Available online at www.sciencedirect.com
ScienceDirect
Journal of Sound and Vibration 312 (2008) 19–38
JOURNAL OF SOUND AND VIBRATION
www.elsevier.com/locate/jsvi

A first-principles derivation procedure for wake-body models in vortex-induced vibration: Proof-of-concept

R.D. Gabbai[a,b], H. Benaroya[a,b,*]

[a]*National Institute of Standards and Technologies, Gaithersburg, MD, USA*
[b]*Department of Mechanical and Aerospace Engineering, Center for Structures in Extreme Environments (csXe), Rutgers University, Piscataway, NJ, USA*

Received 12 December 2005; received in revised form 26 January 2007; accepted 9 July 2007

Abstract

A generalized variational equation based on the extended Hamilton's principle is presented as the starting point for a procedure from which wake-body models for the two-dimensional vortex-induced vibration of a cylinder can be derived. The case of an elastically mounted cylinder with a transverse degree-of-freedom in a uniform flow is offered as a "model problem" through which this derivation procedure is illustrated. A number of wake-body models from the literature are shown to be recoverable from the more general model derived here. The correspondence with these "classical" models is presented as evidence of the feasibility of incorporating this methodology into vortex-induced vibration modeling. Unlike these comparison models, however, which are often assigned an *ad hoc* form, any assumptions made in arriving at the derived wake-body model are explicitly stated.

0. Introduction

Wake-body models have enjoyed much success because they represent an alternative to flow-field models, which require the determination of the hydrodynamic force coefficients before the structural equation of motion can be solved. Wake-body models are based on the premise that the fluctuating hydrodynamic forces (i.e., the lift and drag forces) on a given structure can be regarded as nonlinear oscillators representing some near-wake effect. These nonlinear oscillator equations are solved in tandem with the structural equation of motion. In essence, one is modeling only the effect of the fluid "seen" by the structure. As a consequence, computations are generally faster and extend into Reynolds number regimes not accessible by direct numerical simulation.

The transverse vortex-induced vibration (VIV) of a circular cylinder embedded in a uniform flow is by far the application for which the majority of wake-oscillator models have been derived. The cylinder is assumed rigid and is elastically mounted. In addition, the cylinder is prevented from moving in the flow direction, the

*Corresponding author at. Department of Mechanical and Aerospace Engineering, Center for Structures in Extreme Environments (csXe), Rutgers University, 98 Brett Road, Piscataway, NJ 08854-8058, USA. Tel.: +1 732 445 4408; fax: +1 732 445 3124.
E-mail address: benaroya@rci.rutgers.edu (H. Benaroya).

0022-460X/$ - see front matter
doi:10.1016/j.jsv.2007.07.086

exact details of the restraining system not being important for the development given here. The fluid–structure interaction possessing these characteristics is subsequently referred to as the "model problem".

Additional assumptions, all fairly standard in analytical modeling of vortex-induced vibration, are as follows: (i) the flow remains nominally two dimensional at all times, and (ii) the vortex shedding is fully correlated along the span. This last assumption is rendered less approximate if attention is focused on the synchronization or "lock-in" regime. This is the regime of flow speeds over which the vortex shedding frequency is found to deviate from the Strouhal law and "locks-in" to the cylinder oscillation frequency.

With regards to the model problem, the typical wake-body model would involve a nonlinear equation in the lift coefficient (or in some other appropriately chosen fluid variable) coupled to the linear oscillator governing the transverse motion of the cylinder (subsequently referred to as the "structural oscillator"). The structural and wake oscillators interact via the coupling terms in each equation. Usually, but not always, these are taken to be linear functions. Sometimes, a distributed-parametered structural equation is considered.

A major criticism of the wake-body models is that the formulation of the nonlinear wake-oscillator is completely *ad hoc*, involving no analysis of the flow field. Furthermore, when fluid-mechanical arguments are presented to bolster a given model, these arguments are not altogether convincing. An extensive review of wake-oscillator models can be found in Refs. [1,2].

This paper can be viewed as a proof-of-concept paper for a modeling approach that places wake-oscillator models on a more rigorous theoretical foundation. This modeling approach is based on a general variational framework for fluid–structure interaction problems, the latter of which has been described in some detail in Refs. [3,4]. The modeling approach is as follows: For a given fluid–structure interaction problem, a governing variational equation is derived. The variational equation can then be simplified in a systematic fashion to yield a class[1] of wake-body models. In the context of the model problem, three wake-body models from the literature are shown to be special cases of the more general derived model.

The aim here is not to tout the derived wake-body model as a better model than the comparison models from the literature. Rather, it is shown that they can be integrated into the more rigorous modeling framework. This will go a long way in helping wake-body models gain wider acceptance among the wider fluid dynamics community.

Numerical approaches to fluid–structure interaction problems of the type defined by the model problem are many. As our purpose is to formalize the development of flow-oscillator models, rather than to compare their predictive capabilities with those of numerical approaches, we do not discuss numerical approaches here. Flow-oscillator models are viewed as simplified models that provide the analyst and designer order of magnitude predictions along with an understanding of some of the underlying physics which is gained from having a mathematical model that can be dissected and understood.

The organization of this paper is as follows. The general variational equation for the model problem is presented in Section 1. A brief description of the simplification process applied to this variational equation and the class of wake-body models that arises from said process are also presented in this section. In Section 2, the derived class of wake-body models are nondimensionalized for the purpose of easing the subsequent comparisons. In Section 3, it is demonstrated how the derived wake-body model can be related to the models of Hall [5], Berger [6], and Tamura and Matsui [7]. Finally, some general remarks on the comparisons are given in Section 4.

Before proceeding, it is worthwhile to introduce the basic variables that are pervasive throughout this paper. These are listed in Table 1. The reader is thus freed from the task of looking for definitions as they lay scattered throughout the paper.

1. The extended Hamilton's principle and the reduction process

Consider the system of particles inhabiting the open control volume $R_o(\mathbf{x}, t)$ at time t. This system of particles is referred to as the *open system*. Only instantaneously does it coincide with the *closed system* of particles M. The control volume has a part $B_o(\mathbf{x}, t)$ of its bounding surface $B(\mathbf{x}, t)$ which is open to the flow of particles. The closed part of the bounding surface $B_c(\mathbf{x}, t)$ includes any solid boundaries and portions of the surface in which the local streamline is normal to the surface.

[1]It it a class insofar as it depends on a number of model parameters whose values are determined from experiments. Since all members of the class have the same form, however, it is convenient to just refer to the singular form "derived model".

Table 1
A list of key parameters

Parameter	Defined by	Description
S	$\simeq 0.2$	Strouhal number
m_c	$\cdots$	Cylinder dry mass
L	$\cdots$	Cylinder length
D	$\cdots$	Cylinder diameter
m_{fl}	$\cdots$	Mass of near-wake fluid oscillator
$c^{(\text{vac})}$	$\cdots$	*In vacuo* material damping of the cylinder + supports
$k_s^{(y)}$	$\cdots$	Total stiffness of the supporting springs
U_o	$\cdots$	Magnitude of the uniform freestream velocity
ρ	$\cdots$	Fluid density
C_A	1.0	Potential flow added mass coefficient
C_a	$\cdots$	Added mass coefficient for a cylinder in crossflow
m_d	$\dfrac{\rho\pi L D^2}{4}$	Mass of fluid displaced by the cylinder
$\chi(t)$	$\cdots$	Transverse displacement of the cylinder from equilibrium
$\hat{m}^*$	$\dfrac{m_d}{m_c}$	Reduced mass
ω_{st}	$\dfrac{2\pi S U_o}{D}$	CircularStrouhal vortex shedding frequency

The kinetic energy of the open system is denoted $\mathscr{K}_o$. The sum of the potential energy due to gravitational and/or buoyancy forces $\mathscr{E}_o^{(g/b)}$, the strain energy $\mathscr{E}_o^{(s)}$, and the internal energy $\mathscr{E}_o^{(i)}$ of the open system is denoted $\mathscr{E}_o$.

The extended form of Hamilton's principle for a system of changing mass (e.g., the exhaust jet of a rocket) or a system of constant mass that does not always consist of the same set of particles (e.g., a pipe of constant diameter conveying fluid) can be written as [3]

$$\delta \int_{t_1}^{t_2} \mathscr{L}_o \,\mathrm{d}t + \int_{t_1}^{t_2} \delta W_o \,\mathrm{d}t + \int_{t_1}^{t_2} \iint_{B_o(\mathbf{x},t)} \rho(\mathbf{u}_{\text{rel}} \cdot \delta\mathbf{r})(\mathbf{u}\cdot\mathbf{n})\,\mathrm{d}s\,\mathrm{d}t = 0, \tag{1}$$

where $\mathscr{L}_o = (\mathscr{K} - \mathscr{E})_o$ is the Lagrangian of the open system, δW_o is the virtual work performed by non-potential forces on the same system, and $\mathrm{d}s = \mathrm{d}s(\mathbf{x}, t)$ represents a differential surface element. At position $\mathbf{x}$ and time t, the density is ρ and the velocity is $\mathbf{u}$. Note that Eq. (1) is related to the Reynolds transport theorem.

Consider the model problem introduced previously, which is shown schematically in Fig. 1. Since the cylinder is assumed rigid, its motion in the transverse direction y can be described by a single generalized coordinate χ. In addition, the assumption of perfectly correlated vortex shedding implies that the transverse displacement of all points on the cylinder is the same, that displacement being $\chi = \chi(t)$. The horizontal plane passing through the cylinder's center of mass is chosen as the reference plane and all dynamic variables (i.e., displacement, velocity, acceleration) are defined on this plane.

From Fig. 1, the control volume $R_o(t)$ is defined as the rectangular volume surrounding the nonstationary cylinder. The origin of the coordinate system is at the center of the cylinder when the cylinder is at rest. The coordinate system does *not* move with the cylinder and is also considered to be at rest relative to the free-stream. The open part of the control surface $B(t)$, $B_o(t)$, is the perimeter of the outer rectangle multiplied by a unit projection out of the plane of the paper.

