incident wave and k_1 and ω the wavenumber and the angular frequency of the incident wave, respectively.

Assume that the electromagnetic waves propagate along the positive z-axis and the electromagnetic potential function is proportional to the transverse displacement. Wave mode is generated along z-axis in the plate, while radiate mode is generated outside the plate. So the magnetic potential function can be described as follows:

$$\varphi^{(i)} = [a_1 \cosh(k_1 z) + a_2 \sinh(k_1 z)]W^{(i)} \quad (|z| \leqslant h), \tag{21a}$$

$$\varphi^{(ei)} = a_3 \exp(-k_1 z) W^{(i)} \quad (z > h), \tag{21b}$$

$$\varphi^{(ei)} = a_4 \exp(k_1 z) W^{(i)} \quad (z < -h), \tag{21c}$$

where a_i ($i = 1, 2, 3$) is the coefficients of the mode of electromagnetic wave and can be defined by the boundary conditions. According to the boundary conditions, i.e. Eqs. (9) and (10), one obtains

$$a_1 = \frac{\chi B_0}{\mu_0 \mu_r \Delta}, \quad a_2 = 0, \quad a_3 = a_4 = -a_1 \mu_r \exp(qh) \sinh(qh), \tag{22}$$

where $\Delta = \mu_r \sinh(k_1 h) + \cosh(k_1 h)$.

Therefore the relation between the incident electromagnetic field and incident flexural wave field in the plate can be expressed as

$$\varphi^{(i)} = \frac{\chi B_0}{\mu_0 \mu_r \Delta} \cosh(k_1 z) W^{(i)}. \tag{23}$$

To get the analytic solution, three functions F, f, g are introduced:

$$\Psi_x = \frac{\partial F}{\partial x} + \frac{\partial f}{\partial y}, \tag{24a}$$

$$\Psi_y = \frac{\partial F}{\partial y} - \frac{\partial f}{\partial x}, \tag{24b}$$

$$W = \frac{1}{G}\left[\frac{D}{C}\nabla^2 F - \left(1 + \frac{\rho J}{C}\frac{\partial^2}{\partial t^2}\right)F\right], \tag{24c}$$

$$m_{xx} = \frac{\partial g}{\partial x}, \tag{24d}$$

$$m_{yy} = \frac{\partial g}{\partial y}, \tag{24e}$$

where

$$G = 1 - \frac{2\chi B_0}{C k_1 \mu_r} a_1 [k_1 h \cosh(k_1 h) - 2 \sinh(k_1 h)], \tag{25a}$$

$$g(x, y) = \frac{2\chi B_0}{k_1 \mu_r} a_1 [k_1 h \cosh(k_1 h) - 2 \sinh(k_1 h)] W(x, y, t). \tag{25b}$$

Substituting Eq. (19) into Eq. (17), one can obtain

$$q = \frac{2\chi^2 B_0}{\mu_r} a_1 k_1 \sinh(k_1 h) W(x, y, t). \tag{26}$$

Substituting Eq. (24) into Eq. (18), one can obtain the following expression [10]:

$$D\nabla^2\nabla^2 F + \left[\frac{D}{C}\frac{2\chi^2 B_0}{\mu_r} a_1 k_1 \sinh(k_1 h) - C(1+G) - 2h\rho \frac{D}{C}\frac{\partial^2}{\partial t^2} - \rho J \frac{\partial^2}{\partial t^2}\right]\nabla^2 F + \left[-\frac{2\chi^2 B_0}{\mu_r} a_1 k_1 \sinh(k_1 h) + 2h\rho \frac{\partial^2}{\partial t^2}\right]\left(1 + \frac{\rho J}{C}\frac{\partial^2}{\partial t^2}\right)F = 0, \tag{27a}$$

$$\frac{(1-v)}{2}D\nabla^2 f - Cf = \rho J\frac{\partial^2 f}{\partial t^2}. \tag{27b}$$

Eq. (27) can be transferred into the following forms:

$$\nabla^2 F_1 + k_1^2 F_1 = 0, \tag{28a}$$

$$\nabla^2 F_2 - k_2^2 F_2 = 0, \tag{28b}$$

$$\nabla^2 f - k_3^2 f = 0, \tag{28c}$$

where k_i ($i = 1, 2$) should satisfy the following frequency dispersion equation:

$$k^4 - \left[\frac{2(1+v)\chi^3}{\kappa^2 h\mu_0\mu_r^2\Delta}\frac{B_0^2}{E}k_1\sinh(k_1 h) - \frac{C(1-G)}{D} + \frac{8h^2}{\pi^2(1-v)}k_0^4 + \frac{1}{3}h^2k_0^4\right]k^2 + \left[-\frac{3(1-v^2)\chi^3}{h^3\mu_0\mu_r^2\Delta}\frac{B_0^2}{E}k_1\sinh(k_1 h) - k_0^4\right]\left(1 - \frac{8h^4}{3\pi^2(1-v)}k_0^4\right) = 0 \tag{29a}$$

and

$$k_3^2 = \frac{2(C-\rho J\omega^2)}{D(1-v)} = \frac{2}{1-v}\left[\frac{\pi^2}{8}\frac{(1-v)}{h^2} - \frac{1}{3}h^2k_0^4\right] = \frac{\pi^2}{4}\frac{1}{h^2} - \frac{2}{3(1-v)}k_0^4h^2, \tag{29b}$$

where $k_0 = (2\rho h\omega^2/D)^{1/4}$ is the wavenumber of the incident wave in the thin plate.

