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Vibration analysis of simply supported rectangular plates with unidirectionally, arbitrarily varying thickness

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Abstract

In this paper, a practical analytical method for the free vibration analysis of a simply supported rectangular plate with unidirectional, *arbitrary* thickness variation is proposed. First, the plate is divided into a number of regions of which the values of thickness are assumed to be constant and are given by the known thickness function of the plate. The close-form frequency function that yields the eigenvalues of the plate is extracted by considering the condition of continuity in displacement and slope between the regions and by considering the simply supported boundary condition of the plate. It is shown by several case studies that the proposed method has good convergence characteristics and yields accurate eigenvalues and mode shapes, compared with other analytical methods including FEM (ANSYS).
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1. Introduction

Many investigators have, over the years, studied the free vibration problems of non-homogeneous plates with simply supported edges, which have a theoretical analogy to similarly shaped fixed membranes. The author has also researched various analytical methods to solve the eigenvalue problems of non-homogeneous plates [1–3].

Sakata and Pulmano [4,5] have calculated eigenvalues of rectangular plates with linearly varying thickness using the double Fourier-series expansion method and the finite strip method. Also, Appl [6] studied an analytical method for calculating the fundamental frequencies of simply supported rectangular plates with linearly varying thickness. Chehil [7] dealt with the buckling problem of rectangular plates with general thickness variation.

Recently, the approximate fundamental natural frequencies of composite rectangular membranes have been obtained by Cortinez using classical and optimized Kantrovitch methods [8]. Liew studied symmetric and unsymmetric trapezoidal plates having unidirectional thickness variation [9,10]. Singh [11] has investigated the transverse vibrations of a rectangular plate of bidirectionally, linearly varying thickness. In his work, the first three eigenvalues of plates with different combinations of elementary boundary condition at four edges were

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presented and compared with those by Kobayashi [12]. The fundamental frequencies of non-homogeneous simply supported plates (or fixed membranes) of various dimensions were calculated by various methods [13–16].

Furthermore, the free vibrations of circular, annular composite membranes have been investigated; Laura dealt with a doubly connected annular membrane [17]. Rossit calculated antisymmetric modes of composite annular membranes using the general formulation of the problem for the case of m-discontinuous variations in surface density [18]. Buchanan has verified the results of previous investigators on circular, annular membranes with variable density using the special finite element formulation [19]. Jabareen obtained exact solutions for both the axisymmetric and antisymmetric modes of circular and annular membranes using a power-series solution [20], and Gottlieb found exact solutions for some annular membranes with special density variation in the radial direction [21].

As seen in the above investigation on previous papers, many papers have dealt with the vibration problems of non-homogeneous *rectangular* plates or membranes because rectangular configuration has many applications in practical use. In the paper, the free vibration problems of simply supported rectangular plates with unidirectionally, *arbitrarily* varying thickness are solved using the so-called transfer matrix method. For this, the procedure that the plate of interest is divided into N regions (or elements) is required. The proposed method gives a close-form frequency function extracted by means of assembling the transfer matrices that have information on the condition of continuity in displacement and slope between discretized elements. In case studies, the eigenvalues and mode shapes of the plates of interest in the paper are presented, and the validity and accuracy of the method are shown by comparing the present results to the eigenvalues given from other analytical methods and FEM (ANSYS).

2. Theoretical formulation

2.1. General theory

The equation of motion for the free flexural vibration of a thin plate is written as

$$D\nabla^4 w + \rho_s \frac{\partial^2 w}{\partial t^2} = 0, \tag{1}$$

where $w = w(\mathbf{r}, t)$ is the transverse deflection at position vector $\mathbf{r}$, ρ_s the surface density, and D the flexural rigidity expressed as $D = Eh^3/12(1-v^2)$ in terms of Young's modulus E, Poisson ratio v, and the plate thickness h. Assuming a harmonic motion $w(\mathbf{r}, t) = W(\mathbf{r})e^{j\omega t}$ in which ω denotes the circular frequency, Eq. (1) leads to

$$\nabla^4 W - \Lambda^4 W = 0, \quad \Lambda = (\rho_s \omega^2/D)^{1/4}. \tag{2, 3}$$

There exists an analogy between the vibration of *polygonal* plates with the simply supported boundary condition and similarly shaped membranes with fixed edges [22,23]. Since a solution of Eq. (2) for a simply supported polygonal plate can be obtained by multiplying a solution of a similarly shaped membrane with fixed edges by a constant [22,23], Eq. (2) in the paper dealing with simply supported rectangular plates can be reduced to the membrane equation

$$\nabla^2 W + \tilde{\Lambda}^2 W = 0. \tag{4}$$

In Eq. (4), $\tilde{\Lambda}$ is related to Λ by

$$\Lambda = \tilde{\Lambda}\sqrt{a/b}, \tag{5}$$

where a and b denote the width and height of the plate, respectively.

