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Journal of Sound and Vibration 312 (2008) 754–768
JOURNAL OF SOUND AND VIBRATION
www.elsevier.com/locate/jsvi

Change in instantaneous eigenproperties due to yielding of a structure

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Received 8 March 2007; received in revised form 7 September 2007; accepted 7 November 2007
Available online 21 December 2007

Abstract

This paper presents an insight into the pattern of change in instantaneous eigenproperties of a building as a result of reduction in tangent stiffness due to yielding of different members of the building. Using a perturbation approach, it is found that for a shear building, a reduction in instantaneous stiffness due to a yielding occurring in the bottom story will cause a greater absolute percentage change in eigenvalues (squared frequencies) of a lower mode than that of a higher mode. In contrast, for a reduction in instantaneous stiffness due to a yielding occurring in the top story, this absolute percentage change in the eigenvalues increases with a increase in mode number starting from the fundamental mode. A numerical study with a four-story shear building and a four-story and an eight-story steel moment-resisting frame models representative of existing buildings demonstrates that the trends in change in eigenproperties obtained from the analytical study works well for these buildings. The findings of this study may be utilized to improve the design practices of buildings as well as non-structural components attached to buildings.

1. Introduction

It is well known that in current practice of performance-based design, in order to dissipate vibrational energy through inelastic deformations and reduce force demands, structures are designed to go into their nonlinear range of behavior when subjected to a strong earthquake excitation. Thus, during a strong earthquake excitation, a structure goes into its nonlinear range of behavior from its initial pre-yield linear behavior and comes back to linear behavior again. The cycle continues following a hysteretic constitutive relationship, while keeping an acceptable level of performance of the structure. As the structure goes into its nonlinear range of behavior from its initial linear range due to yielding of one or more of its structural elements, its instantaneous stiffness (tangent stiffness) reduces from its initial linear stiffness, and in consequence, its instantaneous eigenproperties change from its initial eigenproperties. This change in eigenproperties due to a reduction in stiffness results in redistribution of vibrational energy among the instantaneous modes, which plays a major role in governing where the next yield will occur, and therefore plays a critical role in defining the seismic response of a structure. In this paper, by 'yielding' the author

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0022-460X/$ - see front matter
doi:10.1016/j.jsv.2007.11.009

implies reduction in instantaneous stiffness of a member due to nonlinear behavior of that member or formation of plastic hinge.

When a structure undergoes nonlinear deformation, the response of a stiff non-structural component (e.g., chiller, parapet, control panel, electrical, and mechanical equipment) attached to the yielding structure increases when compared with the case where the non-structural component is attached to the same structure but idealized as a linear system and undergoing the same seismic excitation. This increase in response, which was found through linear and nonlinear time-history analyses [1–3], can be explained by the aforementioned redistribution of vibrational energy among modes [4]. A better understanding of this energy redistribution mechanism of a yielding structure, when subjected to strong earthquake excitation, can be used in a response spectrum-based approach to develop better approximate methods for seismic response analysis of non-structural components attached to the structure. In addition, a good understanding of energy redistribution among instantaneous modes and associated energy loss can be used to scrutinize current seismic design philosophy of a yielding structure and may provide a better seismic design practice of the present-day structures. It is noteworthy that which member will yield first and suffer nonlinear deformation (i.e., whether a member situated at the top story of a building, or at the bottom story of the building, or situated at somewhere in between will yield first), when subjected to a strong earthquake excitation, depends on the characteristics of the ground excitation, distributions of relative rigidity of members and mass distribution along the height of a structure. An assessment in the change in eigenproperties related to where the first yield occurs within a structure is therefore necessary.

1.1. Previous research

Significant research has been taken place to characterize eigenproperties of many complex systems and their constituting elements. Perhaps to mechanical, aeronautical, and structural engineers, most studied structural element is a beam, which is often modeled to idealize a building, its foundation, and also some other mechanical and aeronautical structures and components. The fundamental vibration behavior of various types of short, long, slender, prismatic and non-uniform beams has been investigated using the theories of classical shear, Euler–Bernoulli and Timoshenko beams. Typically studied area is the determination of eigenproperties of beams considering the effect of shear and/or flexural deformations, rotational inertia, boundary conditions, and axial or in-plane loading. A comprehensive summary of these studies can be found in Bokaian [5] and Esmailzadeh and Ohadi [6].

A non-uniform beam in which the cross-section is varying along the length of the beam is widely conceived as a simple model of many real-world structures. Significant development has occurred to address the vibration behaviors of a non-uniform beam with different support conditions, i.e., clamped supported, elastically supported, free end and pinned end. Some of these works when listed chronologically are as follows: Mabi and Rogers [7] studied the transverse vibration of double-tapered cantilever beams; Irie et al. [8] studied the steady-state response of a Timoshenko beam with varying cross-section to harmonic point load; Rossi et al. [9] obtained the natural frequencies of a cantilever Timoshenko beam with tip mass for both discontinuous and linear variation of cross-section; Jategaonkar and Chehil [10] gave the natural frequencies of a beam with varying cross-section; Gutierrez et al. [11] studied the vibration of elastically clamped Timoshenko beam; Lee et al. [12] studied the vibration of non-uniform beams subjected to variably distributed axial loads; Esmailzadeh and Ohadi [6] studied the vibration of non-uniform Timoshenko beam subjected to different loading conditions. For discontinuous variation of thickness, the vibration behaviors were finally obtained by numerical solutions. For the cases where the cross section varies smoothly (i.e., varying linearly or in some functional form), solutions are expressed using Bessel's functions or the method of Frobenius is used (e.g., Lee and Kuo [13,14], and Lee and Lin [15]). Numerical techniques including finite element analysis and Galerkin's approach have also been widely used to characterize eigenfrequencies of systems with non-uniform stiffness distribution.