Therefore, the general solution of Eq. (28) is

$$F = F_1 + F_2 = \sum_{n=-\infty}^{+\infty} A_n H_n^{(1)}(k_1 r)\,\mathrm{e}^{\mathrm{i}(n\theta-\omega t)} + \sum_{n=-\infty}^{+\infty} B_n K_n(k_2 r)\,\mathrm{e}^{\mathrm{i}(n\theta-\omega t)}, \tag{30a}$$

$$f = \sum_{n=-\infty}^{+\infty} C_n K_n(k_3 r)\,\mathrm{e}^{\mathrm{i}(n\theta-\omega t)}, \tag{30b}$$

where A_n, B_n and C_n are the mode coefficients of the flexural waves and determined by the boundary conditions, $H_n^{(1)}(\cdot)$ the Hankel function of the first kind and $K_n(\cdot)$ the modified Bessel function.

4. Motivation of the incident wave and total flexural wave field

For the plate problem, the incident wave fields around the cutout are written by

$$F_1^{(i)} = F_0\,\mathrm{e}^{\mathrm{i}(k_1 x-\omega t)}, \tag{31a}$$

$$F_2^{(i)} = 0, \tag{31b}$$

$$f^{(i)} = 0. \tag{31c}$$

And the scattering fields of flexural waves which is motivated by the cutout in the plate are written by

$$F^{(s)} = \sum_{n=-\infty}^{+\infty} A_n H_n^{(1)}(k_1 r)\,\mathrm{e}^{\mathrm{i}(n\theta-\omega t)} + \sum_{n=-\infty}^{+\infty} B_n K_n(k_2 r)\,\mathrm{e}^{\mathrm{i}(n\theta-\omega t)}, \tag{32a}$$

$$f^{(s)} = \sum_{n=-\infty}^{+\infty} C_n K_n(k_3 r)\,\mathrm{e}^{\mathrm{i}(n\theta-\omega t)}. \tag{32b}$$

Therefore, the total fields of flexural waves in the vicinity of the cutout should be superposed by the incident fields and scattering fields, namely

$$F = F^{(i)} + F^{(s)}, \tag{33a}$$

$$f = f^{(i)} + f^{(s)}. \tag{33b}$$

5. Boundary-value condition and the definition of mode coefficients of flexural waves

In the form of generalized displacements, the boundary-value conditions are

$$\psi_n = \bar{\psi}, \tag{34a}$$

$$\psi_t = \bar{\psi}_t, \tag{34b}$$

$$W_n = \bar{W}_n, \tag{34c}$$

where n and t are the normal and tangential directions of the boundary, respectively. While in the form of generalized stresses, the boundary-value conditions are

$$M_n = \bar{M}_n, \tag{35a}$$

$$M_{nt} = \bar{M}_{nt}, \tag{35b}$$

$$Q_n = \bar{Q}_n. \tag{35c}$$

For a circular hole, the boundary-value conditions in the form of generalized displacements are

$$\psi_r = \frac{\partial F}{\partial r} + \frac{1}{r}\frac{\partial f}{\partial \theta} = \bar{\psi}_r, \tag{36a}$$

$$\psi_\theta = \frac{1}{r}\frac{\partial F}{\partial \theta} - \frac{\partial f}{\partial r} = \bar{\psi}_\theta, \tag{36b}$$

$$W_r = \frac{1}{G}\left[\frac{D}{C}\nabla^2 F - \left(1 + \frac{\rho J}{C}\frac{\partial^2}{\partial t^2}\right)F\right] = \bar{W}_r. \tag{36c}$$

And the boundary-value conditions in the form of generalized stresses for a circular hole are

$$M_r = D\left[\frac{\partial \psi_r}{\partial r} + v\left(\frac{1}{r}\frac{\partial \psi_\theta}{\partial \theta} + \frac{\psi_r}{r}\right)\right] = \bar{M}_r = 0, \tag{37a}$$

$$M_{r\theta} = \frac{1}{2}(1-v)D\left[\frac{1}{r}\frac{\partial \psi_r}{\partial \theta} + \frac{\partial \psi_\theta}{\partial r} - \frac{\psi_\theta}{r}\right] = \bar{M}_{r\theta} = 0, \tag{37b}$$

$$Q_r = C\left(\frac{\partial W_r}{\partial r} + \psi_r\right) = \bar{Q}_r = 0. \tag{37c}$$

For convenience, we neglect the time factor $e^{-i\omega t}$. The incident field $F^{(i)}$ in polar coordinates can be expressed as

$$F^{(i)} = F_0 \sum_{n=-\infty}^{+\infty} i^n J_n(k_1 r)\, e^{in\theta}. \tag{38}$$