To deal with the plate whose thickness is varying in the x direction as shown in Fig. 1 (see AA' section), $\tilde{\Lambda}$ is rewritten as

$$\tilde{\Lambda} = \sqrt{\lambda/(h(x)/h_0)}, \quad \lambda = \sqrt{12(1-v^2)\rho\omega^2/Eh_0^2}, \tag{6, 7}$$

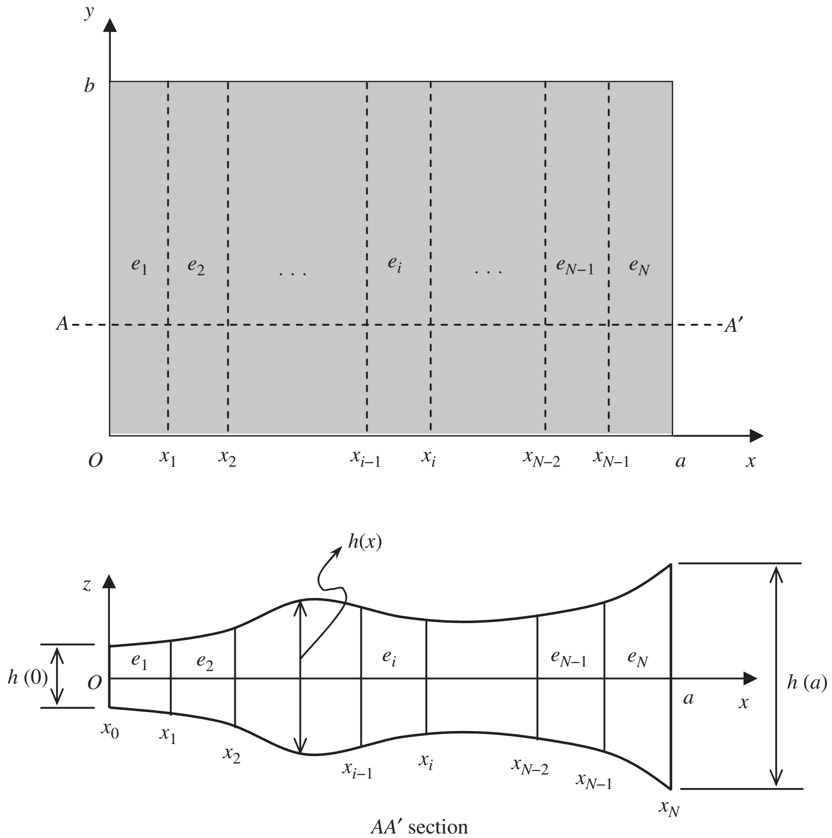

Fig. 1. Discretization of a rectangular plate with arbitrarily varying thickness $h(x)$; the plate is divided into N regions ($e_1, e_2, \ldots, e_N$) with equal space a/N so that $x_i = ia/N$.

where $h(x)$ represents the thickness function as indicated in Fig. 1 (see AA' section), h_0 denotes the thickness of the plate at $x = 0$ (i.e., $h_0 = h(0)$), ρ is the mass per unit volume (i.e., $\rho = \rho_s/h(x)$), and λ is defined as the *frequency parameter*.

To use the separation of variables method, $W(x, y)$ of Eq. (4) is assumed as

$$W(x, y) = f(x)g(y). \tag{8}$$

If Eq. (8) is substituted into Eq. (4) and the simply supported boundary condition at $y = 0$ and b, $W(x, 0) = W(x, b) = 0$, is considered, one can obtain

$$W(x, y) = f^{(m)}(x)g^{(m)}(y), \quad m = 1, 2, \ldots, \tag{9}$$

where

$$f^{(m)}(x) = (A \sin k_{xm}x + B \cos k_{xm}x), \tag{10}$$

$$g^{(m)}(y) = \sin m\pi y/b, \tag{11}$$

$$k_{xm} = \sqrt{\lambda/(h(x)/h_0) - (m\pi/b)^2}. \tag{12}$$

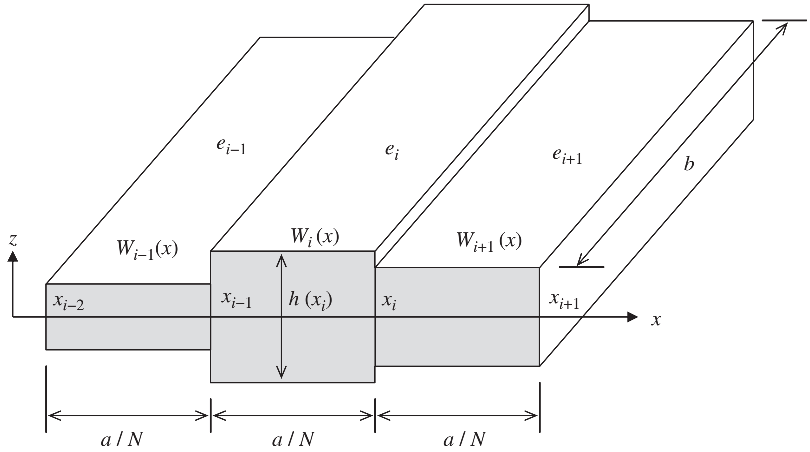

Fig. 2. Partial element model to extract transfer matrices by using the condition of continuity in displacement and slope between adjacent regions ($x_i = ia/N$).

2.2. Transfer matrices between discretized elements

As shown in Fig. 1, the plate is first discretized to N elements, $e_1, e_2, \ldots, e_N$ with equal space a/N. The thickness of the ith element is assumed to be constant and is approximated as $h(x_i)$ by substituting $x_i = ia/N$ into the thickness function $h(x)$. Using Eq. (9), the transverse deflection $W_i(x, y)$ of the ith element e_i shown in Fig. 2 is assumed as

$$W_i(x,y) = f_i^{(m)}(x)g^{(m)}(y), \quad x_{i-1} < x < x_i, \tag{13}$$

where

$$f_i^{(m)}(x) = A_i \sin k_{xm}^{(i)}(x_i - x) + B_i \cos k_{xm}^{(i)}(x_i - x), \tag{14}$$

$k_{xm}^{(i)} = \sqrt{\lambda/(h(x_i)/h_0) - (m\pi/b)^2}$ and $x_i = a_i/N$ (see Fig. 2).

