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Stabilization of linear undamped systems via position and delayed position feedbacks

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Abstract

This paper presents a systematic approach to stabilizing a kind of linear undamped systems of multiple degrees of freedom by using both position and delayed position feedbacks, namely, PDP feedbacks for short. For the fully actuated system, the approach enables one to complete the design of controller directly through the use of modal decoupling and a stability chart. For the under-actuated system, the approach includes two steps. The first step is to move all the eigenvalues of the system on the imaginary axis of the complex plane by using a position feedback, and the second step is to drag all the eigenvalues of the system to the left half open complex plane through the use of a delayed position feedback, which can be determined on the basis of sensitivity analysis of eigenvalues. Two examples, i.e., a fully actuated robotic manipulator and an under-actuated double inverted pendulum, are discussed in the paper to demonstrate the design of controllers for the two different types of systems and to support the efficacy of the proposed approach.

1. Introduction

For slightly damped or undamped mechanical systems, such as a rotary crane and a robotic manipulator, their free vibrations always decay very slowly. Over the past decades, various passive and active control techniques have been developed to suppress the vibrations of those mechanical systems [1,2]. Even though the passive control techniques, including various dampers, distributed damping treatments, dynamic vibration absorbers, etc. [1,3], are widely used in engineering, they have some inherent drawbacks, such as the difficulty in suppressing the vibration of very low frequency. The active control techniques make it possible to achieve the better performance of vibration suppression in the fields where the passive control techniques exhibit their limits [4], especially when a system has negative stiffness and/or damping so that the unstable equilibrium of the system should be stabilized.

The last decade has witnessed an increasing interest toward the stabilization of unstable mechanical systems, such as those without damping or with negative stiffness and damping. For example, Jangid studied the parametric optimization of the multiple tuned mass dampers for an undamped primary system [5]. Coupe proposed a design procedure of relay controller for the vibration reduction of a linear undamped system [6].

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Kobayashi stabilized an infinite-dimensional undamped system by using a low-gain adaptive velocity feedback [7]. Hu presented how to stabilize the periodic vibration of a linear undamped oscillator by using a delayed position feedback (DPF) or a delayed velocity feedback, or both [8], with help of the theory of stability switches of time-delay systems [9]. As a following work, Wang and Hu studied the stabilization of linear undamped systems of multiple degrees of freedom [10]. Moreover, the stabilization of chaotic motions has become a hot topic since the pioneering work of Pyragas [11].

Theoretically speaking, a linear, time-invariant system can always be stabilized by using a full state feedback, such as a linear quadratic regulator (LQR) controller, provided that the system is controllable. However, the realization of an LQR requires the measurements of both position and velocity of the controlled system at each degree of freedom. The velocity sensors may result in cost, space, and malfunction problems [12] while constructing the velocity measurement from other kind of sensors usually introduce heavy noise and deviation, which have to be removed by using additional filters. Hence, the design of controllers in the absence of velocity measurement has become an interesting problem. Nevertheless, it is not possible to stabilize the vibration of linear undamped system by using only a position feedback, such as a positive position feedback controller [13]. Recent studies show that the DPF plays a part of role of the velocity feedback. For example, Atay stabilized an inverted pendulum by using a DPF controller [14]. Jnifene used a DPF controller to suppress the lightly damped flexible link [15]. Masound et al. constructed a nonlinear DPF to reduce the payload pendulations on rotary cranes [16]. These studies, however, have not yet given a systematic approach to the design of a DPF controller for the linear undamped system of multiple degrees of freedom.

The objective of this paper is to propose a systematic approach to stabilizing the unstable or critically stable equilibrium of a kind of linear undamped systems of multiple degrees of freedom by using both position and delayed position feedbacks. The rest part of the paper is organized as follows. In Section 2, the controlled systems of concern are described and classified into two types, namely, the fully actuated systems and under-actuated systems. Then, a systematic approach is presented in Section 3 to stabilize the linear undamped systems of multiple degrees of freedom. In Section 4, two illustrative examples are given to demonstrate the design of controllers and the efficacy of the proposed approach. Finally, some concluding remarks are made after a brief discussion in Section 5.

2. Description of controlled systems

The study focuses on a kind of linear undamped systems of n degrees of freedom under control governed by

$$\mathbf{M}\ddot{\mathbf{x}}(t) + \mathbf{K}\mathbf{x}(t) = \mathbf{B}\mathbf{u}(t), \tag{1}$$

where $\mathbf{x} \in R^n$ is the position vector in a frame of physical coordinates, $\mathbf{u} \in R^m$ the vector of independent control input, $\mathbf{M} \in R^{n\times n}$ the mass matrix, $\mathbf{K} \in R^{n\times n}$ the stiffness matrix, $\mathbf{B} \in R^{n\times m}$ the input matrix of control, respectively. In what follows, the mass matrix $\mathbf{M}$ is assumed to be positive definite as usual, whereas the stiffness matrix $\mathbf{K}$ is assumed to be symmetric only so as to cover the case when the system is unstable. For example, $\mathbf{K}$ is negative definite for a linearized double inverted pendulum such that the four eigenvalues of the system are either positive or negative real numbers.

The controlled systems described by Eq. (1) can be divided into two types, namely, the fully actuated systems if $m = n$ and the under-actuated systems if $m<n$. This classification is essential for the design of controllers in Section 3.

3. Design of PDP feedback

3.1. Fully actuated systems ($m = n$ and $\mathbf{B}$ is invertible)

A fully actuated system is subject to as many independent control inputs as its degrees of freedom. When each degree of freedom of the system is directly driven by an actuator, the input matrix $\mathrm{B} = \mathrm{diag}[b_1, b_2, \ldots, b_n]$ yields $b_1 b_2, \ldots, b_n \neq 0$ such that $\mathbf{B}$ is invertible. For many practical systems, say, the continuous systems such as beams and plates, the input force of an actuator does not correspond to a single degree of freedom, but several or even all of them instead. In such a case, $\mathbf{B}$ is not a diagonal matrix, but should be full in rank. Otherwise, some of actuators are dependent on each other and the others are not necessary so that the definition of fully actuated system is not true. That is, the full rank of input matrix **B** means that the system is fully actuated.

It is well known that the linear undamped system of n degrees of freedom described by Eq. (1) can be decoupled into n systems of single degree of freedom in the modal space since the mass matrix **M** is positive definite, and the stiffness matrix **K** is symmetric. This fact enables one to determine the feedback gains for the n decoupled systems of single degree of freedom independently as following.

Let **T** be the modal matrix of the system described by Eq. (1). Then, one has

$$\mathbf{M}_m \equiv \mathbf{T}^{\mathrm{T}}\mathbf{M}\mathbf{T} = \mathrm{diag}[m_1, m_2, \ldots, m_n], \quad \mathbf{K}_m \equiv \mathbf{T}^{\mathrm{T}}\mathbf{K}\mathbf{T} = \mathrm{diag}[k_1, k_2, \ldots, k_n], \tag{2}$$

where $\mathbf{M}_m$ and $\mathbf{K}_m$ are the modal mass matrix and the modal stiffness matrix, respectively. Substituting the following modal transform

$$\mathbf{x}(t) = \mathbf{T}\mathbf{y}(t), \quad \mathbf{y} \in R^n \tag{3}$$

into Eq. (1) yields

$$\mathbf{M}_m\ddot{\mathbf{y}}(t)+\mathbf{K}_m\mathbf{y}(t) = \mathbf{v}(t), \tag{4}$$

where $\mathbf{v}(t)\equiv\mathbf{T}^{\mathrm{T}}\mathbf{B}\mathbf{u}(t)$ is the control input in the modal space. Eq. (4) governs n decoupled systems of single degree of freedom as follows:

$$m_j\ddot{y}_j(t) + k_j y_j(t) = v_j(t), \quad j = 1, \ldots, n. \tag{5}$$

For simplicity, the subscript j will be neglected hereinafter in this section. Now, the original problem of controller design for a system of n degrees of freedom is converted into the one for n similar systems of single degree of freedom as follows:

$$m\ddot{y}(t) + ky(t) = v(t), \quad v(t) \equiv k_p y(t) + k_{dp} y(t-\tau). \tag{6}$$

The design of controller is to choose two proper feedback gains k_p and k_{dp}, together with a time delay τ, so as to ensure the asymptotic stability of the zero solution of Eq. (6).