It can be shown (see Ref. [8]) that the variational equation corresponding to the model problem is given by

$$\overbrace{\delta \int_{t_1}^{t_2} \iiint_{R_o(t)} \frac{1}{2}\rho(\mathbf{u}\cdot\mathbf{u})\,\mathrm{d}v\,\mathrm{d}t}^{\mathbf{1}} + \delta \int_{t_1}^{t_2} \frac{1}{2} m_c \dot{\chi}^2 \,\mathrm{d}t - \delta \int_{t_1}^{t_2} \frac{1}{2} k_s^{(y)} \chi^2 \,\mathrm{d}t - \int_{t_1}^{t_2} c^{(\text{vac})} \dot{\chi} \delta\chi \,\mathrm{d}t$$
$$- \overbrace{\int_{t_1}^{t_2} \iiint_{R_o(t)} \hat{\sigma}_{ij} \left[\frac{\partial(\delta r_i)}{\partial x_j}\right] \mathrm{d}v\,\mathrm{d}t}^{\mathbf{2}} + \overbrace{\int_{t_1}^{t_2} \iint_{B_o(t)} [\rho\mathbf{u}(\mathbf{u}\cdot\mathbf{n}) + \mathbf{Q}] \cdot \delta\mathbf{r}\,\mathrm{d}s\,\mathrm{d}t}^{\mathbf{3}} = 0, \tag{2}$$

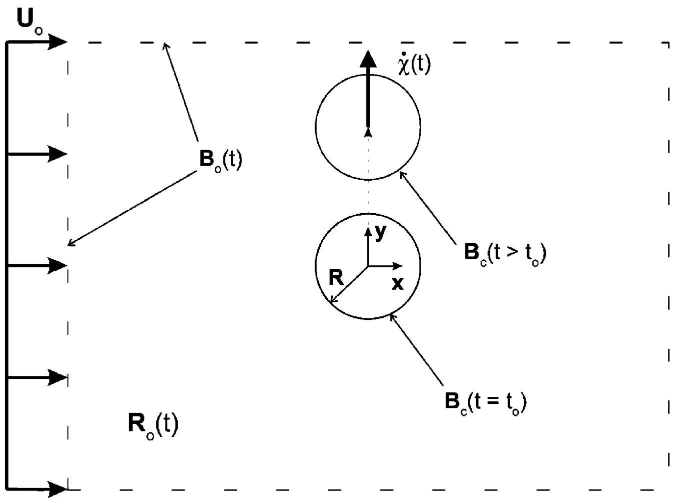

Fig. 1. Plan view of the open control surface $B_o(t)$, closed control surface $B_c(t)$ (at two different instances), and the open control volume $R_o(t)$ for the model problem. The cylinder is only free to move transversely to the uniform incoming flow. The restraining springs are not shown.

where the terms with numbered overbraces are those that are to be simplified. In Eq. (2), $\mathbf{u}(\mathbf{x}, t)$ is the fluid velocity field and $\hat{\sigma}_{ij}$ are the components of the stress dyadic $\underline{\boldsymbol{\sigma}}$ for an incompressible fluid in the x–y rectangular coordinate system of Fig. 1. The boundary traction $\mathbf{Q}(\mathbf{x}, t)$ is a prescribed quantity. Note that the structural damping force is always opposite in direction to the velocity, such that the nonconservative virtual work, $\delta W_{o_{\text{cyl}}} = c^{(\text{vac})}\dot{\chi}\delta\chi$, is negative for positive $\dot{\chi}$.

In order to reduce the complexity of Eq. (2), the control volume $R_o(t)$ is first reduced to a small rectangular region R^{**} incorporating the formation region. The negative damping condition initiating the cylinder motion, as well as the periodic wake feeding the growing amplitudes of the cylinder are generated in the formation region. The existence of a temporal global wake instability (see, for example, Refs. [9–11]) in the formation region allows a second, more crucial simplification to be made. The flow in R^{**} is assumed to be represented by the representative mass m_{fl} whose transverse displacement is $w(t)$. All spatial dependencies are lost.

It emphasized that while $\dot{w}(t)$ and $\dot{\chi}(t)$ ($\ddot{w}(t)$ and $\ddot{\chi}(t)$) are both transverse velocities (accelerations), they need not always have the same sign at any one instant. It is therefore important to talk about *relative* velocities (accelerations).

Term “1” of Eq. (2) is reduced to

$$\delta \int_{t_1}^{t_2} \iiint_{R_o(t)} \frac{1}{2}\rho(\mathbf{u}\cdot\mathbf{u})\,\mathrm{d}v\,\mathrm{d}t \Rightarrow \int_{t_1}^{t_2} \hat{a}_0 m_{\text{fl}}\dot{w}\delta\dot{w}\,\mathrm{d}t, \tag{3}$$

where $\hat{a}_0$ is a dimensionless constant. Term “2” can likewise be reduced. Term “3” is eliminated when the assumption is made that energy is (approximately) evenly exchanged between the structure and the wake. In essence, the near-wake is treated as a “closed” cavity such that the net rate of momentum flux through the boundaries of R^{**} is virtually nil. Note that the energy balance argument has been employed previously by Krenk and Nielsen [12]. A more elaborate discussion on the simplification of the aforementioned terms can be found in Ref. [4].

Suppose the following separation is made:

$$-\int_{t_1}^{t_2} \hat{\sigma}_{ij}\left[\frac{\partial(\delta r_i)}{\partial x_j}\right]\mathrm{d}v\,\mathrm{d}t \Rightarrow -\int_{t_1}^{t_2} \delta W(\dot{w}, \ddot{w}, \chi, \dot{\chi}, \ddot{\chi}, t)\,\mathrm{d}t - \int_{t_1}^{t_2} F(w, t)\delta w\,\mathrm{d}t. \tag{4}$$

The functional $F(w, t) = \hat{a}_1 m_{\text{fl}} f_{\text{st}} U_o w(t)/D$ represents the “fluid stiffness” term, where $\hat{a}_1$ is a dimensionless constant and $f_{\text{st}} = \omega_{\text{st}}/2\pi$. The positive square-root $(\hat{a}_1 m_{\text{fl}} f_{\text{st}} U_o/D)^{0.5}$ represents the natural frequency of the undamped wake-oscillator for small $w(t)$ (no motion of the cylinder). This form is consistent with the observation that the damped (and hence the undamped) natural frequency of the wake-oscillator must change as the flow velocity U_o changes [13].

Using Eqs. (3) and (4), Eq. (2) becomes

$$\int_{t_1}^{t_2} \hat{a}_0 m_{\mathrm{fl}} \dot{w}\delta\dot{w}\,\mathrm{d}t + \delta\int_{t_1}^{t_2} \frac{1}{2} m_c \dot{\chi}^2\,\mathrm{d}t - \delta\int_{t_1}^{t_2} \frac{1}{2} k_s^{(y)} \chi^2\,\mathrm{d}t - \int_{t_1}^{t_2} c^{(\mathrm{vac})}\dot{\chi}\delta\chi\,\mathrm{d}t$$
$$- \int_{t_1}^{t_2} \delta W(\dot{w}, \ddot{w}, \chi, \dot{\chi}, \ddot{\chi}, t)\,\mathrm{d}t - \int_{t_1}^{t_2} F(w,t)\delta w\,\mathrm{d}t = 0. \tag{5}$$

It is assumed that the functional $\delta W(\dot{w}, \ddot{w}, \dot{\chi}, \ddot{\chi}, t)$ can be separated as follows:[2]

$$\delta W(\dot{w}, \ddot{w}, \chi, \dot{\chi}, \ddot{\chi}, t) = -F^{(y)}_{\mathrm{fl/st}}(\dot{w}, \ddot{w}, \dot{\chi}, \ddot{\chi}, t)\delta\chi + F^{(y)}_{\mu/p}(\dot{w}, \ddot{w}, \dot{\chi}, \ddot{\chi}, t)\delta w. \tag{6}$$

$F^{(y)}_{\mathrm{fl/st}}(\dot{w}, \ddot{w}, \dot{\chi}, \ddot{\chi}, t)\delta\chi$ is the instantaneous virtual work done by total transverse hydrodynamic force acting on the cylinder, while $F^{(y)}_{\mu/p}(w, \dot{w}, \ddot{w}, \chi, \dot{\chi}, \ddot{\chi}, t)\delta w$ represents virtual work done by the vertical components of the viscous (μ) and pressure (p) forces within R^{**}, excluding the boundary of the cylinder. The negative sign on the $F^{(y)}_{\mathrm{fl/st}}(\dot{w}, \ddot{w}, \dot{\chi}, \ddot{\chi}, t)\delta\chi$ term is due to the fact that on the surface of the cylinder $\delta\chi = -\delta w$ because of the no-slip condition.

In Eq. (6), the following form is assumed for $F^{(y)}_{\mathrm{fl/st}}$:

$$F^{(y)}_{\mathrm{fl/st}}(\dot{w}, \ddot{w}, \dot{\chi}, \ddot{\chi}, t) = -\tfrac{1}{4}\rho\pi D^2 L C_a \ddot{\chi}(t)$$
$$+ \tfrac{1}{2}\rho D L C_d[\dot{w}(t) - \dot{\chi}(t)]|\dot{w}(t) - \dot{\chi}(t)| + \tfrac{1}{4}\pi\rho D^2 L(1 + C_a)\ddot{w}(t). \tag{7}$$

C_d represents the component of the instantaneous vortex lift coefficient $C_{y_{\mathrm{vortex}}}(t)$ that is out-of-phase with the cylinder displacement.

Eq. (7) is equivalent to the Morison–O'Brien–Johnson–Schaff (MOJS) equation for the fluid force on a cylinder moving parallel to a time-dependent fluid stream [14]. In principle, geometric considerations require that the MOJS equation be modified when the cylinder is moving transversely to the free stream. However, Eq. (7) is retained unaltered with the understanding that said equation can then be only referred to as "MOJS-like."