In order to consider the condition of continuity in displacement and slope at the interface $x = x_i$, the transverse deflection in element e_{i+1} adjacent to element e_i as shown in Fig. 2 is defined by using Eq. (13). Then, the transverse deflection in element e_{i+1} can be expressed as

$$W_{i+1}(x,y) = f_{i+1}^{(m)}(x)g^{(m)}(y), \quad x_i < x < x_{i+1}, \tag{15}$$

where

$$f_{i+1}^{(m)}(x) = A_{i+1} \sin k_{xm}^{(i+1)}(x_{i+1} - x) + B_{i+1} \cos k_{xm}^{(i+1)}(x_{i+1} - x), \tag{16}$$

$k_{xm}^{(i+1)} = \sqrt{\lambda/(h(x_{i+1})/h_0) - (m\pi/b)^2}$ and $x_{i+1} = a(i+1)/N$. Using Eqs. (13) and (15), the condition of continuity in displacement and slope at $x = x_i$ may be written as

$$f_i^{(m)}(x_i) = f_{i+1}^{(m)}(x_i), \quad \mathrm{d}f_i^{(m)}(x_i)/\mathrm{d}x = \mathrm{d}f_{i+1}^{(m)}(x_i)/\mathrm{d}x, \tag{17, 18}$$

where $\mathrm{d}/\mathrm{d}x_i$ denotes the differential at $x = x_i$. Substituting Eqs. (14) and (16) into Eqs. (17,18) gives a matrix equation as follows:

$$\begin{Bmatrix} A_{i+1} \\ B_{i+1} \end{Bmatrix} = \mathbf{T}_{(i)}^{(i+1)} \begin{Bmatrix} A_i \\ B_i \end{Bmatrix}, \tag{19}$$

$$\mathbf{T}_{(i)}^{(i+1)} = (1/k_{xm}^{(i+1)}) \begin{bmatrix} k_{xm}^{(i)} \cos(k_{xm}^{(i+1)} a/N) & \sin(k_{xm}^{(i+1)} a/N) \\ -k_{xm}^{(i)} \sin(k_{xm}^{(i+1)} a/N) & \cos(k_{xm}^{(i+1)} a/N) \end{bmatrix}, \tag{20}$$

where $\mathbf{T}_{(i)}^{(i+1)}$ is the square matrix of order 2 and is called the *transfer matrix* between the two elements, e_i and e_{i+1}.

If considering Eq. (19) for all interfaces, the following N matrix equations are obtained:

$$\begin{Bmatrix} A_2 \\ B_2 \end{Bmatrix} = \mathbf{T}_{(1)}^{(2)} \begin{Bmatrix} A_1 \\ B_1 \end{Bmatrix} \quad \text{for } x = x_1, \tag{21}$$

$$\begin{Bmatrix} A_3 \\ B_3 \end{Bmatrix} = \mathbf{T}_{(2)}^{(3)} \begin{Bmatrix} A_2 \\ B_2 \end{Bmatrix} \quad \text{for } x = x_2, \tag{22}$$

$$\begin{Bmatrix} A_N \\ B_N \end{Bmatrix} = \mathbf{T}_{(N-1)}^{(N)} \begin{Bmatrix} A_{N-1} \\ B_{N-1} \end{Bmatrix} \quad \text{for } x = x_{N-1}. \tag{23}$$

If applying the chain rule to the above N matrix equations, unknown coefficient vectors

$$\begin{Bmatrix} A_2 \\ B_2 \end{Bmatrix}, \begin{Bmatrix} A_3 \\ B_3 \end{Bmatrix}, \ldots, \begin{Bmatrix} A_{N-1} \\ B_{N-1} \end{Bmatrix}$$

can be removed. As a result, one can obtain

$$\begin{Bmatrix} A_N \\ B_N \end{Bmatrix} = \mathbf{T}_{(1)}^{(N)} \begin{Bmatrix} A_1 \\ B_1 \end{Bmatrix}, \tag{24}$$

where

$$\mathbf{T}_{(1)}^{(N)} = \mathbf{T}_{(N-1)}^{(N)} \mathbf{T}_{(N-2)}^{(N-1)} \cdots \mathbf{T}_{(2)}^{(3)} \mathbf{T}_{(1)}^{(2)}. \tag{25}$$

In Eq. (24), $\mathbf{T}_{(1)}^{(N)}$ is a square matrix of order 2 and is called the *transfer matrix* between the first elements e_1 and the last element e_N.