As done in Ref. [14], the study begins with the characteristic equation of Eq. (6):

$$D(s) \equiv ms^2 + (k - k_p) - k_{dp}\exp(-s\tau) = 0. \tag{7}$$

If all the roots of Eq. (7) stay on the left half open complex plane, then the zero solution of Eq. (6) is asymptotically stable. If the zero solution is marginally stable, then $D(s) = 0$ has a pair of pure imaginary roots $s = \pm\mathrm{i}\omega$ with $\omega > 0$ for some positive value of τ. Separating the real and imaginary parts of $D(\pm\mathrm{i}\omega) = 0$ yields

$$k_{dp}\sin(\omega\tau) = 0, \tag{8a}$$

$$(k - k_p) - m\omega^2 = k_{dp}\cos(\omega\tau). \tag{8b}$$

Solving Eq. (8a) gives

$$k_{dp} = 0 \quad \text{or} \quad \omega\tau = j\pi, \quad j = 0, 1, \ldots . \tag{9}$$

Substituting Eq. (9) into Eq. (8b) leads to three possible cases as following.

Case I: $k_{dp} = 0$, i.e., there is no DPF. In this case, Eq. (8b) has a pair of roots $\omega_{1,2} = \pm\sqrt{(k-k_p)/m}$, corresponding to the natural frequency of the decoupled system of single degree of freedom, if and only if $k > k_p$ holds.

Case II: $k_{dp}\neq 0$, $\tau = 0$, i.e., the time delay vanishes so that the DPF joins the position feedback. In this case, Eq. (8b) has a pair of roots $\omega_{1,2} = \pm\sqrt{(k-k_p-k_{dp})/m}$, corresponding to the natural frequency of the decoupled system of single degree of freedom, if and only if $k > k_p + k_{dp}$ holds.

Case III: $k_{dp}\neq 0$ and $\tau\neq 0$. In this case, $\omega = j\pi/\tau$, $j = 0,1,\ldots$ and Eq. (8b) becomes

$$(-1)^j\frac{k_{dp}}{m} = \left(\frac{k-k_p}{m} - \frac{j^2\pi^2}{\tau^2}\right), \quad j = 0, 1, \ldots . \tag{10}$$

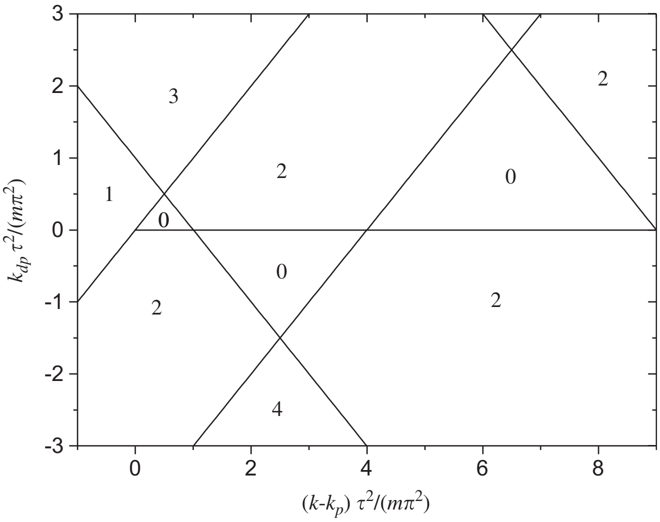

Fig. 1. Stability regions (marked by 0) of Eq. (6) on the (k_p, k_{dp}) plane. The number in each region is the number of roots with positive real part.

Eq. (10) governs a set of lines on the (k_p, k_{dp}) plane as shown in Fig. 1. Hence, a pair of roots is crossing the imaginary axis of the complex plane at $s = \pm i\omega = \pm ij\pi/\tau$, $j>0$ when the combination of feedback gains (k_p, k_{dp}) falls at a line in Fig. 1, except for the line $k = k_p + k_{dp}$ corresponding to the case when a single real root is crossing the imaginary axis at $s = 0$, namely, $j = 0$. The transition direction of the roots can be determined by checking the sign of $\partial[\mathrm{Re}(s)]/\partial k_{dp}$. The above discussion can be summarized as the following lemma depicted in Fig. 1, see also in Ref. [14] for details.

Lemma 1. *The zero solution of Eq.* (6) *is asymptotically stable if and only if the following inequalities hold true*:

$$\tau>0,\ \ \min\left[\frac{(k-k_p)}{m}-\frac{j^2\pi^2}{\tau^2},\ \ \frac{(j+1)^2\pi^2}{\tau^2}-\frac{(k-k_p)}{m}\right]>(-1)^j\frac{k_{dp}}{m}>0, \tag{11}$$

for any nonnegative integer j.

Once the feedback gains for every modal degree of freedom are determined, the control input in the physical coordinates can be constructed as following:

$$\begin{aligned}\mathbf{u}(t) &= \mathbf{B}^{-1}\mathbf{T}^{-\mathrm{T}}\mathbf{v}(t)\\ &= \mathbf{B}^{-1}\mathbf{T}^{-\mathrm{T}}\,\mathrm{diag}[k_{p1},k_{p2},\ldots,k_{pn}]\mathbf{y}(t)+\mathbf{B}^{-1}\mathbf{T}^{-\mathrm{T}}\,\mathrm{diag}[k_{dp1},k_{dp2},\ldots,k_{dpn}]\mathbf{y}(t-\tau).\end{aligned} \tag{12}$$

From Eq. (3), the position and delayed position (PDP) feedback controller given in Eq. (12) can be recast as a simpler form

$$\mathbf{u}(t)=\mathbf{K}_p\mathbf{x}(t)+\mathbf{K}_{dp}\mathbf{x}(t-\tau), \tag{13}$$

where $\mathbf{K}_{dp}\equiv\mathbf{B}^{-1}\mathbf{T}^{-\mathrm{T}}\,\mathrm{diag}[k_{dp1},k_{dp2},\ldots,k_{dpn}]\mathbf{T}^{-1}$, $\mathbf{K}_{dp}\equiv\mathbf{B}^{-1}\mathbf{T}^{-\mathrm{T}}\,[k_{d1},k_{d2},\ldots,k_{dn}]\mathbf{T}^{-1}$.

Remark 1. If the controlled system described by Eq. (1) is proportionally damped, the dynamic equation of the fully actuated system is

$$\mathbf{M}\ddot{\mathbf{x}}(t)+\mathbf{C}\dot{\mathbf{x}}(t)+\mathbf{K}\mathbf{x}(t)=\mathbf{B}\mathbf{u}(t), \tag{14}$$

where the damping matrix $\mathbf{C}$ is positive semi-definite, and can be diagonalized by the modal matrix $\mathbf{T}$ as

$$\mathbf{C}_m\equiv\mathbf{T}^{\mathrm{T}}\mathbf{C}\mathbf{T}=\mathrm{diag}[c_1,c_2,\ldots,c_n],\quad c_j\geqslant 0,\quad j=1,\ldots,n. \tag{15}$$

The corresponding characteristic equation of the damped system of single degree of freedom, with the subscript neglected, reads

$$D(s) \equiv ms^2 + cs + (k - k_p) - k_{dp}\exp(-s\tau) = 0. \tag{16}$$

It is easy to derive the sensitivity of the eigenvalue s with respect to the modal damping c as following:

$$\frac{\partial s}{\partial c} = -\frac{s}{2ms + c + k_{dp}\tau\exp(-s\tau)}. \tag{17}$$

Substituting $s = \alpha + i\omega$ into Eq. (17) gives

$$\mathrm{sgn}\left[\mathrm{Re}\left(\frac{\partial s}{\partial c}\right)\right] = -\mathrm{sgn}\left\{2m(\omega^2 + \alpha^2) + \alpha c + k_{dp}\tau\exp(-\alpha\tau)[\alpha\cos(\omega\tau) - \omega\sin(\omega\tau)]\right\}. \tag{18}$$

Eq. (18) indicates that the increasing damping does not necessarily improve the performance of the PDP feedback controller. As a matter of fact, it is possible to achieve the asymptotic stability of the controlled system through the only use of a position feedback if the system is damped.

3.2. Under-actuated systems ($m<n$)

The under-actuated systems cover a great variety of industrial products, such as space robots, underwater robots, mobile robots, VTOL aircraft, etc. [17,18]. Those systems have fewer control inputs than their degrees of freedom, and give rise to a challenging problem of controller design. Furthermore, when one or more actuators of a fully actuated system happen to fail, the system becomes an under-actuated one and requires an adaptive change of the control law. Thus, the proper design of controllers for under-actuated systems will increase the reliability and fault tolerance of systems. To stabilize an under-actuated system by using a PDP feedback controller, a two-step strategy is presented as following.