From Eqs. (24) and (37a)–(37c), one can get

$$\begin{cases} \dfrac{\partial^2 F}{\partial r^2}+\dfrac{1}{r}\dfrac{\partial^2 f}{\partial r\partial\theta}-\dfrac{1}{r^2}\dfrac{\partial f}{\partial\theta}+v\left(\dfrac{1}{r^2}\dfrac{\partial^2 F}{\partial\theta^2}-\dfrac{1}{r}\dfrac{\partial^2 f}{\partial r\partial\theta}+\dfrac{1}{r}\dfrac{\partial F}{\partial r}+\dfrac{1}{r^2}\dfrac{\partial f}{\partial\theta}\right)=0, \\ \dfrac{1}{r}\dfrac{\partial^2 F}{\partial r\partial\theta}-\dfrac{1}{r^2}\dfrac{\partial F}{\partial\theta}-\dfrac{\partial^2 f}{\partial r^2}+\left(\dfrac{1}{r}\dfrac{\partial^2 F}{\partial r\partial\theta}+\dfrac{1}{r^2}\dfrac{\partial^2 f}{\partial\theta^2}-\dfrac{1}{r^2}\dfrac{\partial F}{\partial\theta}+\dfrac{1}{r}\dfrac{\partial f}{\partial r}\right)=0, \\ \dfrac{1}{G}\left[\dfrac{D}{C}\left(\dfrac{\partial^3 F}{\partial r^3}+\dfrac{1}{r}\dfrac{\partial^2 F}{\partial r^2}-\dfrac{1}{r^2}\dfrac{\partial F}{\partial r}+\dfrac{1}{r^2}\dfrac{\partial^3 F}{\partial r\partial\theta^2}-\dfrac{2}{r^3}\dfrac{\partial^2 F}{\partial\theta^2}\right)-\left(1-\dfrac{\rho J}{C}\omega^2\right)\dfrac{\partial F}{\partial r}\right]+\dfrac{\partial F}{\partial r}+\dfrac{1}{r}\dfrac{\partial f}{\partial\theta}=0. \end{cases} \quad (39)$$

Substituting Eq. (33) into Eq. (39) and comparing the mode coefficients, one can get the matrix equation

$$\begin{bmatrix} L_{11} & L_{12} & L_{13} \\ L_{21} & L_{22} & L_{23} \\ L_{31} & L_{32} & L_{33} \end{bmatrix} \begin{Bmatrix} A_n \\ B_n \\ C_n \end{Bmatrix} = \begin{Bmatrix} -L_{14} \\ -L_{24} \\ -L_{34} \end{Bmatrix}, \quad (40)$$

where

$$L_{11}=k_1^2 H''^{(1)}_n(k_1 r)-vn^2 H^{(1)}_n(k_1 r)+vk_1 H'^{(1)}_n(k_1 r),$$

$$L_{12}=k_2^2 K''_n(k_2 r)-vn^2 K_n(k_2 r)+vk_2 K'_n(k_2 r),$$

$$L_{13}=in(1-v)[k_3 K'_n(k_3 r)-K_n(k_3 r)],$$

$$L_{14}=i^n F_0[k_1^2 J''_n(k_1 r)-vn^2 J_n(k_1 r)+vk_1 J'_n(k_1 r)],$$

$$L_{21}=2in[k_1 H'^{(1)}_n(k_1 r)-H^{(1)}_n(k_1 r)],$$

$$L_{22}=2in[k_2 K'_n(k_2 r)-K_n(k_2 r)],$$

$$L_{23}=k_3 K'_n(k_3 r)-k_3^2 K''_n(k_3 r)-n^2 K_n(k_3 r),$$

$$L_{24}=2ni^{n+1}F_0[k_1 J'_n(k_1 r)-J_n(k_1 r)]$$

$$L_{31}=k_1^3 H'''^{(1)}_n(k_1 r)+k_1^2 H''^{(1)}_n(k_1 r)-k_1 n^2 H'^{(1)}_n(k_1 r)+2n^2 H^{(1)}_n(k_1 r) + \left[\frac{C}{D}\left(\frac{\rho J}{C}\omega^2-1+G\right)-1\right]k_1 H'^{(1)}_n(k_1 r),$$

$$L_{32}=k_2^3 K'''_n(k_2 r)+k_2^2 K''_n(k_2 r)-k_2 n^2 K'_n(k_2 r)+2n^2 K_n(k_2 r) + \left[\frac{C}{D}(\frac{\rho J}{C}\omega^2-1+G)-1\right]k_2 K'_n(k_2 r),$$

$$L_{33}=\frac{GC}{D}K_n(k_3 r)in,$$

$$L_{34} = i^n F_0\{k_1^3 J_n'''(k_1 r) + k_1^2 J_n''(k_1 r) - n^2 k_1 J_n'(k_1 r) + 2n^2 J_n(k_1 r) + \left[\frac{C}{D}\left(\frac{\rho J}{C}\omega^2 - 1 + G\right) - 1\right] k_1 J_n'(k_1 r)\}.$$