2.3. Extraction of frequency function

2.3.1. Consideration of the simply supported boundary condition

The simply supported boundary conditions at $x = 0$ and a of the plate shown in Fig. 1 can be expressed as, respectively,

$$W(0, y) = 0, \quad W(a, y) = 0. \tag{26, 27}$$

Applying Eq. (13) for $i = 1$ and N to Eqs. (26, 27), respectively, leads the simply supported boundary conditions to

$$W_1(0, y) = f_1^{(m)}(0) g^{(m)}(y) = 0, \tag{28}$$

$$W_N(a, y) = f_N^{(m)}(a) g^{(m)}(y) = 0. \tag{29}$$

The above boundary conditions can be simplified as, respectively,

$$f_1^{(m)}(0) = 0, \quad f_N^{(m)}(a) = 0. \tag{30, 31}$$

By substituting Eq. (14) into Eqs. (30, 31), one can obtain two equations as follows:

$$A_1 \sin(k_{xm}^{(1)} a/N) + B_1 \cos(k_{xm}^{(1)} a/N) = 0, \quad B_N = 0. \tag{32, 33}$$

2.3.2. Extraction of the system matrix

Eqs. (24) and (33) lead to

$$\begin{Bmatrix} A_N \\ 0 \end{Bmatrix} = \mathbf{T}_{(1)}^{(N)} \begin{Bmatrix} A_1 \\ B_1 \end{Bmatrix}, \tag{34}$$

which can be expressed as two equations:

$$T_{(1)}^{(N)}(1,1)A_1 + T_{(1)}^{(N)}(1,2)B_1 = A_N, \tag{35}$$

$$T_{(1)}^{(N)}(2,1)A_1 + T_{(1)}^{(N)}(2,2)B_1 = 0, \tag{36}$$

where $T_{(1)}^{(N)}(r,s)$ denotes an element at the rth row and sth column in $\mathbf{T}_{(1)}^{(N)}$.

In order to extract the system matrix of which the determinant gives eigenvalues, Eqs. (32) and (36) are formed in a single matrix equation:

$$\mathbf{SM}\begin{Bmatrix} A_1 \\ B_1 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}, \tag{37}$$

where **SM** is called the *system matrix* that is a function of the frequency parameter λ given by Eq. (7) and

$$\mathbf{SM}(\lambda) = \begin{bmatrix} \sin(k_{xm}^{(1)}a/n) & \cos(k_{xm}^{(1)}a/n) \\ T_{(1)}^{(N)}(2,1) & T_{(1)}^{(N)}(2,2) \end{bmatrix}. \tag{38}$$

Finally, the eigenvalues of the plate of interest can be extracted from the values of λ at which the determinant of the system matrix is equal to zero, i.e.,

$$\det[\mathbf{SM}(\lambda)] = 0 \tag{39}$$

in which $\det[\mathbf{SM}(\lambda)]$ is called the *frequency* function in the paper. From Eq. (39), eigenvalues $\lambda_j^{(m)}$'s for $j = 1,2,\ldots$ and $m = 1,2,\ldots$ can be calculated.

Note that $A_1 = B_1 = 0$ if the determinant of the system matrix is not equal to zero and also $A_i = B_i = 0$ for $i = 2,3,\ldots,N$ by Eqs. (21)–(23). It may be said from this fact that the trivial solution ($W(x,y) = 0$) is obtained by Eq. (37) if the determinant of the system matrix is not equal to zero.

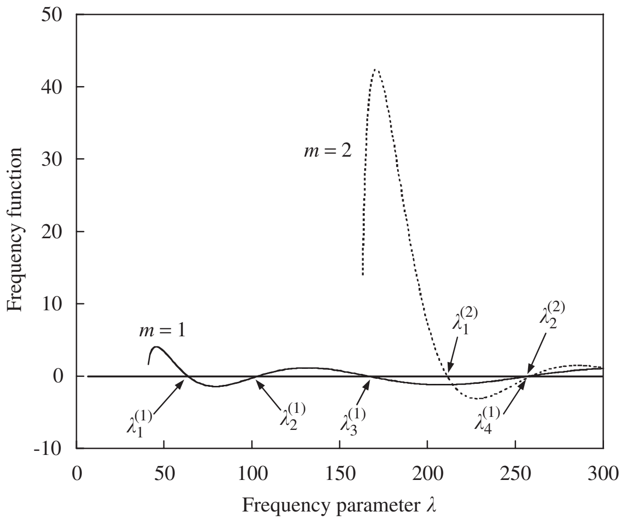

Fig. 3. Frequency function for $h(x) = h_0(1+\alpha(x/a))$, where $h_0 = 0.001\,\text{m}$ and $\alpha = 0.6$.

2.3.3. Extraction of mode shapes

On the other hand, the mode shapes of the plate can be plotted by using Eqs. (13), (19), (37). For this, eigenvalue $\lambda_j^{(m)}$ is substituted into Eq. (37) and a value of B_1 for $A_1 = 1$ is calculated. Then, the values of A_i's and B_i's for $i = 2,3,\ldots,N$ can be successively calculated with Eq. (19). Finally, the mode shape for eigenvalue $\lambda_j^{(m)}$ can be plotted by substituting values of A_i's and B_i's for $i = 2,3,\ldots,N$ into Eq. (13).

3. Numerical work and discussion

3.1. Plate with linearly varying thickness: $h(x) = h_0(1+\alpha(x/a))$

To show the validity and accuracy of the present method, a simply supported rectangular plate with dimensions 1.0 m × 0.5 m is first considered. The analysis results are compared with the results by other methods. In the current case, the thickness function $h(x)$ is expressed as $h(x) = h_0(1+\alpha(x/a))$, where $h_0 = 0.001$ m and $\alpha = 0.6$. For $m = 1$ and 2, the functional values of the frequency function det[**SM**(λ)] are

Table 1
Eigenvalues of the plate with $h(x) = h_0(1+\alpha(x/a))$ where $h_0 = 0.001$ m and $\alpha = 0.6$, obtained by the proposed method, other analytical methods, and FEM (ANSYS); parenthesized values represent errors (%) compared with the FEM results for $N_{ele} = 800$ and N/P denotes 'not presented'