Step 1: Moving all of the eigenvalues of the controlled system onto the imaginary axis of the complex plane by using a position feedback.

Obviously, it is not possible to achieve the asymptotical stability of Eq. (1) by using a position feedback because the position feedback only changes the equivalent stiffness matrix of the controlled system and does not affect the damping matrix of the system. Hence, the best result of a position feedback in stabilization is to reach the marginal stability.

To gain an insight into Step 1, it is better to rewrite Eq. (1) in the form

$$\ddot{\mathbf{x}}(t) = \tilde{\mathbf{A}}\,\mathbf{x}(t) + \tilde{\mathbf{B}}\,\mathbf{u}(t), \tag{19}$$

where $\tilde{\mathbf{A}} \equiv -\mathbf{M}^{-1}\mathbf{K}$ and $\tilde{\mathbf{B}} \equiv \mathbf{M}^{-1}\mathbf{B}$. Substituting the linear position feedback

$$\mathbf{u}_p(t) = -\mathbf{K}_p\mathbf{x}(t) \tag{20}$$

into Eq. (19) yields

$$\ddot{\mathbf{x}}(t) = (\tilde{\mathbf{A}} - \tilde{\mathbf{B}}\,\mathbf{K}_p)\mathbf{x}(t). \tag{21}$$

Eq. (21) is marginally stable or unstable when the real part of any eigenvalue is zero or positive. On the basis of the theory of linear control, it is easy to reach the following Proposition 1.

Proposition 1. *The eigenvalues $s_j, j = 1, 2, \ldots, 2n$ of Eq.* (21) *can be arbitrarily assigned on the imaginary axis if the system described by*

$$\dot{\mathbf{y}}(t) = \tilde{\mathbf{A}}\,\mathbf{y}(t) + \tilde{\mathbf{B}}\,\mathbf{u}(t) \tag{22}$$

is controllable.

Proof. The characteristic equation of Eq. (21) reads

$$D(s) \equiv \det\left[s^2\mathbf{I} - (\tilde{\mathbf{A}} - \tilde{\mathbf{B}}\,\mathbf{K}_p)\right] = 0. \tag{23}$$

Substituting the control input $\mathbf{u}(t) = -\mathbf{K}_p\mathbf{y}(t)$ into Eq. (22) yields

$$\dot{\mathbf{y}}(t) = (\tilde{\mathbf{A}} - \tilde{\mathbf{B}}\,\mathbf{K}_p)\mathbf{y}(t). \tag{24}$$

The corresponding characteristic equation is

$$D(\lambda) \equiv \det\left[\lambda \mathbf{I} - (\tilde{\mathbf{A}} - \tilde{\mathbf{B}}\,\mathbf{K}_p)\right] = 0. \tag{25}$$

The theory of linear control indicates that the roots $\lambda_j, j = 1, 2, \ldots, n$ of Eq. (25) can be arbitrarily assigned on the complex plane if Eq. (22) is controllable. The comparison of Eqs. (23) and (25) shows that it is possible to assign all the roots of Eq. (25) arbitrarily on the negative real axis, namely, $\lambda_j = s_j^2 < 0,\ j = 1, 2, \ldots, n$, under the assumption of the controllable Eq. (22). As a result, the eigenvalues of Eq. (21) can be arbitrarily assigned as n pair of conjugated pure imaginary numbers

$$s_j = \mathrm{i}\sqrt{|\lambda_j|}, \quad s_{n+j} = -\mathrm{i}\sqrt{|\lambda_j|}, \quad j = 1, 2, \ldots, n. \qquad \square$$

Remark 2. In most cases, all the eigenvalues of an undamped system stay on the imaginary axis and Step 1 can be skipped. However, Step 1 becomes necessary in the following cases:

(1) Sometimes, the stiffness matrix of a system is not positive definite and even negative definite so that some eigenvalues of the system may be pairs of positive and negative real numbers [19]. For example, the linearized dynamic equation of a double pendulum around the inverted equilibrium has two eigenvalues with positive real part.
(2) The eigenvalues of uncontrolled systems are on the imaginary axis, but may not at the desired locations.

Remark 3. Under the condition that Eq. (22) is controllable, the eigenvalues of the system can be placed on the imaginary axis by using the suboptimal control method [20] or the pole assignment. Sometimes, the design of a controller by using the pole assignment will encounter indeterminate problem, which makes the choice of feedback gains very difficult due to infinite feasible options. In such a case, it is possible to design a suboptimal control to solve the problem.

Step 2: Dragging all the eigenvalues of the controlled system to the left half open complex plane from the imaginary axis by introducing a DPF into the controller.

In Step 1, all the eigenvalues of Eq. (21) have been placed on the imaginary axis by using the position feedback $\mathbf{u}_p(t) \equiv -\mathbf{K}_p\mathbf{x}(t)$ in Eq. (20). Step 2 is to construct the following controller:

$$\mathbf{u}(t) \equiv \mathbf{u}_p(t) + \mathbf{u}_{dp}(t), \tag{26}$$

where $\mathbf{u}_{dp}(t)$ is the DPF as follows:

$$\mathbf{u}_{dp}(t) \equiv -\mathbf{K}_{dp}[\mathbf{x}(t) - \mathbf{x}(t-\tau)]. \tag{27}$$

The objective of this step is to determine the feedback gain matrix $\mathbf{K}_{dp}$ and the time delay τ to achieve the asymptotic stability of controlled system.

Substituting Eqs. (20), (26) and (27) into Eq. (1) yields

$$\mathbf{M}\ddot{\mathbf{x}}(t) + \mathbf{K}\mathbf{x}(t) = -\mathbf{B}\left\{\mathbf{K}_p\mathbf{x}(t) + \mathbf{K}_{dp}[\mathbf{x}(t) - \mathbf{x}(t-\tau)]\right\}. \tag{28}$$

Eq. (28) corresponds to a pair of adjoint transcendental eigenvalue problem in the form

$$\mathbf{E}(s,\tau)\mathbf{y} = 0, \quad \bar{\mathbf{z}}^{\mathrm{T}}\mathbf{E}(s,\tau) = 0, \tag{29}$$

where

$$\mathbf{E}(s,\tau) \equiv \left[s^2\mathbf{M} + \mathbf{K} + \mathbf{B}(\mathbf{K}_p + \mathbf{K}_{dp}) - \mathbf{B}\mathbf{K}_{dp}\exp(-s\tau)\right], \tag{30}$$

where $\mathbf{y}$ and $\mathbf{z}$ are the right and left eigenvectors corresponding to eigenvalue s. Note that when $\tau = 0$ and $\mathbf{u}_{dp}(t) = 0$, all the eigenvalues $s_j, j = 1, 2, \ldots, n$ are pure imaginary numbers with zero real part. Hence, if the following conditions

$$\mathrm{Re}\left(\frac{\partial s}{\partial \tau}\right)\bigg|_{s=s_j,\tau=0} < 0, \quad j = 1, 2, \ldots, 2n \tag{31}$$

are ensured, then every eigenvalue will come into left half open complex plane as time delay increase from zero to a small positive value, because s depends continuously on τ. Eq. (31) is, by nature, the inverse problem of the sensitivity analysis of a nonlinear eigenvalue problem, for which Bindolino and Mantegazza suggested the following method [21].