The following form is assumed for $F^{(y)}_{\mu/p}(\dot{w}, \ddot{w}, \chi, \dot{\chi}, \ddot{\chi}, t)$:

$$F^{(y)}_{\mu/p}(\dot{w}, \ddot{w}, \dot{\chi}, \ddot{\chi}, t) = \hat{a}_2 m_{\mathrm{fl}} f_{\mathrm{st}}[\dot{w}(t) - \dot{\chi}(t)] + \frac{\hat{a}_3 m_{\mathrm{fl}} f_{\mathrm{st}}}{U_o^2}[\dot{w}(t) - \dot{\chi}(t)]^3. \tag{8}$$

The $\hat{a}_i$'s are again dimensionless constants.

It is easily shown that Eqs. (7) and (8) change sign under the coordinate transformation $y := -y$. This is a manifestation of the invariance of the forces these equations represent to this particular type of coordinate transformation, an invariance that is required in Newtonian mechanics.

The functional $F^{(y)}_{\mu/p}(\dot{w}, \ddot{w}, \dot{\chi}, \ddot{\chi}, t)$ has two distinct roles. In the first place, it is intended to capture the nonlinear damping effects in the wake-oscillator, much like the $\varepsilon f_{\mathrm{st}}(\dot{q}^2(t) - 1)\dot{q}(t)$ term in the Rayleigh equation, or the $\varepsilon f_{\mathrm{st}}(q^2(t) - 1)\dot{q}(t)$ term in the van der Pol equation. This damping term must be such that the wake-oscillator is self-excited and self-limiting.

Self-excitation of the wake is due to amplification by the shear-layers of initial instabilities generated at the separation points, and an upstream influence caused by a region of absolute instability in the near wake. This region of absolute instability, whose downstream boundary is the point in the wake where traveling waves can be reflected, is associated with causing the propagation an upstream traveling wave disturbance which amounts to a "feedback" to the separation points.

In addition, $F^{(y)}_{\mu/p}(\dot{w}, \ddot{w}, \dot{\chi}, \ddot{\chi}, t)$ must represent the nonlinear interaction (i.e., the right-hand side) between the wake-oscillator and the motion of the cylinder.

[2]Any explicit dependence on $\chi(t)$ has been excluded. This is purely for the purpose of simplicity.

First, Eqs. (7) and (8) are substituted in Eq. (6). The result is then replaced in Eq. (5), and the indicated variations performed. The conditions $\delta w|_{t_1}^{t_2} = \delta\chi|_{t_1}^{t_2} = 0$ are imposed and similar terms collected. The independence of the variations $\delta\chi$ and δw leads to the following set of coupled differential equations:

$$\begin{aligned}(m_c + \tfrac{1}{4}\pi\rho D^2 L C_a)\ddot{\chi}(t) + c^{(\mathrm{vac})}\dot{\chi}(t) + k_s^{(y)}\chi(t) \\ = \tfrac{1}{2}\rho D L C_d|\dot{w}(t) - \dot{\chi}(t)|[\dot{w}(t) - \dot{\chi}(t)] + \tfrac{1}{4}\pi\rho D^2 L(C_a + 1)\ddot{w}(t)\end{aligned} \tag{9}$$

and

$$\begin{aligned}\hat{a}_0 m_{\mathrm{fl}}\ddot{w}(t) + \hat{a}_1 m_{\mathrm{fl}}\frac{U_o f_{\mathrm{st}}}{D}w(t) + \frac{m_{\mathrm{fl}} f_{\mathrm{st}}}{U_o^2}[\hat{a}_3\dot{w}^2(t) + \hat{a}_2 U_o^2]\dot{w}(t) \\ = \hat{a}_2 m_{\mathrm{fl}} f_{\mathrm{st}}\dot{\chi}(t) + \frac{\hat{a}_3 m_{\mathrm{fl}} f_{\mathrm{st}}}{U_o^2}\dot{\chi}^3(t) + \frac{3\hat{a}_3 m_{\mathrm{fl}} f_{\mathrm{st}}}{U_o^2}[\dot{w}^2(t)\dot{\chi}(t) - \dot{w}(t)\dot{\chi}^2(t)].\end{aligned} \tag{10}$$

2. Nondimensionalization of the derived wake-body model

Let the dimensionless cylinder and wake displacement variables be

$$\Xi(t) = \frac{\chi(t)}{D}$$

and

$$W(t) = \frac{w(t)}{D},$$

respectively. The dimensionless time variable is defined as

$$T = t\omega_{\mathrm{st}}.$$

The notation $(\cdot)'$ is used to denote time derivatives with respect to T.

2.1. The structural oscillator

Nondimensionalizing Eq. (9) via the above variables yields

$$\begin{aligned}(m_c + m_d C_a)\Xi''(T) + \frac{c^{(\mathrm{vac})}}{\omega_{\mathrm{st}}}\Xi'(T) + \frac{k_s^{(y)}}{\omega_{\mathrm{st}}^2}\Xi(T) \\ = \frac{1}{2}\rho D^2 L C_d|W'(T) - \Xi'(T)|[W'(T) - \Xi'(T)] + m_d(1 + C_a)W''(T).\end{aligned} \tag{11}$$

Eq. (11) can be simplified and rewritten in a number of different ways. These differ fundamentally with respect to the definitions of the natural frequency, the damping ratio, and the mass ratio that are employed. Table 2 summarizes the various definitions that are of importance here. In the interest of making the paper more readable, the various forms of Eq. (11) are not all presented in one place. Rather, they are presented as needed in subsequent sections of the paper.

Table 2
A number of possible characterizations of the cylinder natural frequency, and damping and mass ratios

Parameter	Value	Description
$\omega_n^{(\mathrm{vac})}$	$\sqrt{\frac{k_s^{(y)}}{m_c}}$	Cylinder natural frequency *in vacuo*
$\omega_n^{(\mathrm{true})}$	$\sqrt{\frac{k_s^{(y)}}{(m_c+C_a m_d)}}$	Cylinder true (or *in situ*) natural frequency
$\omega_n^{(\mathrm{st.\ water})}$	$\sqrt{\frac{k_s^{(y)}}{(m_c+C_A m_d)}}$	Cylinder natural frequency in still water
$\zeta^{(\mathrm{vac})}$	$\frac{c^{(\mathrm{vac})}}{2\omega_n^{(\mathrm{vac})} m_c}$	Damping ratio (material/critical) *in vacuo*
$\zeta^{(\mathrm{true})}$	$\zeta^{(\mathrm{vac})}\sqrt{\frac{1}{\hat{m}^* C_a}}$	True (or *in situ*) damping ratio
$\zeta^{(\mathrm{st.\ water})}$	$\zeta^{(\mathrm{vac})}\sqrt{\frac{1}{\hat{m}^* C_A}}$	Damping ratio in still water
μ	$\frac{m^*}{1+m^* C_a}$	Mass ratio including added mass
$V_r^{(\mathrm{vac})}$	$\frac{2\pi U_o}{\omega_n^{(\mathrm{vac})} D}$	Reduced velocity defined using the *in vacuo* natural frequency

2.2. The wake oscillator

Introducing the dimensionless variables $(\Xi, \Xi', \Xi'', W, W', W''; T)$ into Eq. (10) gives

$$\begin{aligned} W''(T) &+ \left[(2\pi S)^2 \frac{\hat{a}_3}{\hat{a}_0} W'^2(T) + \frac{\hat{a}_2}{\hat{a}_0}\right] W'(T) + \frac{1}{(2\pi S)} \frac{\hat{a}_1}{\hat{a}_0} W(T) \\ &= -3(2\pi S)^2 \frac{\hat{a}_4}{\hat{a}_0} [W'(T)\Xi'^2(T) - \Xi'(T) W'^2(T)] \\ &\quad + (2\pi S)^2 \frac{\hat{a}_3}{\hat{a}_0} \Xi'^3(T) + \frac{\hat{a}_2}{\hat{a}_0} \Xi'(T), \end{aligned} \tag{12}$$

where it is assumed $\hat{a}_o \neq 0$.

Consider for the moment the case of a stationary cylinder. In this case, $\Xi(T)$ and all its derivatives are identically zero. Eq. (12) then reduces to

$$W_o''(T) + \left[(2\pi S)^2 \frac{\hat{a}_3}{\hat{a}_0} W_o'^2(T) + \frac{\hat{a}_2}{\hat{a}_0}\right] W_o'(T) + \frac{1}{(2\pi S)} \frac{\hat{a}_1}{\hat{a}_0} W_o(T) = 0, \tag{13}$$

where

$$W_o(T) = \frac{w_o(T)}{D}.$$

The van der Pol and Rayleigh equations are the nonlinear oscillators most commonly used to model the fluctuating nature of the vortex shedding. For a stationary cylinder, they adequately model the self-sustained, quasi-harmonic oscillations seen experimentally in the lift coefficient, for example. For definiteness, the Rayleigh equation is taken as the "reference" nonlinear wake oscillator. The idea is then to construct a Rayleigh-type equation from Eq. (13).

The dimensionless Rayleigh equation,

$$Q''(T) + \varepsilon(Q'^2(T) - 1)Q'(T) + Q(T) = 0,$$

with $0 < \varepsilon \ll 1$, is known to provide a stable quasi-harmonic oscillation of finite amplitude at the frequency

$$\Omega = 1.$$

Eq. (13) reduces to the Rayleigh type provided that

$$\frac{1}{(2\pi S)}\frac{\hat{a}_1}{\hat{a}_0} = 1,$$

$$\frac{\hat{a}_2}{\hat{a}_0} < 0,$$

$$\left|\frac{\hat{a}_2}{\hat{a}_0}\right| \ll 1,$$

$$\frac{\hat{a}_3}{\hat{a}_0} > 0$$

and

$$(2\pi S)^2\left|\frac{\hat{a}_3}{\hat{a}_0}\right| \ll 1.$$

It is evident from the above conditions that if $\hat{a}_0 < 0$, then $\hat{a}_2 > 0$ and $\hat{a}_{1,3} < 0$. On the other hand, if $\hat{a}_0 > 0$, then $\hat{a}_2 < 0$ and $\hat{a}_{1,3} > 0$.