Eigenvalues	Proposed method				Ref. [11]	Ref. [12]	FEM (ANSYS)	
	$N = 10$	$N = 20$	$N = 30$	$N = 40$			$N_{ele} = 200$	$N_{ele} = 800$
$\lambda_1^{(1)}$	64.452 (2.0)	63.706 (0.8)	63.454 (0.4)	63.322 (0.19)	63.488 (0.45)	63.488 (0.45)	63.138 (0.1)	63.203
$\lambda_2^{(1)}$	104.02 (2.2)	102.85 (1.0)	102.46 (0.7)	101.98 (0.19)	102.01 (0.22)	101.99 (0.20)	101.58 (0.2)	101.79
$\lambda_3^{(1)}$	168.75 (2.3)	166.87 (1.2)	166.22 (0.8)	165.21 (0.17)	166.96 (1.23)	165.24 (0.19)	164.44 (0.3)	164.93
$\lambda_1^{(2)}$	213.91 (2.4)	211.32 (1.2)	210.50 (0.8)	209.42 (0.25)	_N/P	_N/P	208.44 (0.2)	208.89
$\lambda_4^{(1)}$	259.39 (2.4)	256.44 (1.2)	255.43 (0.8)	254.17 (0.32)	_N/P	_N/P	252.42 (0.4)	253.35
$\lambda_2^{(2)}$	261.43 (2.3)	258.47 (1.2)	257.45 (0.8)	256.38 (0.35)	_N/P	_N/P	254.56 (0.4)	255.49

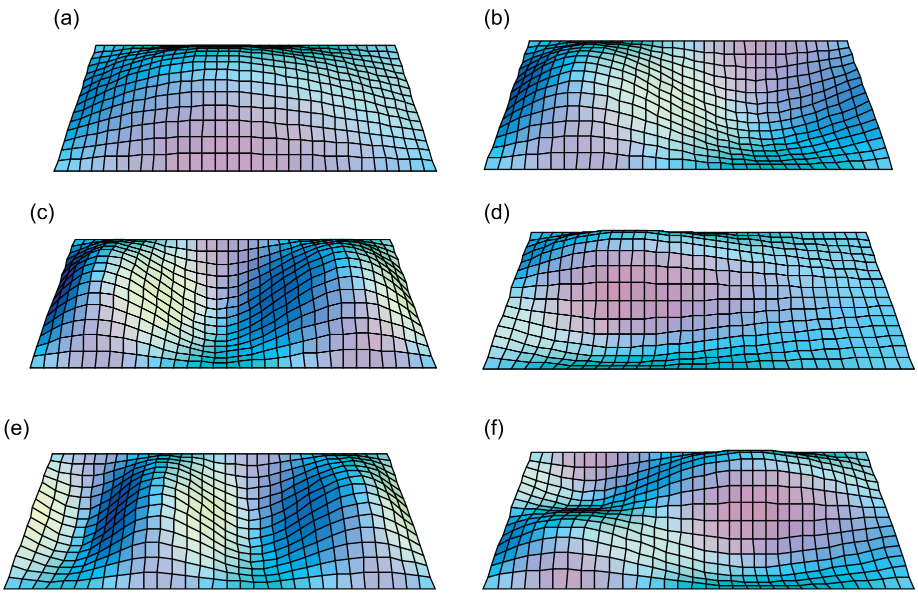

Fig. 4. First six mode shapes of the plate whose thickness function is $h(x) = h_0(1+\alpha(x/a))$, where $h_0 = 0.001$ m and $\alpha = 0.6$.

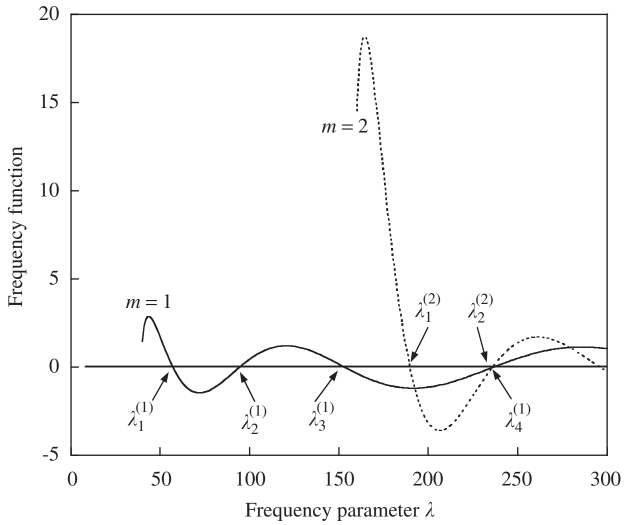

Fig. 5. Frequency function for $h(x) = h_0(1 + \alpha(x/a)^2)$, where $h_0 = 0.001$ m and $\alpha = 0.6$.