Lemma 2. *For a pair of adjoint nonlinear eigenvalue problem*

$$\mathbf{N}(s,p)\mathbf{y}=0, \quad \bar{\mathbf{z}}^{\mathrm{T}}\mathbf{N}(s,p)=0, \tag{32}$$

where p is a real or complex parameter, the sensitivity of eigenvalue s with respect to the parameter p is

$$\frac{\partial s}{\partial p}=-\frac{\bar{\mathbf{z}}^{\mathrm{T}}(\partial \mathbf{N}/\partial p)\mathbf{y}}{\bar{\mathbf{z}}^{\mathrm{T}}(\partial \mathbf{N}/\partial s)\mathbf{y}}. \tag{33}$$

Applying Lemma 2 into Eqs. (29) and (30) leads to

$$\frac{\partial s}{\partial \tau}=-\frac{\bar{\mathbf{z}}^{\mathrm{T}}(\partial \mathbf{E}/\partial \tau)\mathbf{y}}{\bar{\mathbf{z}}^{\mathrm{T}}(\partial \mathbf{E}/\partial s)\mathbf{y}}=\frac{-s\bar{\mathbf{z}}^{\mathrm{T}}\left[\mathbf{B}\mathbf{K}_{dp}\exp(-s\tau)\right]\mathbf{y}}{\bar{\mathbf{z}}^{\mathrm{T}}\left[2\mathbf{M}s+\tau\mathbf{B}\mathbf{K}_{dp}\exp(-s\tau)\right]\mathbf{y}}. \tag{34}$$

Substituting $\tau=0$, $s_j=\mathrm{i}\omega$, $s_{n+j}=-\mathrm{i}\omega$, $j=1, 2, \ldots, n$ into Eq. (34) yields

$$\left.\frac{\partial s_j}{\partial \tau}\right|_{s=s_j,\tau=0}=-\frac{\bar{\mathbf{z}}_j^{\mathrm{T}}\mathbf{B}\mathbf{K}_{dp}\mathbf{y}_j}{2\bar{\mathbf{z}}_j^{\mathrm{T}}\mathbf{M}\mathbf{y}_j}, \quad j=1, 2, \ldots, 2n, \tag{35}$$

where $\mathbf{y}_j$ and $\mathbf{z}_j$ are the right and left eigenvectors corresponding to the jth eigenvalue s_j of Eq. (29) when $\tau=0$, namely, $\mathbf{u}_{dp}(t)=0$. They are just the right and left eigenvectors of Eq. (21). As assumed, the mass matrix $\mathbf{M}$ is positive definite so that $\bar{\mathbf{z}}_j^{\mathrm{T}}\mathbf{M}\mathbf{y}_j>0$, $j=1,2,\ldots,2n$ can hold true. Then, the conditions in Eq. (31) become

$$\bar{\mathbf{z}}_j^{\mathrm{T}}\mathbf{B}\mathbf{K}_{dp}\mathbf{y}_j>0, \quad j=1,2,\ldots,2n. \tag{36}$$

Because $\mathbf{B}$, all $\mathbf{y}_j$ and $\mathbf{z}_j$ are already known after $\mathbf{K}_p$ is determined, Eq. (36) represents $2n$ inequalities with only $\mathbf{K}_{dp}$, whereas the first n inequalities are identical to the last n inequalities. As shown in Appendix A, Eq. (36) has solutions provided that $\mathbf{z}_j^{\mathrm{T}}\mathbf{B}\neq 0$, $j=1,2,\ldots,n$ hold and $\{\mathbf{y}_i,\ i=1,2,\ldots,n\}$ is a linear independent family of vector. In most cases these two conditions hold. If they happened to be false, one can always make them be satisfied by adjusting $\mathbf{K}_p$ since Eq. (22) is controllable. Solving the first n inequalities gives the stability region of Eq. (28) in the parameter space of $\mathbf{K}_{dp}$ when the time delay τ is slightly larger than zero.

As mentioned at the end of last paragraph, it is possible to guarantee the stability of controlled system only if the time delay is short. Nevertheless, a longer time delay will make the system unstable even though the elaborately designed $\mathbf{K}_{dp}$ yields Eq. (36). Therefore, it is necessary to figure out the tolerable upper bound of the time delay to guarantee the stability of the controlled system. The consideration of this problem begins with the characteristic equation of Eq. (28) as following:

$$D(s,\tau)\equiv \det \mathbf{E}(s,\tau)=0, \tag{37}$$

which can always be recast in the form

$$D(s,z)\equiv \sum_{k=0}^{m} Q_k(s)z^k=0, \quad z\equiv \exp(-s\tau), \tag{38}$$

where $\tau\geqslant 0$, $Q_0(s),\ldots,Q_m(s)$ are $m+1$ polynomials of real coefficients that have even orders only, and $\deg[Q_0(s)]=2n>\deg[Q_k(s)]$, $k=1,\ldots,m$. As done in Refs. [10,22], one defines

$$D^{(j)}(s,z)\equiv Q_0^{(j-1)}(-s)D^{(j-1)}(s,z)-Q_{m+1-j}^{(j-1)}(s)z^{m+1-j}D^{(j-1)}(-s,1/z), \tag{39}$$

where $Q_k^{(j)}(s)$ is the coefficient of $D^{(j)}(s,z)$ with $D^{(0)}(s,z)\equiv D(s,z)$ and $Q_m^{(0)}(s)=Q_m(s)$. Through a straightforward calculation, it is possible to see that $D^{(m)}(s,z)\equiv D^{(m)}(s)$ is independent of z and if $D(s,\tau)$ has a pair of roots $s_{1,2} = \pm i\omega$ for $\tau > 0$, then $D^{(m)}(\pm i\omega) = 0$ is true and can be simplified as a polynomial equation in ω

$$F(\omega) \equiv d_0\omega^{2(2^m n)} + \cdots + d_i\omega^{2(2^m n-i)} + \cdots + d_{2^m n} = 0. \tag{40}$$

For $\tau = 0$, all the eigenvalues of the system are pure imaginary. Then $\mathbf{K}_{dp}$ should be selected to make sure that the derivative of the real part of an eigenvalue with respect to τ is negative so that each of the eigenvalues has a negative real part with an increase in τ from zero to a small positive value. As the time delay increases gradually, Eq. (37) will have a pair of pure imaginary roots $\pm i\omega$ again and lead to Eq. (40). Once a root ω_c of $F(\omega)$ is in hand, many routines, say, those in Ref. [10] and the references therein, are available to determine the critical time delays τ_k such that $D(i\omega_c, \tau_k) = 0$ holds. In fact, if $F(\omega)$ has r positive real roots $\{\omega_c\} = \{\omega_{c1}, \omega_{c2}, \ldots, \omega_{cr}\}$, substituting $s = i\omega_{cj}$ into Eq. (38) and separating the real and the imaginary parts, one obtains two linear equations of m orders

$$f(\gamma, \beta) = 0, \quad g(\gamma, \beta) = 0, \tag{41}$$

where $\gamma = \sin(\omega_{cj}\tau)$ and $\beta = \cos(\omega_{cj}\tau)$. Eq. (41) is easy to be solved. Having γ and β in hand, one can determine the critical time delays associated with ω_{cj} by

$$\tau_{j,l} = \frac{\theta}{\omega_{cj}} + \frac{2l\pi}{\omega_{cj}}, \quad l = 0, 1, 2, \ldots, \tag{42}$$

where $\theta \in [0, 2\pi)$, $\sin(\theta) = \gamma$ and $\cos(\theta) = \beta$. Undoubtedly, zero is a critical time delay with n positive roots of Eq. (40). When $\tau = 0$, Eq. (37) degenerates to Eq. (23). Thus, the n pairs of pure imaginary roots $\pm i\omega_{cj}, j = 1, \ldots, n$ of Eq. (23) are the roots of Eq. (37) with critical time delays $\tau_{j,0} = 0, j = 1, \ldots, n$. When τ equals to the next critical time delay, denoted by $\tau_{n+1,0}$, at least one pair of eigenvalues will cross the imaginary axis from the left to the right. The controlled system becomes unstable again. As discussed in Refs. [9,10], the time-delay system has the property of stability switches as the time delay increases. One can only focus on the first interval $(0, \tau_{n+1,0})$ of time delay, where the controlled system is asymptotically stable.

In practice, it may not be enough if only the time-delay region is known to claim the asymptotic stability of a controlled system. It may be desirable for the controlled system to be convergent quickly. Therefore, it is an essential issue to properly select the time delay. This issue will be discussed through the illustrative examples in Section 4.