Next, define

$$\hat{b}_i = \left|\frac{\hat{a}_i}{\hat{a}_0}\right|,$$

where $i = 2, 3$. The sign of the model constant $\hat{a}_4$ is not known *a priori,* and, therefore, there are no constraints to determine the sign of the ratio $\hat{a}_4/\hat{a}_0$. As a result, said ratio is represented as b_4, where $b_4 \gtreqless 0$.

Eq. (12) can now be written as

$$\begin{aligned} &W''(T) + [(2\pi S)^2\hat{b}_3 W'^2(T) - \hat{b}_2]W'(T) + W(T) \\ &\quad = -3(2\pi S)^2 b_4[W'(T)\Xi'^2(T) - \Xi'(T)W'^2(T)] + (2\pi S)^2\hat{b}_3\Xi'^3(T) - \hat{b}_2\Xi'(T). \end{aligned} \tag{14}$$

Upon examining Eqs. (11) and (14), it is apparent that the number of model parameters is reduced to five, $(\hat{b}_2, \hat{b}_3, b_4, C_a, C_d)$. However, since $\hat{b}_2$, $\hat{b}_3$, and b_4 are not all independent, the true number of independent model parameters is actually six, $(\hat{a}_0, \hat{a}_2, \hat{a}_3, \hat{a}_4, C_a, C_d)$.

3. The comparison models

3.1. Hall (1981)

As part of his doctoral dissertation, Hall [5] considers a modified Blevins model for the transverse VIV of a spring-mounted rigid cylinder in uniform fluid flow. When said fluid is water, the model equations are given in dimensionless form[3]

$$\begin{aligned} &\Xi''(T) + \frac{1}{(1+\eta a_3)}\left[2\zeta^{(\text{st. water})}\frac{\omega_n^{(\text{st. water})}}{\omega_{\text{st}}} + \frac{a_4\eta}{2\pi S}\right]\Xi'(T) + \left(\frac{1}{\sqrt{(1+\eta a_3)}}\frac{\omega_n^{(\text{st. water})}}{\omega_{\text{st}}}\right)^2\Xi(T) \\ &\quad = \left[\frac{a_4\eta}{2\pi S(1+\eta a_3)}\right]Z'(T) + \left[\frac{(a_3+a_5)\eta}{(1+\eta a_3)}\right]Z''(T) \end{aligned} \tag{15}$$

[3]Hall defines his mechanical parameters per unit length. The necessary invariance of the equations to scaling, however, means they can simply be rewritten using the relevant parameters in Table 2 without compromising their validity.

and

$$Z''(T) - \left[\frac{(a_1 - a_4)}{2\pi S(a_0 + a_3 + a_5)}\right] Z'(T) + \left[\frac{2\pi S a_2}{(a_0 + a_3 + a_5)}\right] Z'^3(T) + Z(T) = \left[\frac{a_4}{2\pi S(a_0 + a_3 + a_5)}\right] \Xi'(T) + \left[\frac{a_3}{(a_0 + a_3 + a_5)}\right] \Xi''(T). \tag{16}$$

In the above equations, the a_i's are independent model constants and

$$\eta = \frac{\rho D^2 L}{(m_c + m_d)}$$

is a mass ratio based on the potential flow added mass.[4]

Note that there exists the following relationship between η and $\hat{m}^*$:

$$\eta = \frac{\rho D^2 L}{(m_c + m_d)} = \frac{\rho D^2 L}{m_c(1 + \hat{m}^*)} = \frac{4}{\pi} \frac{\hat{m}^*}{(1 + \hat{m}^*)}. \tag{17}$$

Eq. (15) can now be written as

$$\Xi''(T) + \frac{1}{(1 + \eta a_3)} \left[2\zeta^{(\text{st. water})} \frac{\omega_n^{(\text{st. water})}}{\omega_{\text{st}}} + \frac{a_4 \eta}{2\pi S} \right] \Xi'(T) + \left(\frac{1}{\sqrt{(1 + \eta a_3)}} \frac{\omega_n^{(\text{st. water})}}{\omega_{\text{st}}} \right)^2 \Xi(T) = \left[\frac{a_4 \eta}{2\pi S(1 + \eta a_3)} \right] Z'(T) + \left[\frac{(a_3 + a_5)\eta}{(1 + \eta a_3)} \right] Z''(T). \tag{18}$$

T and Ξ have the same meaning in Eqs. (16) and (18) as they do in Eqs. (11) and (14). In principle, however, $Z(T)$ has a slightly different interpretation than does $W(T)$. $Z'(T)$ is in fact related to the dimensionless form of the Blevins' [16] "hidden fluid variable," $\dot{z}(t)$.

It is crucial to point out that Hall derives his model for a horizontal cylinder. The "Hall hidden fluid variable" is then defined as

$$\dot{z}_{\text{Hall}}(t) = D\omega_{\text{st}} Z'(T) = \frac{1}{a_0 \rho D^2} \iint_{A_o} \rho u_2(y, z, t)\, \mathrm{d}y\, \mathrm{d}z,$$

where A_o is the cross-sectional area of the rectangular control volume of unit axial (x) dimension that surrounds the cylinder.

The distinction between Hall's $\dot{z}_{\text{Hall}}(t)$ and Blevins' $\dot{z}(t)$ vanishes when Hall's model is formulated for a *vertical* cylinder. This is accomplished by rotating Hall's coordinate system about the y-axis by an angle $\alpha = -\pi/2$ radians (i.e., counterclockwise), and then setting a_5 equal to zero. The order of these operations is immaterial.

Is $Z'(T)$ related to $W'(T)$? To find the answer to this question, suppose $u_2(y, z, t)$ is approximately constant over some subset A_o^* of A_o. As a result,

$$\dot{z}(t) \simeq \frac{1}{a_0 \rho D^2} u_2(t) \iint_{A_o^*} \rho\, \mathrm{d}x_2\, \mathrm{d}x_3 = \frac{1}{a_0 \rho D^2} u_2(t) \beta,$$

where β is simply a constant as ρ is assumed constant (the flow is incompressible). From the assumption of fully correlated shedding in the synchronization regime, A_o^* is the same at each axial station. As a result,

$$\beta L = \tilde{m}_{\text{fl}},$$

where $\tilde{m}_{\text{fl}}$ is the total mass of fluid contained in the volume $A_o^* L$. It is then possible to write

$$\dot{z}(t) \simeq \frac{1}{a_0 \rho D^2 L} u_2(t) \tilde{m}_{\text{fl}}.$$

[4]Hall does not directly define his mass in this way. He merely refers to the "mass per unit length." The same ambiguity exists in the Blevins model: In Ref. [15] reference is also made to the "mass per unit length m" , while in Ref. [16] the same mass m includes "the entrained mass of fluid."

If A_o^* coincides with the x–y projection of the region R^{**}, then $\tilde{m}_{\rm fl} = m_{\rm fl}$ and $u_2(t)$ coincides with the wake degree-of-freedom $\dot{w}(t)$. It follows that

$$\dot{z}(t) \simeq \frac{1}{a_0 \rho D^2 L} \dot{w}(t) m_{\rm fl}.$$

In addition, if $m_{\rm fl} = \rho D^2 L$, as is Ref. [12], then

$$\dot{z}(t) \simeq \frac{1}{a_0} \dot{w}(t).$$

It is clear that $Z'(T)$ is indeed related to $W'(T)$, at least in a limiting sense. This supports the notion that model represented by Eqs. (9) and (14) can be meaningfully compared with Hall's model equations.

To that end, consider first rewriting Eq. (11) as

$$\begin{aligned} &\Xi''(T) + \frac{2}{\lambda}\zeta^{(\text{st. water})}\frac{\omega_n^{(\text{st. water})}}{\omega_{\rm st}}\Xi'(T) + \left(\frac{1}{\sqrt{\lambda}}\frac{\omega_n^{(\text{st. water})}}{\omega_{\rm st}}\right)^2 \Xi(T) \\ &\quad = \frac{2}{\pi}\frac{\hat{m}^*}{\lambda(1+\hat{m}^*)} C_d |W'(T) - \Xi'(T)| [W'(T) - \Xi'(T)] \\ &\quad + \frac{\hat{m}^*}{\lambda(1+\hat{m}^*)} (C_a^* + 2) W''(T), \end{aligned} \tag{19}$$

where the parameters $\omega_n^{(\text{st. water})}$ and $\zeta^{(\text{st. water})}$ are defined in Table 1, and $\lambda = [1 + (\hat{m}^*/(1+\hat{m}^*))C_a^*]$. C_a^* represents the deviation of the added mass coefficient from its potential flow value of $C_A = 1$.