Table 2
Eigenvalues of the plate with $h(x) = h_0(1 + \alpha(x/a)^2)$ where $h_0 = 0.001$ m and $\alpha = 0.6$, obtained by the proposed method and FEM (ANSYS); parenthesized values represent errors (%) compared with the FEM results for $N_{ele} = 800$

Eigenvalues	Proposed method			FEM (ANSYS)	
	$N = 10$	$N = 20$	$N = 30$	$N_{ele} = 200$	$N_{ele} = 800$
$\lambda_1^{(1)}$	57.859	57.169	56.930 (0.3)	56.725	56.778
$\lambda_2^{(1)}$	95.335	94.126	93.758 (0.1)	93.685	93.878
$\lambda_3^{(1)}$	155.21	153.27	152.73 (0.5)	151.55	152.00
$\lambda_1^{(2)}$	191.60	189.57	188.96 (0.0)	188.65	189.03
$\lambda_2^{(2)}$	238.67	235.71	234.74 (0.6)	232.52	233.38
$\lambda_4^{(1)}$	238.83	235.90	234.93 (0.4)	233.12	233.98

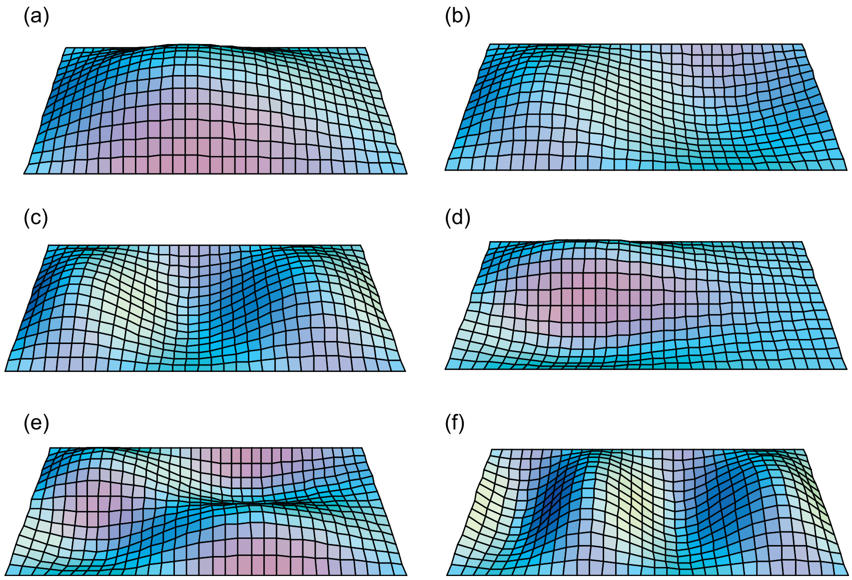

Fig. 6. First six mode shapes of the plate whose thickness function is $h(x) = h_0(1 + \alpha(x/a)^2)$, where $h_0 = 0.001$ m and $\alpha = 0.6$.

plotted as a function of λ in Fig. 3 where the values of λ at which the frequency function satisfies zero represent the eigenvalues of the plate ($\lambda_1^{(1)}, \lambda_2^{(1)}, \lambda_3^{(1)}, \lambda_1^{(2)}, \lambda_4^{(1)}$, and $\lambda_2^{(2)}$ denote the eigenvalues).

In Table 1, the eigenvalues obtained from the present method are compared with those given by other analytical methods [11,12] and FEM (ANSYS). It may be said that the proposed method in the case of $N = 40$ gives more accurate results than the two references' results when the errors of the proposed method for $N = 40$ are compared with the errors of the references' results. In addition, it should be noticed in Table 1 that the proposed method has good convergence characteristics, because the errors decrease as N increases. (Note that the good convergence feature is one of the most important requirements when new methods are developed for solving eigenvalue problems.)

On the other hand, the first six modes plotted by the proposed method are shown in Fig. 4, and the mode shapes and the positions of their nodal lines are in good agreement with the FEM mode shapes, which are omitted in the paper.

3.2. Plate with quadratic thickness function: $h(x) = h_0(1+\alpha(x/a)^2)$

In order to show the validity of the proposed method for plates with more general thickness variation, a rectangular plate with the same dimensions as in Section 3.1 is considered with the thickness function $h(x) = h_0(1 + \alpha(x/a)^2)$ for two cases, $\alpha = 0.6$ and 1.0.

Table 3
Eigenvalues of the plate with $h(x) = h_0(1 + \alpha(x/a)^2)$ where $h_0 = 0.001$ m and $\alpha = 1.0$, obtained by the proposed method and FEM (ANSYS); parenthesized values represent errors (%) compared with the FEM results for $N_{ele} = 800$

Eigenvalues	Proposed method			FEM (ANSYS)	
	$N = 10$	$N = 20$	$N = 30$	$N_{ele} = 200$	$N_{ele} = 800$
$\lambda_1^{(1)}$	62.495	61.405	61.058 (0.6)	60.617	60.674
$\lambda_2^{(1)}$	105.16	103.23	102.66 (0.5)	102.94	103.15
$\lambda_3^{(1)}$	171.29	168.26	167.28 (0.5)	165.99	166.48
$\lambda_1^{(2)}$	202.28	199.27	198.29 (0.2)	198.31	198.75
$\lambda_2^{(2)}$	261.94	257.25	255.73 (0.3)	254.09	255.06
$\lambda_4^{(1)}$	263.41	258.67	257.25 (0.8)	254.17	255.09

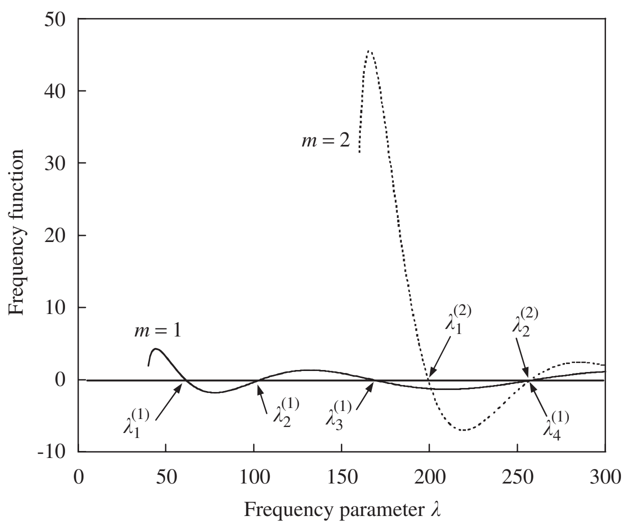

Fig. 7. Frequency function for $h(x) = h_0(1 + \alpha(x/a)^2)$, where $h_0 = 0.001$ m and $\alpha = 1.0$.