6. Dynamical stress concentration factor

According to the definition of dynamical stress concentration factor, the dynamical stress concentration factor around the hole in the plate can be expressed as

$$\mathrm{DMCF} = \left|\frac{\mathrm{Re}(M_\theta)}{M_0}\right|, \tag{41a}$$

$$\mathrm{DQCF} = \left|\frac{\mathrm{Re}(Q_\theta)}{Q_0}\right|, \tag{41b}$$

where the dynamic bending moment and dynamic shear stress are, respectively, expressed as

$$M_\theta = D\left(\frac{1}{r}\frac{\partial \psi_\theta}{\partial \theta} + \frac{\psi_r}{r} + v\frac{\partial \psi_r}{\partial r}\right) = D\left(\frac{1}{r}\frac{\partial^2 F}{\partial \theta^2} + \frac{1}{r}\frac{\partial F}{\partial r} + v\frac{\partial^2 F}{\partial r^2} - \frac{1}{r}(1-v)\frac{\partial^2 f}{\partial r \partial \theta} + (1-v)\frac{1}{r^2}\frac{\partial f}{\partial \theta}\right), \tag{42a}$$

$$Q_\theta = C\left(\frac{1}{r}\frac{\partial W}{\partial \theta} + \psi_\theta\right) = \frac{1}{G}\left[D\frac{1}{r}\nabla^2\frac{\partial F}{\partial \theta} - \frac{1}{r}\left(C + \rho J\frac{\partial^2}{\partial t^2}\right)\frac{\partial F}{\partial \theta} + C\left(\frac{1}{r}\frac{\partial F}{\partial \theta} - \frac{\partial f}{\partial r}\right)\right], \tag{42b}$$

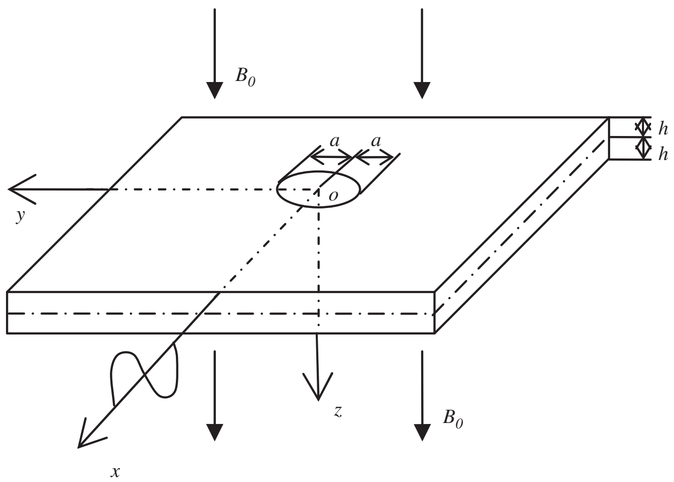

Fig. 1. Bending waves propagate along x-axis and magnetic field.