Setting $a_5 = 0$ in Eq. (18) gives

$$\begin{aligned} &\Xi''(T) + \frac{1}{(1+\eta a_3)}\left[2\zeta^{(\text{st. water})}\frac{\omega_n^{(\text{st. water})}}{\omega_{\rm st}} + \frac{a_4 \eta}{2\pi S}\right]\Xi'(T) + \left(\frac{1}{\sqrt{(1+\eta a_3)}}\frac{\omega_n^{(\text{st. water})}}{\omega_{\rm st}}\right)^2 \Xi(T) \\ &\quad = \left(\frac{a_4 \eta}{2\pi S(1+\eta a_3)}\right) Z'(T) + \left[\frac{a_3 \eta}{(1+\eta a_3)}\right] Z''(T). \end{aligned} \tag{20}$$

Suppose now that the model parameters a_3 and a_4 are related to C_a and C_d, respectively, in the following way:

$$a_3 = \frac{\pi}{4}\frac{(\hat{m}^* C_a - \hat{m}^*)}{\hat{m}^*} = \frac{\pi}{4}(C_a - 1) = \frac{\pi}{4} C_a^*,$$

and

$$a_4 = \frac{\pi}{4} C_d.$$

Using Eq. (17), the following additional relations are obtained:

$$a_3 \eta = \frac{\hat{m}^*}{(1+\hat{m}^*)} C_a^*$$

and

$$a_4 \eta = \frac{\hat{m}^*}{(1+\hat{m}^*)} C_d.$$

Defining

$$1 + a_3 \eta = \left[1 + \frac{\hat{m}^*}{(1+\hat{m}^*)} C_a^*\right] = \lambda,$$

it follows that

$$\frac{a_3 \eta}{(1+a_3 \eta)} = \frac{\hat{m}^*}{\lambda(1+\hat{m}^*)} C_a^*$$

and

$$\frac{a_4\eta}{(1+a_3\eta)} = \frac{\hat{m}^*}{\lambda(1+\hat{m}^*)} C_d.$$

Substituting the above relations into Eq. (20) yields

$$\begin{aligned} &\Xi''(T) + \frac{1}{\lambda}\left[2\zeta^{(\text{st. water})}\frac{\omega_n^{(\text{st. water})}}{\omega_{\text{st}}} + \frac{1}{2\pi S}\frac{\hat{m}^*}{(1+\hat{m}^*)}C_d\right]\Xi'(T) \\ &+ \left(\frac{1}{\sqrt{\lambda}}\frac{\omega_n^{(\text{st. water})}}{\omega_{\text{st}}}\right)^2 \Xi(T) = \frac{1}{2\pi S}\frac{\hat{m}^*}{\lambda(1+\hat{m}^*)}C_d Z'(T) \\ &+ \frac{\hat{m}^*}{\lambda(1+\hat{m}^*)}C_a^* Z''(T). \end{aligned} \quad (21)$$

When Eq. (19) is rewritten as

$$\begin{aligned} &\Xi''(T) + \frac{2}{\lambda}\zeta^{(\text{st. water})}\frac{\omega_n^{(\text{st. water})}}{\omega_{\text{st}}}\Xi'(T) + \left(\frac{1}{\sqrt{\lambda}}\frac{\omega_n^{(\text{st. water})}}{\omega_{\text{st}}}\right)^2 \Xi(T) \\ &= \frac{2}{\pi}\frac{\hat{m}^*}{\lambda(1+\hat{m}^*)}C_d|W'(T) - \Xi'(T)|[W'(T) - \Xi'(T)] \\ &+ \frac{\hat{m}^*}{\lambda(1+\hat{m}^*)}C_a^* W''(T) + \frac{2\hat{m}^*}{\lambda(1+\hat{m}^*)}W''(T), \end{aligned} \quad (22)$$

its similarity with Eq. (21) becomes apparent.

Eqs. (21) and (22) differ primarily with respect to the drag term. This difference is attributable to the fact that Hall's formulation is based on a linear drag term, proportional to $C_d[Z'(T) - \Xi'(T)]$. The presence of this linear drag term is manifested in both the fluid-added damping term on the left-hand side (LHS) of Eq. (21),

$$\frac{1}{2\pi S}\frac{\hat{m}^*}{\lambda(1+\hat{m}^*)}C_d\Xi'(T)$$

and in the forcing term proportional to $Z'(T)$ on the right-hand side (RHS) of said equation. Such a separation is not possible in Eq. (22).

In addition to the aforementioned difference, Eq. (22) possesses the additional term

$$\begin{aligned} \frac{2\hat{m}^*}{\lambda(1+\hat{m}^*)}W''(T) &= \frac{2\hat{m}^*}{\left[1+\frac{\hat{m}^*}{(1+\hat{m}^*)}C_a^*\right](1+\hat{m}^*)}W''(T) \\ &= \frac{2\hat{m}^*}{(1+\hat{m}^*C_a)}W''(T). \end{aligned} \quad (23)$$

The absence of this term in Eq. (21) can be attributed to the fact that the $Z''(T)$ and the $\Xi''(T)$ terms have the same coefficient, $a_3\eta = \hat{m}^*C_a^*/(1+\hat{m}^*)$, in the forcing function[5] Hall assumes for his structural oscillator. On the other hand, the coefficients of the $W''(T)$ and $\Xi''(T)$ terms in Eq. (19) differ, the difference between them being precisely equal to this extra term.

It is of interest to point out that Iwan and Blevins [15] set $a_3 = 0$. They argue that doing so "... does not imply that there is no added mass effect since the flow forward of the separation point remains attached to the cylinder, effectively increasing the oscillating mass of the cylinder." Furthermore, "... the best agreement between model predictions and experimental response data ... is obtained when $a_3 = 0$." If the added mass effect is not represented by the a_3 term, then this suggests that it is already captured in the effective mass which is computed using the potential flow added mass.

The assumption $a_3 = 0$ is suitable when the fluid medium is air, in which case $\hat{m}^*$ is small. The additional term of Eq. (22) can then be neglected. In a water medium, for which $\hat{m}^*$ may no longer be small, added mass

[5]See Eqs. (2.2.7) and (3.2.16) of Ref. [5].

effects are important. Indeed, the potential flow added mass becomes an increasingly inexact representation of the actual added mass as $\hat{m}^*$ decreases. Under these circumstances, the assumption $a_3 = 0$ no longer seems reasonable, and the term represented by Eq. (23) cannot be neglected.

It is difficult to make a comparison between Eqs. (14) and (16) without reinterpreting the model constants a_i $(i = 1, \ldots, 5)$. This is because in Hall's model, the a_i's appear directly in both equations. This is not a feature of the present model. With $a_5 = 0$, Eq. (16) becomes

$$Z''(T) + \frac{2\pi S}{(a_0 + a_3)}\left[a_2 Z'^2(T) - \frac{(a_1 - a_4)}{(2\pi S)^2}\right] Z'(T) + Z(T) = \frac{a_4}{2\pi S(a_0 + a_3)}\Xi'(T) + \frac{a_3}{(a_0 + a_3)}\Xi''(T). \tag{24}$$

Noting that a $\Xi''(T)$ term is absent from the of Eq. (14), one can also set

$$a_3 = 0.$$

Next, suppose

$$\frac{a_2}{(a_0 + a_3)} = \frac{a_2}{a_0} = 2\pi S \hat{b}_3$$

and

$$\left(\frac{a_1 - a_4}{a_0 + a_3}\right) = \frac{(a_1 - a_4)}{a_0} = 2\pi S \hat{b}_2. \tag{25}$$

Eq. (24) then becomes

$$Z''(T) + [(2\pi S)^2 \hat{b}_3 Z'^2(T) - \hat{b}_2] Z'(T) + Z(T) = \frac{a_4}{2\pi S a_0}\Xi'(T).$$

Solving for a_4 from Eq. (25) and then substituting the result in the above equation yields

$$Z''(T) + [(2\pi S)^2 \hat{b}_3 Z'^2(T) - \hat{b}_2] Z'(T) + Z(T) = \frac{1}{2\pi S}\left[\frac{a_1}{a_0} - (2\pi S)\hat{b}_2\right]\Xi'(T). \tag{26}$$

A comparison of Eqs. (14) and (26) reveals that the main difference lies in the forcing function on the right-hand side of each equation. The forcing function in Eq. (14) is the sum of the nonlinear function,

$$f_{\mathrm{NL}}(W'(T), \Xi'(T)) = (2\pi S)^2 \sum_{i=0}^{2} \alpha_i \Xi'^{3-i}(T) W'^{i}(T) (\alpha_0 = \hat{b}_3, \alpha_1 = \alpha_2 = \hat{b}_4) \tag{27}$$

and the linear function

$$f_L(\Xi'(T)) = -\hat{b}_2 \Xi'(T).$$

The nonlinear function $f_{\mathrm{NL}}(W'(T), \Xi'(T))$ represents the autoparametric excitation.

In contrast, the forcing function in Eq. (26) is purely a linear function of the cylinder velocity, and $f_{\mathrm{NL}}(W'(T), \Xi'(T)) = 0$. This type of forcing has been extensively used in wake oscillator models (e.g., Refs. [12,17]).

In summary, it is apparent that the model represented by Eqs. (19) and (14) compares well with the model equations derived by Hall [5]. The major discrepancy between each of the model equations and their counterparts in Hall's formulation is in the form of the forcing function. Hall's forcing functions are in each case linear; the forcing functions in the model derived here include additional nonlinear terms.

3.2. Berger (1988)

Berger's [6] model represents a departure from the previous comparison model in that it uses the lift coefficient as the fluid internal degree-of-freedom. Nonetheless, Berger's model serves a useful comparison because it is generally believed to be one of the most successful improvements on the original Hartlen–Currie model [2]. Its touted success lies primarily in its ability to correctly predict, both quantitatively and qualitatively, oscillation hysteresis for certain values of $\zeta^{(\mathrm{vac})}$.

Berger derives the following pair of dimensionless coupled equations as a model for the transverse VIV of an elastically mounted rigid circular cylinder lying in a uniform stream of air:

$$\Xi''(\tau) + \delta(\Xi'(\tau))\Xi'(\tau) + \Xi(\tau) = \frac{1}{2\pi^3 L}\hat{m}^* V_r^{(\mathrm{vac})^2} C_y(\tau) = a\Omega^2 C_y(\tau) \tag{28}$$

and

$$C_y''(\tau) + f^*(C_y'(\tau))C_y'(\tau) + \Omega^2 C_y(\tau) = b(\Xi'(\tau))\Xi'(\tau). \tag{29}$$

$C_y(\tau)$ is the lift coefficient,[6] $\delta(\Xi'(\tau))$ is the nonlinear structural damping function, a is a coupling constant, $f^*(C_y'(\tau))$ is the nonlinear damping function for the wake oscillator, and $b(\Xi'(\tau))$ is a nonlinear feedback parameter. According to Berger, the $\delta(\Xi'(\tau))$ term is used to model the effects of viscous forces. The reduced velocity $V_r^{(\mathrm{vac})}$ is defined in Table 2.