Fig. 5 shows frequency function curves where $\lambda_1^{(1)}, \lambda_2^{(1)}, \lambda_3^{(1)}, \lambda_1^{(2)}, \lambda_2^{(2)}$, and $\lambda_4^{(1)}$ represent the eigenvalues of the plate in the case of $\alpha = 0.6$. The eigenvalues are compared with the FEM results in Table 2 where it is confirmed that the proposed method for $N = 30$ yields accurate results within 0.6% error compared with the FEM results for $N_{\text{ele}} = 800$. Fig. 6 shows the first six mode shapes that agree well with the FEM mode shapes, which are omitted in the paper. Interestingly, it may be noticed from the comparison of Fig. 6 with Fig. 4 that the sequence of the fifth and sixth modes is changed. The reason may be that an equivalent stiffness in the x direction is increased in the case of the quadratic thickness function.

Fig. 7 shows frequency function curves for $\alpha = 1.0$. Eigenvalues obtained from the frequency function curves are summarized in Table 3 where the results by the proposed method are compared with the FEM results. Note that the results ($N = 30$) by the proposed method are within 0.8% error compared with the FEM results for $N_{\text{ele}} = 800$.

On the other hand, Fig. 8 shows the first six modes plotted by the proposed method. The mode shapes and the positions of their nodal lines are in good agreement with the FEM mode shapes, which are omitted in the paper.

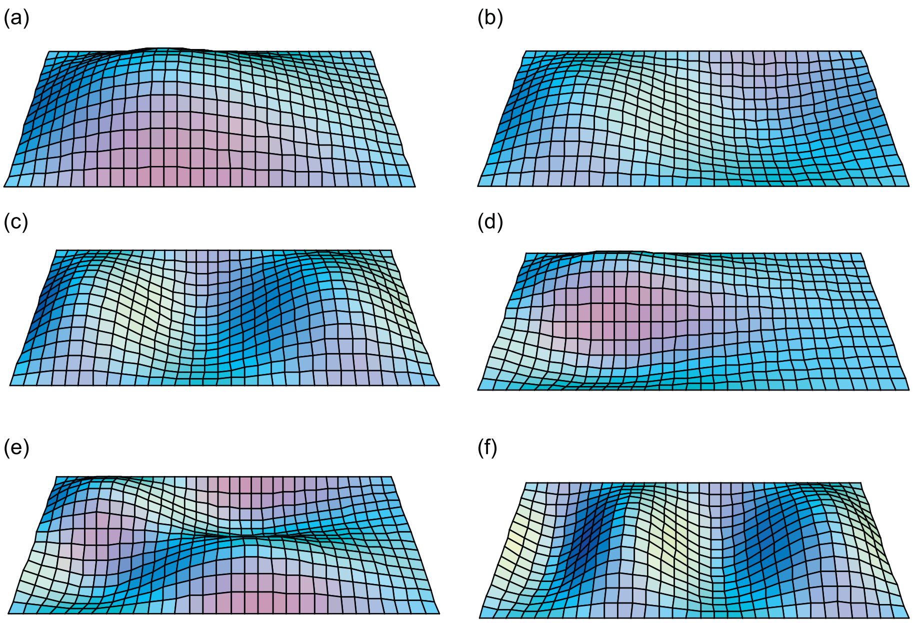

Fig. 8. First six mode shapes of the plate whose thickness function is $h(x) = h_0(1+\alpha(x/a)^2)$, where $h_0 = 0.001\,\text{m}$ and $\alpha = 1.0$.

Table 4
Eigenvalues of the plate with $h(x) = h_0(1+\alpha \sin \pi x/a)$ or $h(x) = h_0(1+\alpha\sqrt{x/a})$, where $h_0 = 0.001\,\text{m}$ for $\alpha = 0.6$, 0.8 and 1.0 ($N = 30$)

Eigenvalues	$h(x) = h_0(1+\alpha \sin \pi x/a)$			$h(x) = h_0(1+\alpha\sqrt{x/a})$		
	$\alpha = 0.6$	$\alpha = 0.8$	$\alpha = 1.0$	$\alpha = 0.6$	$\alpha = 0.8$	$\alpha = 1.0$
$\lambda_1^{(1)}$	73.7	81.6	89.3	69.6	76.2	82.7
$\lambda_2^{(1)}$	109.6	119.4	129.0	111.1	121.6	132.1
$\lambda_3^{(1)}$	176.5	191.9	207.1	180.0	197.0	213.8
$\lambda_1^{(2)}$	247.4	271.5	294.3	233.6	254.0	274.0
$\lambda_2^{(2)}$	270.3	293.5	316.3	276.6	302.5	328.2
$\lambda_4^{(1)}$	272.3	295.8	319.0	278.7	305.6	332.4

3.3. Plate with other arbitaty thickness functions

For further research of researchers with interest in the proposed method, the eigenvalues of plates with arbitrary thickness functions $h(x) = h_0(1 + \alpha \sin \pi x/a)$ and $h(x) = h_0(1 + \alpha\sqrt{x/a})$ are summarized in Table 4. The eigenvalues have been obtained with $N = 30$ for $\alpha = 0.6$, 0.8, and 1.0. It has been revealed that the results in Table 4 are within 0.7% error compared with the FEM results for $N_{ele} = 800$ although the FEM results are omitted in the paper.