Table 1
Material characteristics of the plate

Elastic modulus, E (GPa)	Poisson's ratio, v	Magnetization factor, χ
196	0.3	70

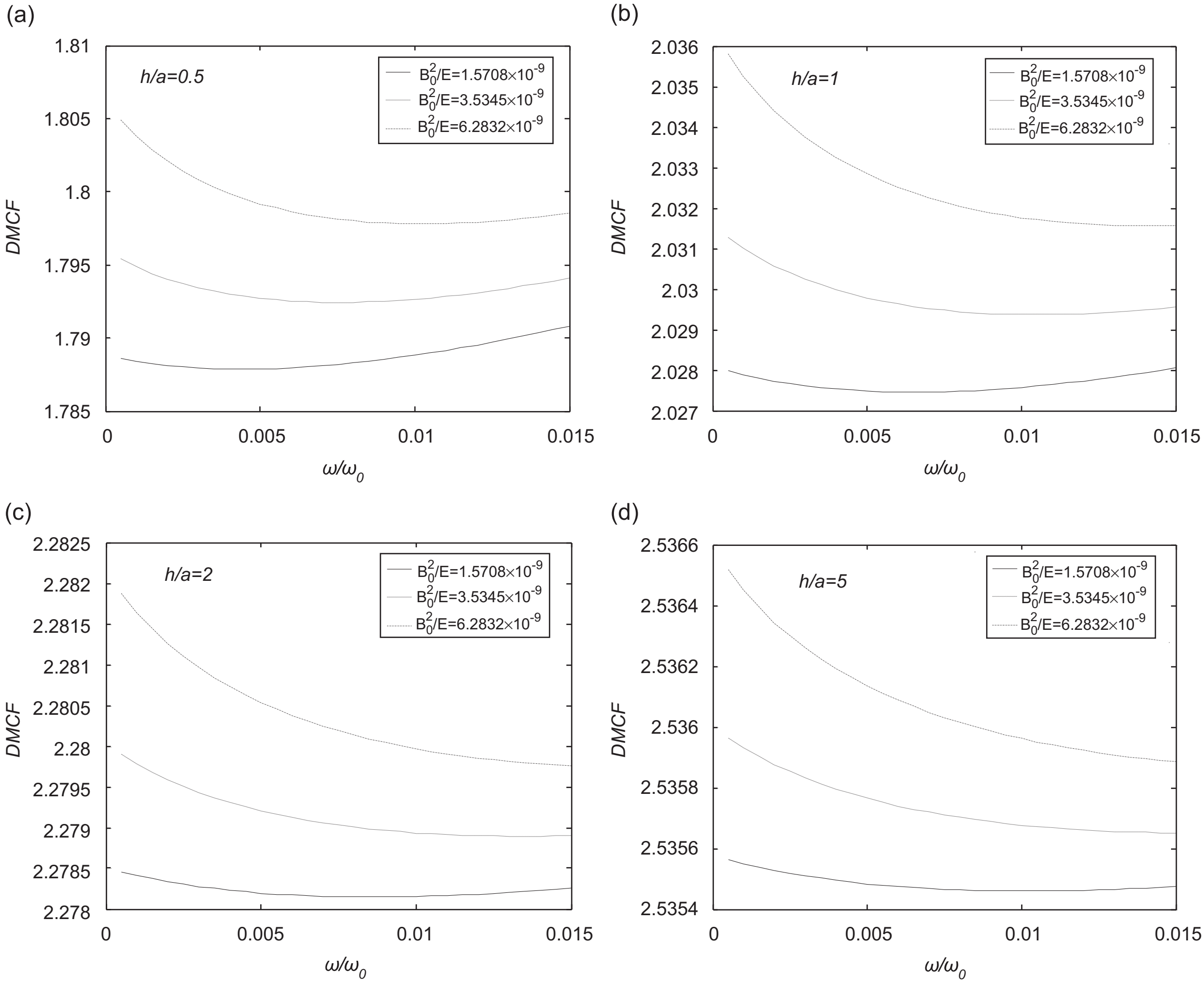

Fig. 2. Dynamic moment and shearing stress vs. incident wave frequency ((a) $h/a = 0.5$; (b) $h/a = 1$; (c) $h/a = 2$; and (d) $h/a = 5$).

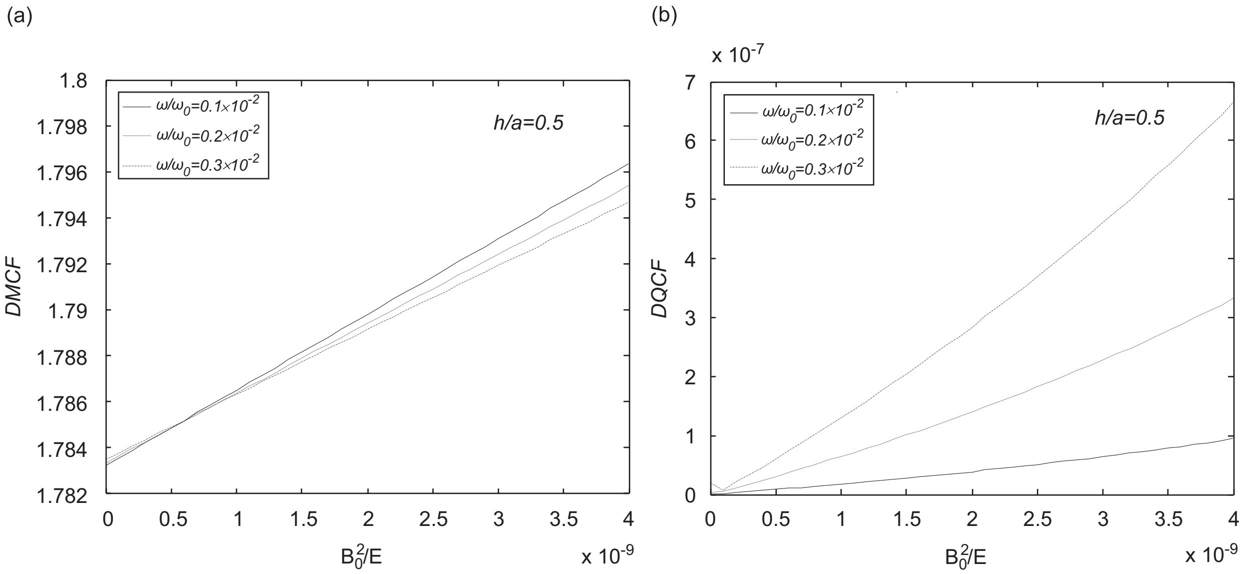

Fig. 3. Dynamic moment and shearing stress vs. magnetic induction density ($h/a = 0.5$).

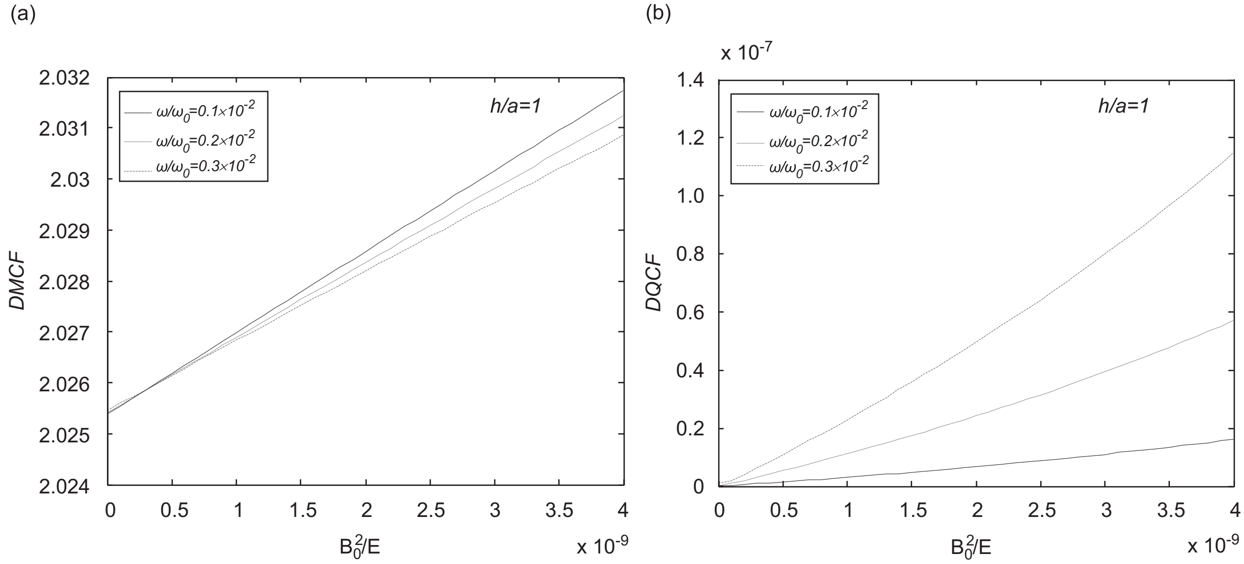

Fig. 4. Dynamic moment and shearing stress vs. magnetic induction density ($h/a = 1$).