The model parameters are defined as

$$\delta(\Xi'(\tau)) = \sum_{k=0}^{m} \delta_{2k}\Xi'^{2k}(\tau),$$

$$f^*(C_y'(\tau)) = \sum_{k=0}^{m} \alpha_{2k} C_y'^{2k}(\tau), \text{ with } \alpha_0 < 0 \text{ and } \alpha_{2m} > 0,$$

$$b(\Xi'(\tau)) = \sum_{k=0}^{m} b_{2k}\Xi'^{2k}(\tau)$$

and

$$\Omega = \frac{f_{\mathrm{st}}}{f_n^{(\mathrm{vac})}} = 2\pi S V_r^{(\mathrm{vac})}. \tag{30}$$

The coefficient set $(\delta_{2k}, \alpha_{2k}, b_{2k})$ represents the model constants that are to be determined from experimental data. Note that the expansions for $\delta(\Xi'(\tau)), f^*(C_y'(\tau))$ and $b(\Xi'(\tau))$ are all necessarily symmetric with respect to the neutral position of the cylinder, $\Xi(\tau) = 0$. This guarantees a system of equations, Eqs. (28) and (29), that is invariant to the sign changes $y := -y$ and $C_y := -C_y$.

Berger defines the dimensionless cylinder displacement as $\Xi(t) = y(t)/D$ and the dimensionless time as $\tau = \omega_n^{(\mathrm{vac})} t$. His equations are now modified so that they are in terms of the dimensionless time T.

To than end, the following relationship between the time derivatives is used:

$$\frac{\mathrm{d}^n(\cdot)}{\mathrm{d}\tau^n} = \left(\frac{\omega_{\mathrm{st}}}{\omega_n^{(\mathrm{vac})}}\right)^n \frac{\mathrm{d}^n(\cdot)}{\mathrm{d}T^n}.$$

In order to distinguish the parenthesis indicating a functional relationship, $f^*(C_y'(\tau))$ for example, from those used solely for grouping purposes, it is convenient to introduce the variable

$$\kappa = \frac{\omega_n^{(\mathrm{vac})}}{\omega_{\mathrm{st}}}. \tag{31}$$

Using the above relationships and the definition of κ, the dimensionless structural equation of motion, Eq. (28), can be written as

$$\Xi''(T) + \kappa\delta(\kappa^{-1}\Xi'(T))\Xi'(T) + \kappa^2\Xi(T) = a\Omega^2\kappa^2 C_y(T). \tag{32}$$

[6]The fluid force term $C_y(\tau)$ implicitly contains the added mass effects.

Effecting the change of variables $\tau \to T$ in the wake oscillator, Eq. (29), and again using the definition of κ yields

$$C_y''(T) + \kappa f^*(\kappa^{-1}C_y'(T))C_y'(T) + \Omega^2\kappa^2 C_y(T) = \kappa b(\kappa^{-1}\Xi'(T))\Xi'(T). \tag{33}$$

Following Berger and Plaschko [18], only the first two terms are retained in each of the power series representations for $\delta(\Xi'(\tau))$, $f^*(C_y'(\tau))$, and $b(\Xi'(\tau))$. Accordingly,

$$\delta(\Xi'(\tau)) = \delta_0 + \delta_2\Xi'^2(\tau),$$

$$f^*(C_y'(\tau)) = \alpha_0 + \alpha_2 C_y'^2(\tau), \text{ with } \alpha_0 < 0 \text{ and } \alpha_2 > 0,$$

$$b(\Xi'(\tau)) = b_0 + b_2\Xi'^2(\tau).$$

In terms of T, these become

$$\delta(\kappa^{-1}\Xi'(T)) = \delta_0 + \delta_2\kappa^{-2}\Xi'^2(T),$$

$$f^*(\kappa^{-1}C_y'(T)) = \alpha_0 + \alpha_2\kappa^{-2}C_y'^2(T), \text{ with } \alpha_0 < 0 \text{ and } \alpha_2 > 0$$

and

$$b(\kappa^{-1}\Xi'(T)) = b_0 + b_2\kappa^{-2}\Xi'^2(T).$$

Substitution of the truncated expansions into Eqs. (32) and (33) yields

$$\Xi''(T) + \kappa[\delta_0 + \delta_2\kappa^{-2}\Xi'^2(T)]\Xi'(T) + \kappa^2\Xi(T) = a\Omega^2\kappa^2 C_y(T)$$

and

$$C_y''(T) + \kappa[\alpha_0 + \alpha_2\kappa^{-2}C_y'^2(T)]C_y'(T) + \Omega^2\kappa^2 C_y(T) = \kappa[b_0 + b_2\kappa^{-2}\Xi'^2(T)]\Xi'(T),$$

respectively. After rearrangement, one finds

$$\Xi''(T) + [\delta_0\kappa + \delta_2\kappa^{-1}\Xi'^2(T)]\Xi'(T) + \kappa^2\Xi(T) = a\Omega^2\kappa^2 C_y(T) \tag{34}$$

and

$$C_y''(T) + [\alpha_0\kappa + \alpha_2\kappa^{-1}C_y'^2(T)]C_y'(T) + \Omega^2\kappa^2 C_y(T) = b_0\kappa\Xi'(T) + b_2\kappa^{-1}\Xi'^3(T), \tag{35}$$

respectively.

Noting that

$$a\Omega^2\kappa^2 = a\left(\frac{f_{\text{st}}}{f_n^{(\text{vac})}}\right)^2\left(\frac{2\pi f_n^{(\text{vac})}}{2\pi f_{\text{st}}}\right)^2 = a, \tag{36}$$

Eq. (34) can be rewritten as

$$\Xi''(T) + \delta_0\kappa\Xi'(T) + \kappa^2\Xi(T) + \delta_2\kappa^{-1}\Xi'^3(T) = aC_y(T). \tag{37}$$

Returning to Eq. (11), consider rewriting it in the form

$$\begin{aligned} &m_c\ddot{\chi}(t) + c^{(\text{vac})}\dot{\chi}(t) + k_s^{(y)}\chi(t) \\ &\quad = \tfrac{1}{2}\rho DLC_d|\dot{w}(t) - \dot{\chi}(t)|[\dot{w}(t) - \dot{\chi}(t)] + m_d(1 + C_a)\ddot{w}(t) - m_d C_a\ddot{\chi}(t). \end{aligned} \tag{38}$$

The right-hand side of Eq. (38) simply manifests the assumed form for

$$F_{\text{fl/st}}^{(y)}(t) = \tfrac{1}{2}\rho U_o^2 DLC_y(t)$$

(see Eq. (7)). Thus, Eq. (38) is equivalent to

$$m_c\ddot{\chi}(t) + c^{(\text{vac})}\dot{\chi}(t) + k_s^{(y)}\chi(t) = \tfrac{1}{2}\rho U_o^2 DLC_y(t).$$

Nondimensionalization of this equation yields

$$\Xi''(T) + 2\zeta^{(\mathrm{vac})}\left(\frac{\omega_n^{(\mathrm{vac})}}{\omega_{\mathrm{st}}}\right)\Xi'(T) + \left(\frac{\omega_n^{(\mathrm{vac})}}{\omega_{\mathrm{st}}}\right)^2 \Xi(T) = \frac{1}{2}\frac{\rho U_o^2 L}{m_c \omega_{\mathrm{st}}^2} C_y(T). \tag{39}$$

Upon using the definitions of $\hat{m}^*$ and $V_r^{(\mathrm{vac})}$, Eq. (39) can be written as

$$\Xi''(T) + 2\zeta^{(\mathrm{vac})}\left(\frac{\omega_n^{(\mathrm{vac})}}{\omega_{\mathrm{st}}}\right)\Xi'(T) + \left(\frac{\omega_n^{(\mathrm{vac})}}{\omega_{\mathrm{st}}}\right)^2 \Xi(T) = \frac{1}{2\pi^3}\hat{m}^* V_r^{(\mathrm{vac})^2}\left(\frac{\omega_n^{(\mathrm{vac})}}{\omega_{\mathrm{st}}}\right)^2 C_y(T). \tag{40}$$

Finally, upon introducing Ω from Eq. (30) and κ from Eq. (31) into Eq. (40), one obtains

$$\Xi''(T) + 2\zeta^{(\mathrm{vac})}\kappa\Xi'(T) + \kappa^2\Xi(T) = a\Omega^2\kappa^2 C_y(T) = aC_y(T), \tag{41}$$

where the final equality follows from Eq. (36).

Comparing Eqs. (37) and (41), it is evident that the right-hand sides are the same. That the right-hand side of each equation lacks terms proportional to $C_y''(T)$ and/or $C_y'(T)$ is of little surprise. The lift coefficient is, after all, a dimensionless force. If its time derivatives were included, then the left-hand side of Eq. (38) would necessarily involve terms proportional to $\Xi^{iv}(T)$ and/or $\Xi'''(T)$ when nondimensionalized. These terms are not physically reasonable.

The cubic damping term of Eq. (37) is not reproduced in Eq. (38). As was previously mentioned, the form of the drag term in Eq. (38) must be such that it guarantees the invariance of said equation to a rotation of the coordinate system about the axis of symmetry x. Acceptable modifications to the current form of the drag term are then: (i) a linearization of the drag term, resulting in a term proportional to $[W'(T) - \Xi'(T)]$, (ii) a representation in higher-order odd powers of $[W'(T) - \Xi'(T)]$, and (iii) a superposition of (i) and (ii).

Additionally, a term consisting of the superposition of the existing drag term and (i) or (ii), or the existing drag term and both (i) and (ii), can be constructed. The addition of a term that is proportional to $\Xi'^3(T)$ would therefore constitute an acceptable modification to the right-hand side of Eq. (38). This term might be introduced by assuming that the lift coefficient is represented by $\hat{C}_y(t) = C_y(t) - \kappa\Xi'^3(T)$, where $C_y(t)$ is determined from Eq. (7). The $\kappa\Xi'^3(T)$ term would then appear in Eq. (41), thus cementing its correspondence with Eq. (37). Given the complexity of the existing drag term in Eq. (38), such a modification is not carried out here.