Remark 4. The design of controller proposed in this section is also feasible to the linear damped systems. For a linear damped system, the position feedback controller can asymptotically stabilize the system by using the sub-optimal control. The corresponding eigenvalue problem of the controlled system becomes

$$\left[s^2\mathbf{M} + s\mathbf{C} + (\mathbf{K} + \mathbf{BK}_p)\right]\mathbf{y} = 0, \quad \bar{\mathbf{z}}^{\mathrm{T}}\left[s^2\mathbf{M} + s\mathbf{C} + (\mathbf{K} + \mathbf{BK}_p)\right] = 0. \tag{43}$$

Eq. (43) has n pairs of conjugated eigenvalues with negative real part, and n pairs of conjugated eigenvectors. Meanwhile, in view of $\mathbf{z}_{j+n} = \bar{\mathbf{z}}_j$, $\mathbf{y}_{j+n} = \bar{\mathbf{y}}_j$ and $s_{j+n} = \bar{s}_j$ in Eq. (43), the counterpart of condition (31) turns to be

$$\mathrm{Re}\left(\frac{\partial s}{\partial \tau}\right)\bigg|_{s_j, \tau=0} = -\mathrm{Re}\left(\frac{s_j\bar{\mathbf{z}}_j^{\mathrm{T}}\mathbf{BK}_{dp}\mathbf{y}_j}{\bar{\mathbf{z}}_j^{\mathrm{T}}(2s_j\mathbf{M} + \mathbf{C})\mathbf{y}_j}\right) < 0, \quad j = 1, 2, \ldots, n. \tag{44}$$

The DPF $\mathbf{u}_{dp}(t) = -\mathbf{K}_{dp}[\mathbf{x}(t) - \mathbf{x}(t-\tau)]$ can improve the stability of controlled system as in the case of undamped systems.

4. Illustrative examples

4.1. A fully actuated system

Fig. 2 presents a robotic manipulator of two degrees of freedom under direct drives similar to the model discussed in Refs. [17,23]. The system is a fully actuated and can be described by

$$\mathbf{M}(\boldsymbol{\theta}(t))\ddot{\boldsymbol{\theta}}(t) + \mathbf{p}(\boldsymbol{\theta}(t), \dot{\boldsymbol{\theta}}(t)) = \mathbf{u}(t), \tag{45}$$

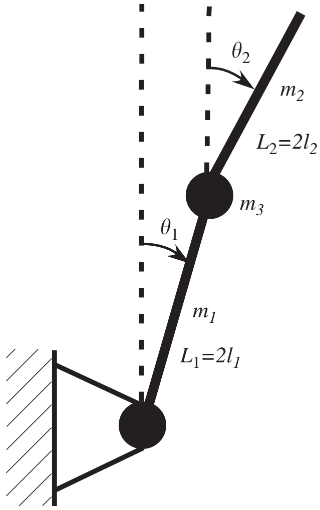

Fig. 2. Schematic of a robotic manipulator.

Table 1
Physical parameters of the robotic manipulator

Parameters	Values	Descriptions
m_1	0.0350 kg	Mass of the first pendulum
m_2	0.1302 kg	Mass of the second pendulum
m_3	0.2230 kg	Mass of the lump mass
$L_1 = 2l_1$	0.185 m	Length of the first pendulum
$L_2 = 2l_2$	0.510 m	Length of the second pendulum
G	$9.8\,\mathrm{m\,s^{-2}}$	Gravitational acceleration

where $\boldsymbol{\theta}(t)$ is the vector of angular position, $\mathbf{u}(t)$ is the vector of driving torque, $\mathbf{M}(\boldsymbol{\theta}(t))$ and $\mathbf{p}(\boldsymbol{\theta}(t), \dot{\boldsymbol{\theta}}(t))$ are the inertia matrix and the vector of the Coriolis force, which yield

$$\mathbf{M}(\boldsymbol{\theta}(t)) \equiv \begin{bmatrix} 4(m_1 + 3m_2 + 3m_3)l_1^2 & 6m_2 l_1 l_2 \cos(\theta_1 - \theta_2) \\ 6m_2 l_1 l_2 \cos(\theta_1 - \theta_2) & 4m_2 l_2^2 \end{bmatrix},$$

$$\mathbf{p}(\boldsymbol{\theta}(t), \dot{\boldsymbol{\theta}}(t)) \equiv \begin{bmatrix} 6m_2 l_1 \dot{\theta}_2^2 \sin(\theta_1 - \theta_2) - 3(m_1 + 2m_2 + 2m_3) l_1 g \sin\theta_1 \\ -6m_2 l_2 \dot{\theta}_1^2 \sin(\theta_1 - \theta_2) - 3m_2 l_2 g \sin\theta_2 \end{bmatrix}. \tag{46}$$

The linearization of Eq. (45) at the upright equilibrium $[\boldsymbol{\theta}, \dot{\boldsymbol{\theta}}]_{uu}^{\mathrm{T}} = [0, 0, 0, 0]^{\mathrm{T}}$ gives

$$\mathbf{M}_{uu}\ddot{\boldsymbol{\theta}}(t) + \mathbf{K}_{uu}\boldsymbol{\theta}(t) = \mathbf{u}(t), \tag{47}$$

where

$$\mathbf{M}_{uu} \equiv \begin{bmatrix} 4(m_1 + 3m_2 + 3m_3)l_1^2 & 6m_2 l_1 l_2 \\ 6m_2 l_1 l_2 & 4m_2 l_2^2 \end{bmatrix}, \quad \mathbf{K}_{uu} = -\begin{bmatrix} 3(m_1 + 2m_2 + 2m_3) l_1 g & 0 \\ 0 & 3m_2 l_2 g \end{bmatrix}. \tag{48}$$

Substituting the system parameters in Table 1 into Eq. (48) and decoupling it by using the following modal transform:

$$\boldsymbol{\theta}(t) = \mathbf{T}\mathbf{x}(t) \equiv \begin{bmatrix} -1.6775 & -5.7995 \\ -4.3073 & 4.6655 \end{bmatrix} \mathbf{x}(t), \tag{49}$$

one arrives at two decoupled systems of single degree of freedom

$$\ddot{x}_1(t) - 23.7834x_1(t) = v_1(t), \quad \ddot{x}_2(t) - 89.0607x_2(t) = v_2(t), \tag{50}$$

where $[v_1(t), v_2(t)]^T \equiv \mathbf{T}^T\mathbf{u}(t)$ is the vector of control input in modal space. Referring to the stability chart in Fig. 1, one can set $\tau = 0.1$ s and take the feedback gains in the leftmost zero zone in Fig. 1 so that the feedback law in modal space reads

$$v_1(t) = -200x_1(t) + 80x_1(t-0.1), \quad v_2(t) = -240x_2(t) + 100x_2(t-0.1). \tag{51}$$

By recalling Eq. (13), one comes to the following control law in physical space:

$$\mathbf{u}(t) = \mathbf{K}_p\boldsymbol{\theta}(t) + \mathbf{K}_{dp}\boldsymbol{\theta}(t-0.1), \tag{52}$$

where

$$\mathbf{K}_p \equiv \begin{bmatrix} -8.1821 & -3.4168 \\ -3.4168 & -6.8776 \end{bmatrix}, \quad \mathbf{K}_{dp} \equiv \begin{bmatrix} 3.3418 & 1.3399 \\ 1.3399 & 2.7615 \end{bmatrix}. \tag{53}$$

Fig. 3 shows the stabilization of the robotic manipulator at the upright equilibrium and verifies the efficacy of the PDP feedback controller. In fact, the time delays chosen for different modal coordinates are not necessarily the same. Of course, the final feedback matrix becomes a little bit complicated if the time delays are not identical.

In practical application, there are several ways to implement the delayed control part, for example, using various delay-lines and the method of recording and reproduction. For very short delays, from several nano-seconds to several micro-seconds, analog implementations may actually use circuitry made up of 'sample-and-hold' or 'bucket brigade' devices. Those devices are commonly seen in audio products and some of them, say SAD-1024, even can produce delays up to hundreds milli-seconds. If a long time delay is desired, one may cascade enough of these devices together. Another way of obtain an analog delay is recording the incoming signal to a tape, and reading it by a playback head at another point on the tape. One can adjust the time delay by changing either the tape speed, or the distance between the recording and playback heads. Actually, it is relatively simple to realize a digital delay by adopting the *circular buffer* technique. In each sampling interval, a previously stored value is read from a location in memory, and then the current value of the signal is stored into the same memory location. The next sampling period, one applies the same operation to the next location. When the end of the memory is reached, one loops around to the first memory location. Obviously, the time delay is N times of sample period where N is the length of the buffer.