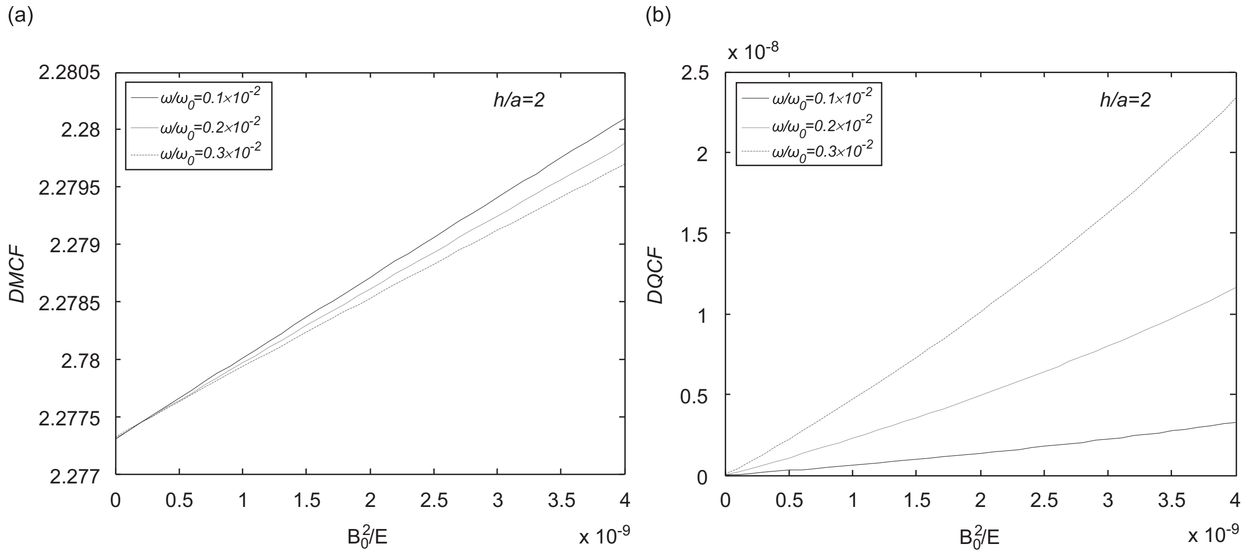

Fig. 5. Dynamic moment and shearing stress vs. magnetic induction density.

where M_θ and Q_θ are the hoop dynamic bending moment and hoop dynamic shearing stress at arbitrary point around the hole. M_0 and Q_0 are the amplitudes of dynamic bending moment and dynamic shearing stress of the incident waves. For the bending waves transmitting along x-axis, one has

$$M_0 = Dk_1^2 F_0, \quad Q_0 = 2\kappa^2 \mu h k_1 i \left\{ \frac{1}{G} \left[k_1^2 \frac{D}{C} + \left(1 - \frac{\rho J}{\omega^2} \right) \right] - 1 \right\} F_0. \tag{43}$$

7. Numerical examples and discussions

Consider the conditions that elastic waves propagate in the positive x direction at the infinite distance of a Mindlin's plate with a cutout and that a magnetic field whose magnetic induction