Turning to the wake oscillators, a comparison of Eq. (14),

$$\begin{aligned} &W''(T) + [(2\pi S)^2\hat{b}_3 W'^2(T) - \hat{b}_2]W'(T) + W(T) \\ &\quad = -3(2\pi S)^2 b_4[W'(T)\Xi'^2(T) - \Xi'(T)W'^2(T)] \\ &\qquad + (2\pi S)^2\hat{b}_3\Xi'^3(T) - \hat{b}_2\Xi'(T) \end{aligned} \tag{42}$$

and Eq. (35) reveals many similarities. Noting again that

$$\Omega^2\left(\frac{\omega_n^{(\mathrm{vac})}}{\omega_{\mathrm{st}}}\right)^2 = \Omega^2\kappa^2 = \left(\frac{2\pi\omega_{\mathrm{st}}}{2\pi\omega_n^{(\mathrm{vac})}}\right)^2\left(\frac{\omega_n^{(\mathrm{vac})}}{\omega_{\mathrm{st}}}\right)^2 = 1,$$

Eq. (35) can be rewritten as

$$C_y''(T) + [\alpha_0\kappa + \alpha_2\kappa^{-1}C_y'^2(T)]C_y'(T) + C_y(T) = b_0\kappa\Xi'(T) + b_2\kappa^{-1}\Xi'^3(T). \tag{43}$$

Suppose that

$$\alpha_0\frac{\omega_n^{(\mathrm{vac})}}{\omega_{\mathrm{st}}} = \alpha_0\kappa = -B_0$$

and

$$\alpha_2\frac{\omega_{\mathrm{st}}}{\omega_n^{(\mathrm{vac})}} = \alpha_2\kappa^{-1} = (2\pi S)^2 B_2,$$

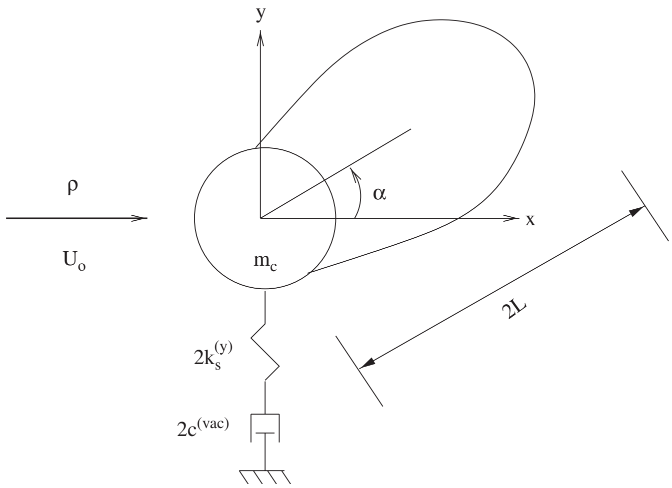

Fig. 2. The relevant physical parameters of the Tamura-Matsui model.

where B_0, $B_2 > 0$. Note that the signs are preserved since $\alpha_0 < 0$ and $\alpha_2 > 0$. Upon substituting these definitions into Eq. (43), one obtains

$$C_y''(T) + [(2\pi S)^2 B_2 C_y'^2(T) - B_0]C_y'(T) + C_y(T) = b_0\kappa \Xi'(T) + b_2\kappa^{-1}\Xi'^3(T). \tag{44}$$

The left-hand side of this equation now has the same form as the left-hand side of Eq. (42).

Consider the case in which

$$\alpha_0 = b_0$$

and

$$\alpha_2 = b_2.$$

It follows, then, that

$$\alpha_0\kappa = b_0\kappa = -B_0$$

and

$$\alpha_2\kappa^{-1} = b_2\kappa^{-1} = (2\pi S)^2 B_2.$$

Under this special set of conditions, Eq. (44) becomes

$$C_y''(T) + [(2\pi S)^2 B_2 C_y'^2(T) - B_0]C_y'(T) + C_y(T) = -B_0\Xi'(T) + (2\pi S)^2 B_2 \Xi'^3(T).$$

This equation is identical in form to Eq. (42 when $f_{NL}(W'(T), \Xi'(T)) = 0$.

Thus, the general form of Berger's equations can be obtained as a special case of the model equations presented here. It is again found, however, that $f_{NL}(W'(T), \Xi'(T))$ must zero for the best agreement.

3.3. Tamura and Matsui (1979)

In Ref. [7], Tamura and Matsui (TM) present a modified Birkhoff wake-oscillator and the equation of motion for an elastically-mounted cylinder as a system of simultaneous equations. The length $L_{w.o.}$ of the wake-oscillator is variable[7] and this translates into the presence of a parametric damping term in the fluid-oscillator equation.

[7]However, the time rate-of-change of the length, $\dot{L}_{w.o.}$, is neglected for simplicity.

Letting α denote the angular displacement of the fluid-oscillator (see Fig. 2), the time evolution equation in α is given by

$$\overline{I}\ddot{\alpha} - \overline{C}(1 - P\alpha^2)\dot{\alpha} + \overline{K}\left(\alpha + \frac{\dot{\chi}}{U_o}\right) = -\frac{\overline{I}}{(0.5D + \overline{L}_{\mathrm{w.o.}})}\ddot{\chi}. \tag{45}$$

$\overline{I}$ is the mean moment of inertia of the wake-oscillator about the center of the cylinder, $\overline{C}$ is the mean damping coefficient representing the effects of discharged vortices, $\overline{K}$ is the mean coefficient of the fluid dynamic restoring force, P is the coefficient of the nonlinear damping mechanism, and $\overline{L}_{\mathrm{w.o.}}$ is the mean length of the wake-oscillator. The rationale behind the $\overline{K}(\alpha + (\dot{\chi}/U_o))$ term is that horizontal motion of the cylinder changes the relative angle over which the fluid dynamic restoring force must act.

The equation of motion for the elastically mounted cylinder is given as

$$m_c\ddot{\chi} + c^{(\mathrm{vac})}\dot{\chi} + k_s^{(y)}\chi(t) = -\frac{1}{2}f\rho U_o^2 DL\left(\alpha + \frac{\dot{\chi}}{U_o}\right) - \frac{1}{2}\rho\sqrt{U_o^2 + \dot{\chi}^2}DLC_d\dot{\chi}, \tag{46}$$

where $f \simeq 1.16$ is a proportionality constant related to the experimentally observed relationship[8] between the Magnus effect lift and wake angular displacement for a rotating cylinder, and $C_d = 1.2$ is the drag coefficient. For a stationary cylinder, the lift coefficient C_y is assumed to be $C_y = -f\alpha$, while for the self-excited cylinder $C_y = -f(\alpha + (\dot{\chi}/U_o))$.

Physically, the first term on the right-hand side of Eq. (46) represents the lift force, while the second term represents the transverse (y) component of the drag force.

TM apply the simplification

$$-\tfrac{1}{2}\rho\sqrt{U_o^2 + \dot{\chi}^2}DLC_d\dot{\chi} \simeq -\tfrac{1}{2}\rho U_o DLC_d\dot{\chi},$$

which is based on the assumption $U_o \gg \dot{\chi}$. Implementing this approximation in Eq. (46) yields

$$m_c\ddot{\chi} + c^{(\mathrm{vac})}\dot{\chi} + k_s^{(y)}\chi(t) = -\frac{1}{2}f\rho U_o^2 DL\left(\alpha + \frac{\dot{\chi}}{U_o}\right) - \frac{1}{2}\rho U_o DLC_d\dot{\chi}. \tag{47}$$

Nondimensionalizing Eqs. (45) and (47) yields

$$\overline{I}\omega_{\mathrm{st}}^2\alpha''(T) - \overline{C}[1 - P\alpha^2(T)]\omega_{\mathrm{st}}\alpha'(T) + \overline{K}\left[\alpha(T) + \frac{D\omega_{\mathrm{st}}\Xi'(T)}{U_o}\right]$$
$$= -\frac{\overline{I}}{(0.5D + \overline{L}_{\mathrm{w.o.}})}D\omega_{\mathrm{st}}^2\Xi''(T)$$

and

$$m_c D\omega_{\mathrm{st}}^2\Xi''(T) + c^{(\mathrm{vac})}D\omega_{\mathrm{st}}\Xi'(T) + k_s^{(y)}D\Xi(T)$$
$$= -\frac{1}{2}f\rho U_o^2 DL\left(\alpha + \frac{D\omega_{\mathrm{st}}\Xi'(T)}{U_o}\right) - \frac{1}{2}\rho U_o DLC_d D\omega_{\mathrm{st}}\Xi'(T),$$

respectively. Upon simplifying the above equations, one obtains

$$\alpha''(T) - \frac{\overline{C}}{\omega_{\mathrm{st}}\overline{I}}[1 - P\alpha^2(T)]\alpha'(T) + \frac{\overline{K}}{\omega_{\mathrm{st}}^2\overline{I}}\alpha(T) = -\frac{\overline{K}D}{U_o\omega_{\mathrm{st}}\overline{I}}\Xi'(T)$$
$$-\frac{1}{(0.5D + \overline{L}_{\mathrm{w.o.}})}D\Xi''(T), \tag{48}$$

[8]The reader may wish to peruse Ref. [19]. Curiously, the authors find that $f = 1.15$ (C_s in their notation) for the case of a spinning football whose trajectory is tracked in three dimensions.

and

$$\Xi''(T) + \left[2\zeta^{(\text{vac})}\frac{\omega_n^{(\text{vac})}}{\omega_{\text{st}}} + 2\hat{m}^*(f + C_d)\frac{U_o}{D\pi\omega_{\text{st}}}\right]\Xi'(T) + \frac{\omega_n^{(\text{vac})^2}}{\omega_{\text{st}}^2}\Xi(T) = -\frac{1}{2}\frac{f\rho U_o^2 L}{\omega_{\text{st}}^2 m_c}\alpha, \tag{49}$$

respectively.