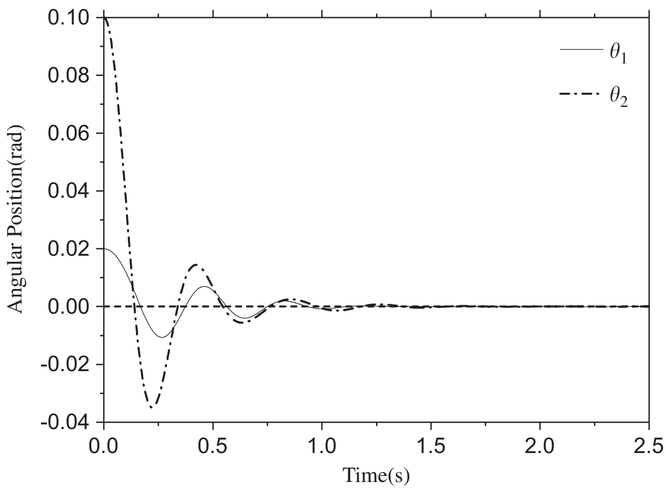

Fig. 3. Angular position of the manipulator under a PDP feedback control.

4.2. An under-actuated system

This subsection focuses on an under-actuated system as shown in Fig. 4, namely, the famous double inverted pendulum on a cart governed by

$$\mathbf{M}(\boldsymbol{\theta})\ddot{\boldsymbol{\theta}}(t) + \mathbf{p}(\boldsymbol{\theta}(t), \dot{\boldsymbol{\theta}}(t)) = \mathbf{B}(\boldsymbol{\theta})\ddot{x}(t), \tag{54}$$

where the acceleration $\ddot{x}(t)$ of the cart plays the role of control input. As done in previous example, by linearizing Eq. (54) at the upright equilibrium $[\boldsymbol{\theta}, \dot{\boldsymbol{\theta}}]_{uu}^{\mathrm{T}} = [0, 0, 0, 0]^{\mathrm{T}}$ and multiplying both right and left sides by $\mathbf{M}_{uu}^{-1}$, one obtains

$$\ddot{\boldsymbol{\theta}}(t) + \tilde{\mathbf{K}}_{uu}\,\boldsymbol{\theta}(t) = \tilde{\mathbf{B}}_{uu}\,\ddot{x}(t), \tag{55}$$

where

$$\begin{aligned} \tilde{\mathbf{K}}_{uu} &\equiv -\frac{g}{4m_1 + 3m_2 + 12m_3}\begin{bmatrix} 3(m_1 + 2m_2 + 2m_3)/l_1 & -9m_2/2l_1 \\ -9(m_1 + 2m_2 + 2m_3)/2l_2 & 3(m_1 + 3m_2 + 3m_3)/l_2 \end{bmatrix}, \\ \tilde{\mathbf{B}}_{uu} &\equiv \frac{1}{2(4m_1 + 3m_2 + 12m_3)}\begin{bmatrix} 3(2m_1 + m_2 + 4m_3)/l_1 \\ -3m_1/l_2 \end{bmatrix}. \end{aligned} \tag{56}$$

For $\ddot{x} = 0$, substituting the system parameters in Table 1 into Eq. (55) and solving the corresponding characteristic equation gives two eigenvalues with positive real part. Thus, Step 1 described in Section 3.2 is needed. To determine the gain matrix $\mathbf{K}_p$ of the position feedback by using suboptimal control method, let the input weight $r\equiv 1$, the state weight matrix $\mathbf{Q}\equiv\mathbf{I}_{4\times 4}$, and the output matrix $\mathbf{C}_s \equiv [\mathbf{I}_{2\times 2}, \quad \mathbf{0}]$. First, one should calculate the gain matrix $\mathbf{K}_{sf}$ of the optimal state feedback by LQR algorithm

$$\mathbf{K}_{sf} = \mathrm{LQR}(\mathbf{A}_s, \mathbf{B}_s, \mathbf{Q}, r), \tag{57}$$

where

$$\mathbf{A}_s \equiv \begin{bmatrix} \mathbf{0} & \mathbf{I}_{2\times 2} \\ -\tilde{\mathbf{K}}_{uu} & \mathbf{0} \end{bmatrix}, \quad \mathbf{B}_s \equiv \begin{bmatrix} \mathbf{0} \\ \tilde{\mathbf{B}}_{uu} \end{bmatrix}. \tag{58}$$

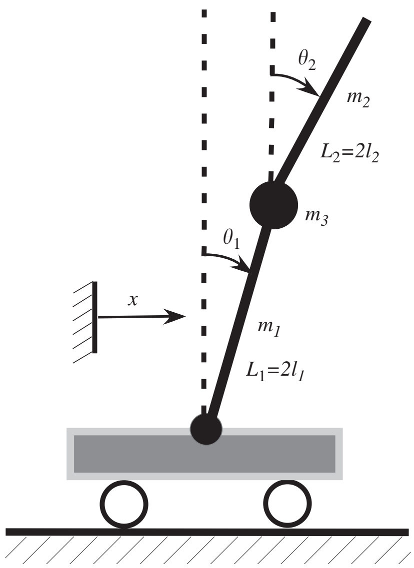

Fig. 4. Schematic of a double inverted pendulum on a cart.

Then, following Ref. [20], one calculates the gain matrix of the output (position) feedback as following:

$$\mathbf{K}_p = \mathbf{K}_{sf}\mathbf{V}\mathbf{C}_s^{\mathrm{T}}(\mathbf{C}_s\mathbf{V}\mathbf{C}_s^{\mathrm{T}})^{-1}, \tag{59}$$

where $\mathbf{V}$ is the solution of the following Lyapunov equation:

$$(\mathbf{A}_s - \mathbf{B}_s\mathbf{K}_{sf})\mathbf{V} + \mathbf{V}(\mathbf{A}_s - \mathbf{B}_s\mathbf{K}_{sf})^{\mathrm{T}} + \mathbf{Q} = \mathbf{0}. \tag{60}$$

Substituting the chosen weight matrices into Eqs. (59) and (60) leads to the following position feedback law:

$$\mathbf{u}_p(t) \equiv -\mathbf{K}_p\boldsymbol{\theta}(t) \tag{61}$$

with the feedback gain

$$\mathbf{K}_p = \begin{bmatrix} 41.0562 & -32.4579 \end{bmatrix}. \tag{62}$$

Corresponding eigenpairs of the controlled system are

$$\{s_1, \mathbf{y}_1, \mathbf{z}_1\} = \left\{ 1.5112i \quad \begin{bmatrix} 1.0 \\ 0.2474 \end{bmatrix} \quad \begin{bmatrix} 0.9442 \\ -1.0 \end{bmatrix} \right\},$$
$$\{s_2, \mathbf{y}_2, \mathbf{z}_2\} = \left\{ 10.6642i \quad \begin{bmatrix} 1.0 \\ 0.9442 \end{bmatrix} \quad \begin{bmatrix} -0.2474 \\ 1.0 \end{bmatrix} \right\}. \tag{63}$$

Now, one can start to figure out the stability diagram in $\mathbf{K}_{dp}$ parameter plane. Substituting $\mathbf{y}_i$, $\mathbf{z}_i$, $i = 1, 2$ and $\mathbf{B}$ into inequality (36), one obtains the following two inequalities:

$$5.26k_{dp1} + 1.29k_{dp2} > 0, \quad -1.42k_{dp1} - 1.34k_{dp2} > 0. \tag{64}$$

Fig. 5 shows the diagrammatic presentation of Eq. (64).

To achieve the high control performance, $\mathbf{K}_{dp}$ should be properly chosen. Otherwise, an excessively large or small gain matrix will lead to the poor control performance no matter how to adjust the time delay. Fig. 6 shows the locus of maximal real part of the eigenvalue of the controlled system for different $\mathbf{K}_{dp}$ with an increase in the time delay.

In many cases, an increase of time delay in the feedback path makes the dominant eigenvalue of closed-loop system move rightward even cross the imaginary axis and accordingly cause the instability of the system. Therefore, time delay makes a bad impression of deteriorating control performance of the system on most people. However, it is not always the case. If the feedback gains are properly designed, the deliberately added time delays in the feedback path will lead the right most eigenvalue to move leftward. Therefore, the stability

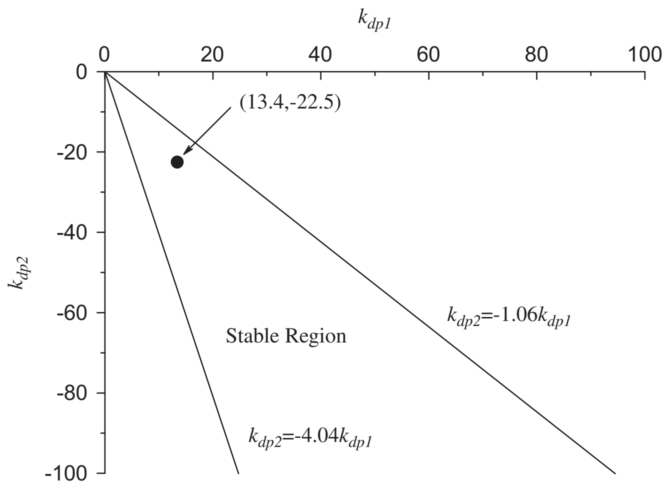

Fig. 5. Stability region of controlled system on (k_{dp1}, k_{dp2}) plane.