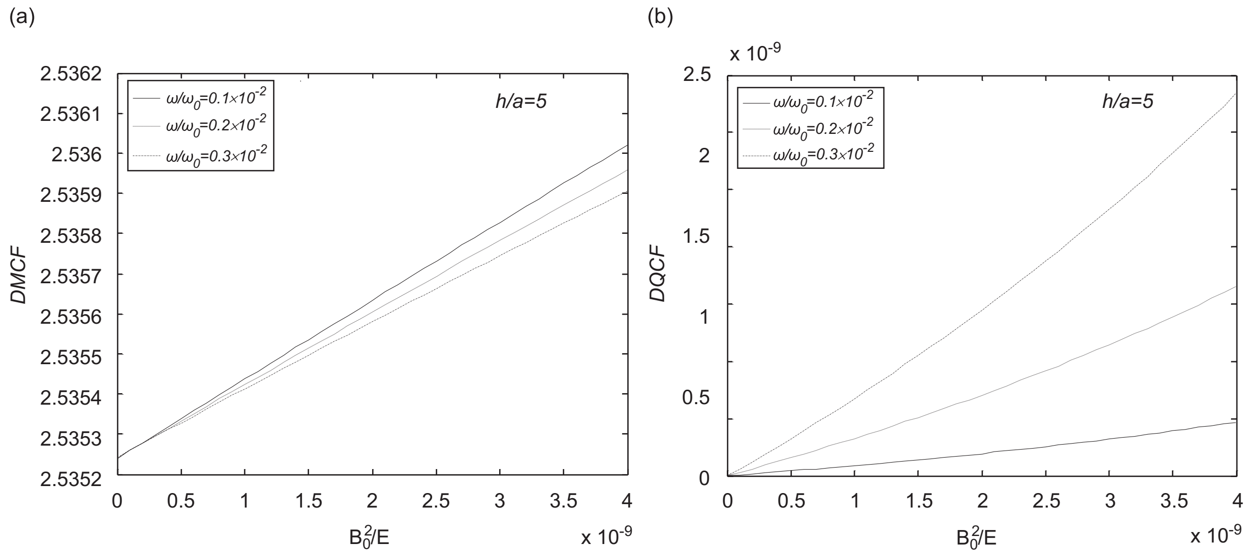

Fig. 6. Dynamic moment and shearing stress vs. magnetic induction density.

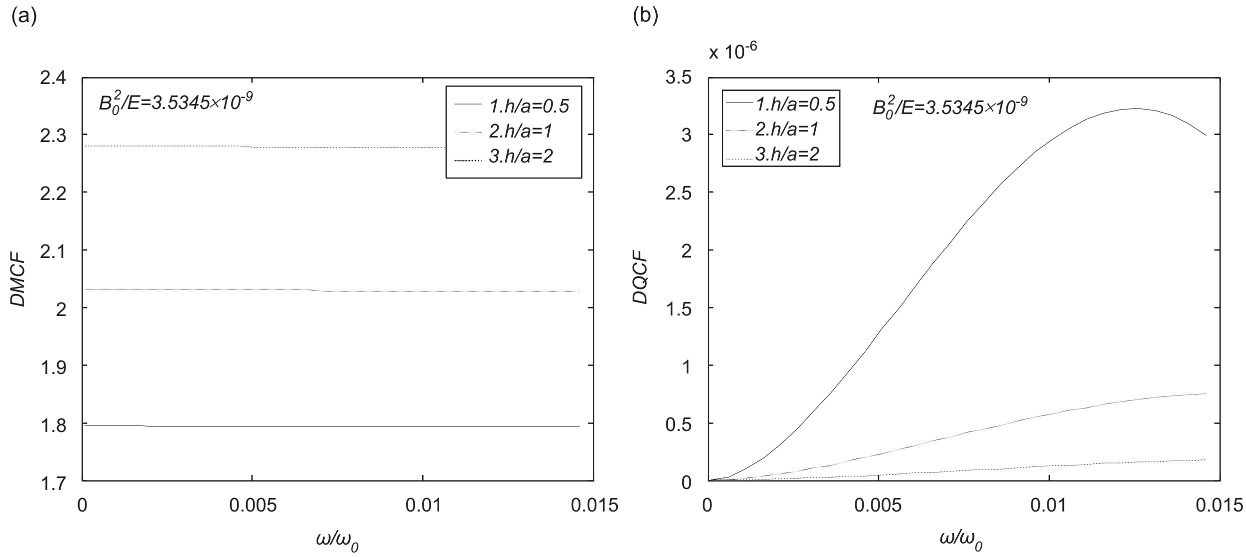

Fig. 7. Dynamic moment and shearing stress vs. incident frequency ($B_0^2/E = 3.5345 \times 10^{-9}$).

density is B_0 is vertical to the plate's surface as shown in Fig. 1. The material properties of the plate are listed in Table 1.

Fig. 2 shows that for different ratios of the square of magnetic induction density to the elastic modulus B_0^2/E, the concentration factor of dynamic bending moment varies with the ratio of the incident wave frequency ω/ω_0 ($\theta = \pi/2$) for certain ratios of h/a. Figs. 3–6 display that for different ratios of the incident wave frequency ω/ω_0, the concentration factors of dynamic bending moment and dynamic shearing stress ($\theta = \pi/2$) vary with the ratio of the square of magnetic induction density to the modulus of elasticity B_0^2/E for certain ratio of h/a. Fig. 7 shows that for different ratios of the half-thickness of the plate to the radius h/a, the concentration factors of dynamic bending moment and dynamic shearing stress ($\theta = \pi/2$) vary with the ratio of the incident wave frequency ω/ω_0 for certain ratios of B_0^2/E. Figs. 8 and 9 display that the concentration factors of the dynamical bending moment and the dynamic shearing stress around the hole vary with the ratio