Defining

$$\omega_{\text{st}} = \sqrt{\frac{\overline{K}}{\overline{I}}},$$

$$\zeta_{\text{w.o.}} = \frac{\overline{C}}{2\omega_{\text{st}}\overline{I}},$$

and

$$\upsilon = \frac{1}{(0.5D + \overline{L}_{\text{w.o.}})}D.$$

Eq. (48) can be rewritten as

$$\alpha''(T) - 2\zeta_{\text{w.o.}}[1 - P\alpha^2(T)]\alpha'(T) + \alpha(T) = -\frac{D\omega_{\text{st}}}{U_o}\Xi'(T) - \upsilon\Xi''(T). \tag{50}$$

However,

$$-\frac{D\omega_{\text{st}}}{U_o} = -2\pi S$$

and Eq. (50) can also be written as

$$\alpha''(T) - 2\zeta_{\text{w.o.}}[1 - P\alpha^2(T)]\alpha'(T) + \alpha(T) = -2\pi S\Xi'(T) - \upsilon\Xi''(T). \tag{51}$$

Turning to Eq. (47), one finds that it can be expressed in the form

$$\Xi''(T) + [2\zeta^{(\text{vac})}\kappa + 4\hat{m}^*(f + C_d)S]\Xi'(T) + \kappa^2\Xi(T) = -\frac{f\hat{m}^*}{2\pi^2 S^2}\alpha, \tag{52}$$

where κ of Eq. (31) has been used.

In order to compare Eqs. (52) and (11), the latter is first rewritten in the following way by using the corresponding parameters from Table 2[9]

$$\Xi''(T) + \frac{2\zeta^{(\text{vac})}}{(1 + \hat{m}^* C_a)}\left(\frac{\omega_n^{(\text{vac})}}{\omega_{\text{st}}}\right)\Xi'(T) + \left(\sqrt{\frac{1}{(1 + \hat{m}^* C_a)}}\frac{\omega_n^{(\text{vac})}}{\omega_{\text{st}}}\right)^2\Xi(T) = \frac{2}{\pi}\frac{\hat{m}^*}{(1 + \hat{m}^* C_a)}C_d|W'(T) - \Xi'(T)|[W'(T) - \Xi'(T)] + \frac{\hat{m}^*}{(1 + \hat{m}^* C_a)}(1 + C_a)W''(T). \tag{53}$$

Next, let $\hat{m}^* C_a \simeq 0$ (i.e., assume the experiments are conducted in air). In this case, the Eq. (53) reduces to

$$\Xi''(T) + 2\zeta^{(\text{vac})}\kappa\Xi'(T) + \kappa^2\Xi(T) = \frac{2}{\pi}\hat{m}^* C_d|W'(T) - \Xi'(T)|[W'(T) - \Xi'(T)] + \hat{m}^* W''(T). \tag{54}$$

[9]Use the relation $\omega_n^{(\text{true})}/\omega_n^{(\text{vac})} = \sqrt{1/(1 + \hat{m}^* C_a)}$, which can be easily derived.

It is apparent that (i) a linearization of the drag term in Eq. (54) would lead to a left-hand side that closely resembles the left-hand side of Eq. (52), (ii) the displacement coupling present in Eq. (52) is not a feature of Eq. (54), and the acceleration coupling present in Eq. (54) is not a feature of Eq. (52).

Comparing Eqs. (51) and (14), it is evident that if only the linear term on the right-hand side of the latter is retained, the expressions are rendered quite similar. Indeed,

$$W''(T) + [(2\pi S)^2 \hat{b}_3 W'^2(T) - \hat{b}_2] W'(T) + W(T) = -\hat{b}_2 \Xi'(T) \tag{55}$$

is qualitatively similar to Eq. (51). However, that acceleration coupling of Eq. (52) is not a feature that is reproduced by Eq. (55).

4. Discussion

Nonlinear wake-oscillator models have been shown to be leading order approximations for the vortex shedding instability from a fixed cylinder in uniform flow [20], while wake-body models have been shown to represent the same type of leading order approximation for forced oscillations of circular cylinders in uniform flows [21]. These findings imply that these models have, at least to some degree, fluid dynamical origins. It is precisely because of these fluid dynamical origins that wake-body models have been successful. However, by the very nature of being leading order approximations to a very complex interaction, they have limitations. The methodology presented in this paper serves to address both of these aspects. The fluid dynamical origins can be accounted for since the starting variational principle is rigorous. The limitations are accounted for because any assumptions made in reducing the variational principle are explicitly stated.

In Section 3 it is shown that the wake-body model derived from the proposed methodology shares many qualitative features with the three comparison models chosen from the literature. One can argue that the comparison models are special cases of the derived models. This follows from the fact that the derived model is found to involve terms that do not appear in the comparison models. These additional terms are, for the most part, the autoparametric terms. It is not the aim of this paper to weigh in on the issue of whether or not these terms should be retained. Suffice it to say that many authors have previously addressed the inadequacy of linear coupling terms in wake-body models (e.g., Refs. [12,13]).

There are terms in the comparison models that are not captured in the derived model. This is simply a manifestation of the assumed forms in Eqs. (7) and (8). Subject to a different set of assumptions, these equations could conceivably be modified such that the "missing" terms appear in the derived model. It cannot be stressed enough that these modifications would need to be justified. This is in essence the embodiment of the advantage of the method presented in this paper: That while the wake-body models still contain arbitrary coefficients, their forms are arrived at by a line of reasoning, rather than a "hit or miss" approach.

The authors believe that this approach can be implemented in other fluid–structure interaction problems. The possibility of applying it to derive wake-body models for elastic structures in uniform and shear flows is something that is currently being given further consideration. To the authors' knowledge, there exists only a small body of literature that deals with the application of wake-body models (a two equation model is still implied) to these particular problems (i.e., Refs. [22,23]). Again, the aim would be to show how these existing models could be made part of a more rigorous deductive process.

Acknowledgments

This work is supported by the Office of Naval Research Grant No. N00014-97-1-0017. We would like to thank our program manager Dr. Thomas Swean for his continued interest and financial support.

References

[1] R.D. Gabbai, H. Benaroya, An overview of modeling and experiments of vortex-induced vibration of circular cylinders, *Journal of Sound and Vibration* 282 (3–5) (2005) 575–616.

[2] G. Parkison, Phenomena and modelling of flow-induced vibrations of bluff bodies, *Progress in Aerospace Sciences* 26 (2) (1989) 169–224.

[3] H. Benaroya, T. Wei, Hamilton's principle for external viscous fluid structure interactions, *Journal of Sound and Vibration* 238 (1) (2000) 113–145.
[4] R.D. Gabbai, Hamilton's Principle for Fluid–Structure Interaction and Applications to the Free-Vibration of an Elastically-Mounted Cylinder, PhD Dissertation, Rutgers, The State University of New Jersey, 2006.
[5] S.A. Hall, Vortex-Induced Vibrations of Structures, PhD Dissertation, California Institute of Technology, 1981.
[6] E. Berger, On a mechanism of vortex excited oscillations of a cylinder, *Journal of Wind Engineering and Industrial Aerodynamics* 28 (1-3) (1988) 301–310.
[7] Y. Tamura, G. Matsui, Wake-oscillator model for vortex-induced oscillation of a circular cylinder, in: J.E. Cermak (Ed.), *Proceedings of the Fifth International Conference on Wind Engineering (Fort Collins, CO, USA)*, Pergamon Press, New York, 1980, pp. 1085–1094.
[8] R.D. Gabbai, H. Benaroya, Hamilton's principle for fluid–structure interaction and applications to reduced-order modeling, *Proceedings of the 2005 ASME International Design Engineering Technical Conferences and Computers and Information in Engineering Conferences*, Paper No. DETC2005-84145, Long Beach, CA, USA, 2005.
[9] G.S. Triantafyllou, K. Kupfer, A. Bers, Absolute instabilities and self-sustained oscillations in the wake of circular cylinders, *Physical Review Letters* 59 (17) (1987) 1914–1917.
[10] P. Huerre, P.A. Monkewitz, Local and global instabilities in spatially developing flows, *Annual Review of Fluid Mechanics* 22 (1990) 473–537.
[11] H. Oertel Jr., Wakes behind blunt bodies, *Annual Review of Fluid Mechanics* 22 (1990) 539–564.
[12] S. Krenk, S.R.K. Nielsen, Energy balanced double oscillator model for vortex-induced vibrations, *Journal of Engineering Mechanics* 125 (1999) 263–271.
[13] K.Y.R. Billah, A Study of Vortex-Induced Vibration, PhD Dissertation, Princeton University, 1989.
[14] T. Sarpkaya, A critical review of the intrinsic nature of vortex-induced vibrations, *Journal of Fluids and Structures* 19 (4) (2004) 389–447.
[15] W.D. Iwan, R.D. Blevins, A model for vortex induced oscillation of structures, *Journal of Applied Mechanics* 41 (1974) 581–586.
[16] R.D. Blevins, *Flow-Induced Vibration*, Van Nostrand Reinhold, New York, 1977.
[17] R.A. Skop, S. Balasubramanian, A new twist on an old model for vortex-excited vibrations, *Journal of Fluids and Structures* 11 (4) (1997) 395–412.
[18] E. Berger, P. Plaschko, Hopf bifurcations and hysteresis in flow-induced vibrations of cylinders, *Journal of Fluids and Structures* 7 (8) (1993) 849–866.
[19] I. Griffiths, C. Evans, N. Griffiths, Tracking the flight of a spinning football in three dimensions, *Measurement Science and Technology* 16 (10) (2005) 2056–2065.
[20] P. Albarède, P.A. Monkewitz, A model for the formation of oblique shedding and "chevron" patterns in cylinder wakes, *Physics of Fluids A: Fluid Dynamics* 4 (4) (1992) 744–756.
[21] R.A. Skop, S. Balasubramanian, A nonlinear oscillator model for vortex shedding from a forced cylinder. Part 1: Uniform flow and model parameters, *International Journal of Offshore and Polar Engineering* 5 (4) (1995) 251–255.
[22] M.L. Facchinetti, E. de Langre, F. Biolley, Vortex-induced travelling waves along a cable, *European Journal of Mechanics—B/Fluids* 23 (1) (2004) 199–208.
[23] L. Mathelin, E. de Langre, Vortex-induced vibrations and waves under shear flow with a wake oscillator model, *European Journal of Mechanics—B/Fluids* 24 (4) (2005) 478–490.