4. Conclusions

An effective, analytical method for the free vibration analysis of a simply supported rectangular plate with *arbitrarily* varying thickness was proposed in this paper. It was revealed that the method gives the close-form frequency function and that, as a result, it shows an excellent convergence feature and yields accurate eigenvalues and mode shapes, when compared with other methods including FEM (ANSYS). It is expected that the application region of the method comes up to the free vibration analyses of non-homogeneous rectangular plates with various combinations of the elementary boundary conditions (simply supported, clamped, and free boundary conditions).

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References

[1] S.W. Kang, J.M. Lee, Free vibration analysis of composite rectangular membranes with an oblique interface, *Journal of Sound and Vibration* 251 (3) (2002) 505–517.
[2] S.W. Kang, Free vibration analysis of composite rectangular membranes with a bent interface, *Journal of Sound and Vibration* 272 (1) (2004) 39–53.
[3] S.W. Kang, J.M. Lee, Free vibration analysis of an unsymmetric trapezoidal membrane, *Journal of Sound and Vibration* 272 (2) (2004) 450–460.
[4] T. Sakata, Eigenvalues of rectangular plates with linearly varying thickness, *Theoretical and Applied Mechanics* 22 (1974) 329–337.
[5] V.A. Pulmano, R.K. Gupta, Vibration of tapered plates by finite strip method, *Journal of the Engineering Mechanics Division—American Society of Civil Engineers* 102 (1976) 553–559.
[6] F.C. Appl, N.F. Byers, Fundamental frequency of simply supported rectangular plates with varying thickness, *Journal of Applied Mechanics* 32 (1965) 163–168.
[7] D.S. Chehil, S.S. Dua, Buckling of rectangular plates with general variation in thickness, *Journal of Applied Mechanics* 40 (1973) 745–751.
[8] V.H. Cortinez, P.A.A. Laura, Vibration of non-homogeneous rectangular membranes, *Journal of Sound and Vibration* 156 (1992) 217–225.
[9] K.M. Liew, M.K. Lim, Transverse vibration of trapezoidal plates of variable thickness: symmetric trapezoids, *Journal of Sound and Vibration* 165 (1993) 45–67.
[10] K.M. Liew, C.W. Lim, M.K. Lim, Transverse vibration of trapezoidal plates of variable thickness: unsymmetric trapezoids, *Journal of Sound and Vibration* 177 (1994) 479–501.
[11] B. Singh, V. Saxena, Transverse vibration of a rectangular plate with bidirectional thickness variation, *Journal of Sound and Vibration* 198 (1) (1996) 51–65.
[12] H. Kobayashi, K. Sonoda, Vibration and buckling of tapered rectangular plates with two opposite edges simply supported and the other two edges elastically restrained against rotation, *Journal of Sound and Vibration* 146 (1991) 323–337.
[13] K. Akiyama, M. Kuroda, Fundamental frequencies of rectangular plates with linearly varying thickness, *Journal of Sound and Vibration* 205 (3) (1997) 380–384.
[14] P.A.A. Laura, R.E. Rossi, R.H. Gutierrez, The fundamental frequency of non-homogeneous rectangular membranes, *Journal of Sound and Vibration* 204 (2) (1997) 373–376.
[15] C.Y. Wang, Fundamental frequencies of a membrane strip with periodic boundary constraints, *Journal of Sound and Vibration* 214 (2) (1998) 389–393.
[16] C.Y. Wang, Fundamental modes of a circular membrane with radial constraints on the boundary, *Journal of Sound and Vibration* 220 (3) (1999) 559–563.
[17] P.A.A. Laura, D.V. Bambill, R.H. Gutierrez, A note on transverse vibrations of circular, annular, composite membranes, *Journal of Sound and Vibration* 205 (5) (1997) 692–697.

[18] C.A. Rossit, S. La Malfa, P.A.A. Laura, Antisymmetric modes of vibrations of composite, doubly-connected membranes, *Journal of Sound and Vibration* 217 (1) (1998) 191–195.
[19] G.R. Buchanan, J. Peddieson Jr., Vibration of circular, annular membranes with variable density, *Journal of Sound and Vibration* 226 (2) (1999) 379–382.
[20] M. Jabareen, M. Eisenberger, Free vibrations of non-homogeneous circular and annular membranes, *Journal of Sound and Vibration* 240 (3) (2001) 409–429.
[21] H.P.W. Gottlieb, Exact solutions for vibrations of some annular membranes with inhomogeneous radial densities, *Journal of Sound and Vibration* 233 (1) (2000) 165–170.
[22] R.D. Blevins, *Formulas for Natural Frequency and Mode Shape*, Van Nostrand Reinhold, New York, 1979, pp. 224–239.
[23] H.D. Conway, K.A. Karnham, The free flexural vibration of triangular, rhombic and parallelogram plates and some anologies, *International Journal of Mechanical Sciences* 7 (1965) 811–816.