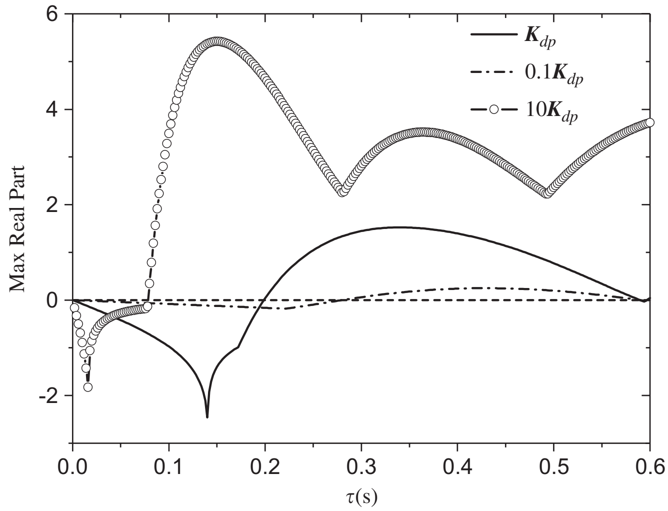

Fig. 6. Locus of real part of the dominant root.

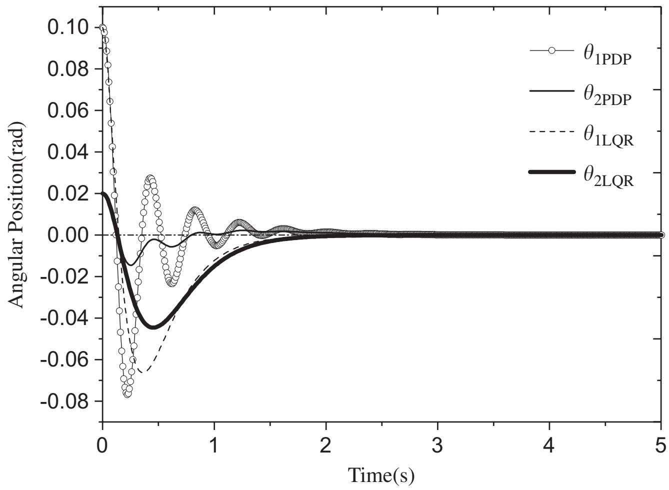

Fig. 7. Comparison between the PDP controller and LQR in the noise-free environment.

margin is enlarged as shown in Fig. 6 when τ varies from 0 to 0.14 and $\mathbf{K}_{dp} = [\,13.4 \quad -22.5\,]$. It is worth being mentioned that during the numerical calculation the Pade approximation [24] can be used to estimate the transcendental term $\exp(-s\tau)$ so as to reduce the computational cost significantly without losing high precision. In the following numerical simulation, $\mathbf{K}_{dp} = [\,13.4 \quad -22.5\,]$ and $\tau = 0.14\,\mathrm{s}$ are chosen.

In Figs. 7 and 8, the PDP controller is compared with the corresponding LQR controller $\mathbf{u} = -\mathbf{K}_{sf}[\boldsymbol{\theta}^{\mathrm{T}}, \dot{\boldsymbol{\theta}}^{\mathrm{T}}]^{\mathrm{T}}$. Fig. 7 shows that both PDP controller and LQR can stabilize the system very well in the noise-free environment. LQR is a little better than PDP, because LQR produces a smaller overshoot without oscillation. However, Figs. 8(a) and (b) show that when the system is exposed to a noisy environment, the PDP controller gives a better steady state performance than LQR. This supports the conjecture that the PDP feedback control is more robust to the noise in feedback signals than LQR. Furthermore, implementation of PDP controller requires only position signals while LQR needs both position and velocity signals.

As the double inverted pendulum is a renowned benchmark for under-actuated systems, the proposed PDP feedback control can be expected to work well for other under-actuated systems.

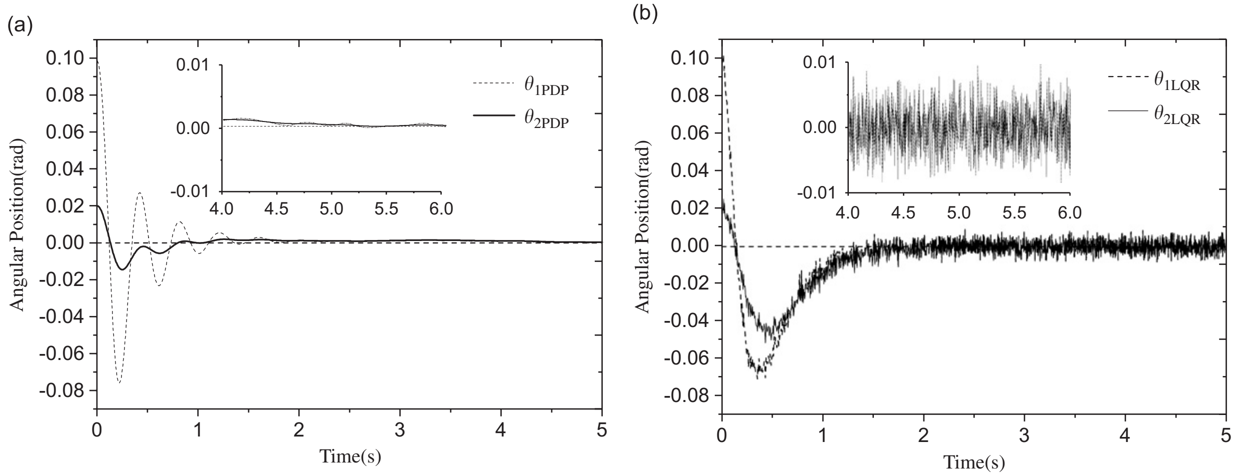

Fig. 8. Comparison between the PDP controller and LQR with band-limited white noise in the feedback signals: (a) PDP feedback control and (b) LQR control.

5. Concluding remarks

If no velocity measurement is available to construct the state feedback, the PDP feedback control is a good alternative so as to avoid any state estimators and to make the controlled system robust with respect to noise. This paper presents a systematic approach to the design of a PDP feedback controller for linear undamped systems of multiple degrees of freedom.

In the implementation of the proposed approach, two types of the controlled systems, i.e., the fully actuated systems and the under-actuated systems, are treated separately. For a fully actuated system, the stabilization design is relatively simple and includes the simultaneous design for the gain matrices of both position feedback and DPF, with an arbitrarily chosen time delay. For an under-actuated system, however, the stabilization design is relatively complex and includes the successive design for the gain matrix of position feedback and the gain matrix of DPF, with a specifically chosen time delay, in order to place all the eigenvalues of the controlled system on the imaginary axis of the complex plane first and then to drag them to the left half open complex plane. Of course, the design of a PDP feedback controller for under-actuated systems is more general and also applicable to fully actuated systems.

Two illustrative examples presented well demonstrate the design procedure of PDP feedback controllers and their efficacy for a fully actuated system and an under-actuated system, respectively. Though the proposed methods are titled to tackle the stabilization of linear undamped systems, they are also applicable to any linear, proportionally damped systems of multiple degrees of freedom.

Acknowledgments

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Appendix A

Proposition 2. *The solutions of Eq.* (36) *exist provided that*

$$\mathbf{z}_j^{\mathrm{T}}\mathbf{B}\neq 0, \quad j=1,2,\ldots,n. \tag{A.1}$$

$$\{\mathbf{y}_i,\ i=1,2,\ldots,n\}\ \textit{is a linear independent family of vectors.} \tag{A.2}$$

Proof. When $\tau = 0$, Eq. (29) degenerates to the following form:

$$\begin{cases} \bar{\mathbf{z}}^{\mathrm{T}}\left[s^2\mathbf{M} + \mathbf{K} + \mathbf{B}\mathbf{K}_p\right] = 0, \\ \left[s^2\mathbf{M} + \mathbf{K} + \mathbf{B}\mathbf{K}_p\right]\mathbf{y} = 0. \end{cases} \tag{A.3}$$

As the position feedback control has made the closed-loop system marginally stable, s^2 in Eq. (A.3) is a real negative number. Consequently, the eigenvectors **y** and **z** in Eq. (A.3) can be real. The proof of the existence of solutions of Eq. (36) under conditions (A.1) and (A.2) can be divided into two steps.