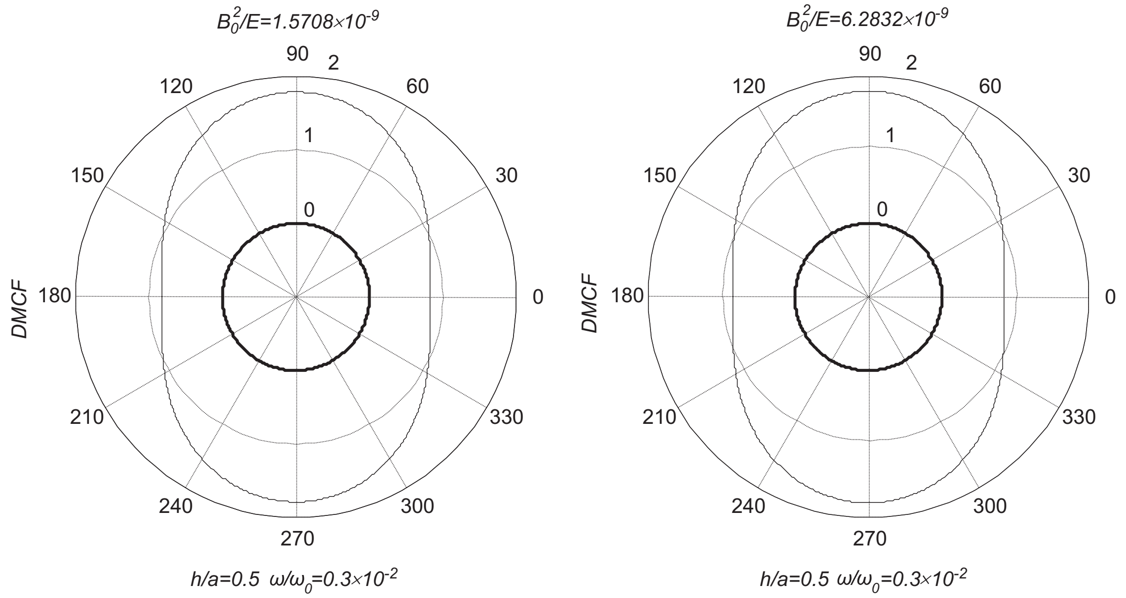

Fig. 8. Dynamic moment vs. magnetic induction density ($h/a = 0.5$, $\omega/\omega_0 = 0.3 \times 10^{-2}$).

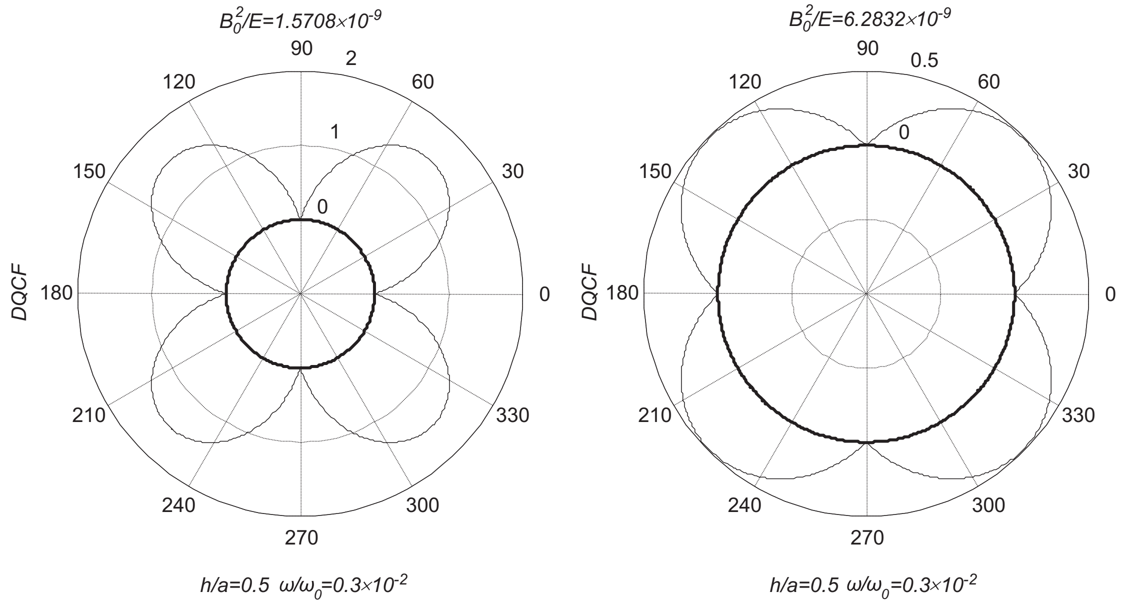

Fig. 9. Dynamic shearing stress vs. magnetic induction density ($h/a = 0.5$, $\omega/\omega_0 = 0.3 \times 10^{-2}$).

of the square of magnetic induction density to the modulus of elasticity B_0^2/E for certain ratios of h/a and ω/ω_0.

From Fig. 2 we can see that when the proportionality factors of the half-thickness of plate h/a and the frequency of incident wave are defined, the dynamic bending moment concentration factor ($\theta = \pi/2$) becomes bigger as the ratio of the square of magnetic induction density to elastic modulus B_0^2/E increases. Figs. 3(a), 4(a), 5(a) and 6(a) show that the concentration factors of dynamic bending moment ($\theta = \pi/2$) become smaller as ω/ω_0 increases. While Figs. 3(b), 4(b), 5(b) and 6(b) show that the cases of the concentration factors of dynamic shearing stress is just opposite.

Fig. 7(a) shows that the dynamic bending moment concentration factor ($\theta = \pi/2$) becomes bigger as h/a increases. While Fig. 7(b) shows that the case of the dynamic shearing stress concentration coefficient is just opposite. Fig. 9 shows that the dynamic shearing stress concentration factor is generally decreased as B_0^2/E increases.

8. Conclusions

In this paper, the problems of scattering of flexural waves and dynamic stress concentration in Mindlin's plate of magneto-elastic interaction are studied using the wave function expansion method. An analytical solution is obtained. By calculating the concentration factors of the dynamic bending moment and dynamic shearing stress, we can find that the magnetic induction intensity can enhance the dynamic bending moment factors.

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