The first step is to show the proposition when $m = 1$, i.e., when u is a scalar control input and $\mathbf{z}^{\mathrm{T}}\mathbf{B}$ is a nonzero real number. According to Eq. (A.1), Eq. (36) can be simplified and re-ordered as following:

$$\begin{cases} \mathbf{K}_{dp}\mathbf{y}_i > 0, & i = 1, 2, \ldots, l, \\ \mathbf{K}_{dp}\mathbf{y}_j < 0, & j = l+1, l+2, \ldots, n. \end{cases} \tag{A.4}$$

One can rewrite Eq. (A.4) into the following matrix form:

$$\mathbf{K}_{dp}\mathbf{N} \equiv \mathbf{K}_{dp}\begin{bmatrix} \mathbf{h} & -\mathbf{g} \end{bmatrix} > 0, \tag{A.5}$$

where $\mathbf{h} = \begin{bmatrix} \mathbf{y}_1 & \cdots & \mathbf{y}_l \end{bmatrix}$, $\mathbf{g} = \begin{bmatrix} \mathbf{y}_{l+1} & \cdots & \mathbf{y}_n \end{bmatrix}$. In view of condition (A.2), **N** is invertible. Supposing that $\mathbf{a} = [a_1, a_2, \ldots, a_n]$ and $a_i > 0$, $i = 1, 2, \ldots, n$, one can see that $\mathbf{K}_{dp} = \mathbf{a}\mathbf{N}^{-1}$ yields Eq. (36). Therefore, the existence of the solutions for Eq. (36) is confirmed.

The second step is to check the cases for $m > 1$. Now $\mathbf{z}^{\mathrm{T}}\mathbf{B}$ and $\mathbf{K}_{dp}\mathbf{y}$ can be recast into the form $[\mathbf{z}^{\mathrm{T}}\mathbf{B}_1 \quad \mathbf{z}^{\mathrm{T}}\mathbf{B}_2 \quad \cdots \quad \mathbf{z}^{\mathrm{T}}\mathbf{B}_m]$ and $[\mathbf{K}_{dp1}\mathbf{y} \quad \mathbf{K}_{dp2}\mathbf{y} \quad \cdots \quad \mathbf{K}_{dpm}\mathbf{y}]$, where $\mathbf{z}^{\mathrm{T}}\mathbf{B}_i$ $i = 1, 2, \ldots, m$ are real numbers and at least one of them is nonzero because $\mathbf{z}^{\mathrm{T}}\mathbf{B} \neq 0$. Thus, Eq. (36) can be rewritten into the form of

$$\mathbf{z}_j^{\mathrm{T}}\mathbf{B}_1\mathbf{K}_{dp1}\mathbf{y}_j + \mathbf{z}_j^{\mathrm{T}}\mathbf{B}_2\mathbf{K}_{dp2}\mathbf{y}_j + \cdots + \mathbf{z}_j^{T}\mathbf{B}_m\mathbf{K}_{dpm}\mathbf{y}_j > 0, \quad j = 1, 2, \ldots, n. \tag{A.6}$$

Now two families of sets are defined as

$$P_j = \left\{ i | \mathbf{z}_j^{\mathrm{T}}\mathbf{B}_i \neq 0, \ i = 1, 2, \ldots, m \right\}, \quad j = 1, 2, \ldots, n, \tag{A.7}$$

$$Q_i = \left\{ j | \mathbf{z}_j^{\mathrm{T}}\mathbf{B}_i \neq 0, \ j = 1, 2, \ldots, n \right\}, \quad i = 1, 2, \ldots, m. \tag{A.8}$$

Obviously, $P_j \subset \{1, 2, \ldots, m\}$, $Q_i \subset \{1, 2, \ldots, n\}$. Thus, Eq. (A.6) is equivalent to the inequality

$$\sum_{i \in P_j} \mathbf{z}_j^{\mathrm{T}}\mathbf{B}_i\mathbf{K}_{dpi}\mathbf{y}_j > 0, \quad j = 1, 2, \ldots, n. \tag{A.9}$$

Eq. (A.9) is a substitute for Eq. (A.6) with those '0' terms deducted. Apparently, if all of the following equalities:

$$\mathbf{z}_j\mathbf{B}_i\mathbf{K}_{dpi}\mathbf{y}_j > 0, \quad i \text{ is ergodic on } P_j, \ j = 1, 2, \ldots, n \tag{A.10}$$

hold, then Eq. (A.9) holds true. Sorting Eq. (A.10) into m groups according to m unknown feedback vectors $\mathbf{K}_{dpi}$ $i = 1, 2, \ldots, m$, one can see the first group is

$$\mathbf{z}_j^{\mathrm{T}}\mathbf{B}_1\mathbf{K}_{dp1}\mathbf{y}_j > 0, \ j \text{ is ergodic on } Q_1. \tag{A.11}$$

As stated before, every $\mathbf{z}_j^{\mathrm{T}}\mathbf{B}_1$, $j \in Q_1$ is a nonzero real numbers. Eq. (A.11) can be simplified and re-ordered as below:

$$\begin{cases} \mathbf{K}_{dp1}\mathbf{y}_k > 0, & k \text{ is ergodic on } Q_{1a}, \\ \mathbf{K}_{dp1}\mathbf{y}_l < 0, & l \text{ is ergodic on } Q_{1b}, \end{cases} \tag{A.12}$$

where $Q_{1a} \cup Q_{1b} = Q_1$, $Q_{1a} \cap Q_{1b} = \varnothing$. A comparison of Eq. (A.12) with Eq. (A.4) shows that they have the same number of indeterminate variables, but Eq. (A.12) has fewer (or at most the same number of) inequality constraints than Eq. (A.4). Therefore, the existence of the solutions for Eq. (A.4) guarantees the existence of the solutions for Eq. (A.12). In a similar way, one can prove the existence of the solutions for the other $m-1$ groups

$$\mathbf{z}_j^{\mathrm{T}}\mathbf{B}_i\mathbf{K}_{dpi}\mathbf{y}_j > 0 \text{ is ergodic on } Q_i, \quad i = 2, \ldots, m. \tag{A.13}$$

This completes the proof.

Remark 5. If the two conditions in Proposition 2 do not hold, one can always make them be satisfied by adjusting $\mathbf{K}_p$ since Eq. (22) is controllable.

Proof. To begin with, one defines the uncontrolled system

$$\mathbf{M}\ddot{\mathbf{x}}(t) + \mathbf{K}\mathbf{x}(t) = 0. \tag{A.14}$$

If the left eigenvector of closed-loop system $\mathbf{z}$ makes $\mathbf{z}^{\mathrm{T}}\mathbf{B} = 0$ hold, one has

$$\mathbf{z}^{\mathrm{T}}\left[s^2\mathbf{M} + \mathbf{K} + \mathbf{B}\mathbf{K}_p\right] = 0, \tag{A.15}$$

$$\mathbf{z}^{\mathrm{T}}\left[s^2\mathbf{M} + \mathbf{K}\right] = 0. \tag{A.16}$$

This implies that s, $\mathbf{z}$ is an eigenpair of both controlled system (21) and uncontrolled system (A.14). Hence, if Eqs. (21) and (A.14) have no common eigenpairs, then $\mathbf{z}^{\mathrm{T}}\mathbf{B} \neq 0$ must hold true. □

As Eq. (22) is controllable, every eigenvalue of Eq. (21) can be assigned arbitrarily on the imaginary axis. Therefore, it is easy to guarantee that Eqs. (21) and (A.14) have no common eigenvalue, and that the eigenvalues of Eq. (21) are different by adjusting $\mathbf{K}_p$. The first fact here is responsible for condition (A.1), while the second fact is for condition (A.2). Actually, condition (A.2) is also needed to ensure $\mathbf{z}_j^{\mathrm{T}}\mathbf{M}\mathbf{y}_j \neq 0, \quad j = 1, 2, \ldots, 2n$, which has been used to derive Eq. (36) from Eq. (35). □

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