Since the presence of cracks in a beam modifies its eigenfrequencies, researchers have tried to identify cracks in a beam by measuring the eigenfrequencies with the aid of ultrasonic fatigue test and comparing the test results with the computed one. Gudmundson [16] used the first-order approximation of perturbation approach to determine the effect of size and position of a crack on eigenfrequencies. Further, using transverse vibration of cantilever Bernoulli–Euler beams, he found that the results of the first four resonance frequencies match well with the experimental results for a crack less than 40% of the width of the beam. He also used the longitudinal vibration of a beam to compare his results with those obtained from a finite element analysis. This study was conducted to predict the eigenfrequencies of a cracked beam. Several other researchers have also studied cracks of joints and edge-cracked beam. Later, Gudmundson [17] used flexibility matrix to represent the cracks. Zheng and Fan [18] studied a non-uniform beam with an arbitrary number of open cracks using modified Fourier series in association with the finite element method. Stetson [19] used perturbation approach to find the change in eigenproperties due to a small change in mass and stiffness of a beam. Later the same technique has been used to formulate the inverse problem, i.e., what change is necessary for mass and stiffness properties of members in design in order to achieve a desired change in eigenfrequencies [20,21].

1.2. Scope of this work

In this paper, an approximate closed form relationship between the initial linear (pre-yield) and the instantaneous eigenproperties of a yielding structure after yielding occurred is established by means of a well-known perturbation approach. A uniform two-term expansion is considered for the perturbation approach to show the relative strength of these terms. Although during a strong earthquake excitation, the members of a structure yield one by one and consequently the eigenproperties of the structure change, in this study, the change in stiffness between the initial linear range (pre-yield) and when the structure is in its nonlinear zone of behavior is considered. Then approximating the vibration characteristics of a building structure as that of an equivalent shear beam, closed form expressions are obtained to get an insight into the change in instantaneous eigenproperties for different patterns of yielding, i.e., where within a building, the first yielding occurs or plastic hinge forms. In particular, the focus is placed on understanding the pattern of change in eigenvalues and coupling matrix of initial linear to post-yield eigenvectors with respect to the mass matrix. A numerical study considering a four-story shear building and two steel moment-resisting frames, one four-story and the other eight-story, has been conducted to investigate the applicability of the proposed theory.

2. Perturbation formulation

Consider a n degree-of-freedom (dof) lumped mass model representing a typical building structure. Let $[m]$, $[k]$ and $[\hat{k}]$, respectively, denote the mass matrix, the initial stiffness matrix, and the post-yield instantaneous stiffness (tangent stiffness) matrix of the structure, where $[\hat{k}]-[k]=[\Delta k]$ implies the change in stiffness matrix of the structure when it goes into its nonlinear range from its linear range of behavior. Therefore, the instantaneous eigenproperties of the structure in the nonlinear range can be obtained from the following equation:

$$[m]\{\ddot{x}(t)\}+[\hat{k}]\{x(t)\}=\{0\}. \tag{1}$$

Let us consider the coordinate transformation

$$\{x(t)\}=[\phi]\{q(t)\}, \tag{2}$$

where $[\phi]$ is the modal matrix of the linear system obtained by considering n orthonormal modes. In view of this transformation and pre-multiplying by $[\phi]^{\mathrm{T}}$, Eq. (1) becomes

$$[I]\{\ddot{q}(t)\}+[[\lambda]+[\phi]^{\mathrm{T}}[\Delta k][\phi]]\{q(t)\}=\{0\}, \tag{3}$$

where $[\lambda]$ is a diagonal matrix with the diagonal elements representing the squares of the frequencies (eigenvalues) of the linear structure. Eq. (3) leads to the eigenvalue problem in normal coordinates as given below

$$[[\lambda]+[\phi]^{\mathrm{T}}[\Delta k][\phi]]\{\hat{\psi}^{(j)}\}=\hat{\lambda}_j\{\hat{\psi}^{(j)}\}, \tag{4}$$

where $\hat{\lambda}_j$ and $\{\hat{\psi}^{(j)}\}$, respectively, denote the square of the jth natural frequency and the jth orthonormal mode shape of the structure in the nonlinear range.

Let us consider that the change in stiffness due to yielding is very small or, in other words, that the order of each term of $[\Delta k]$ is very small compared to the corresponding term of $[k]$. Therefore, assuming different orders of magnitudes of the elements of $[[\lambda] + [\phi]^{\mathrm{T}}[\Delta k][\phi]]$, one can express it as

$$[[\lambda] + [\phi]^{\mathrm{T}}[\Delta k][\phi]] = [A_0] + [A_1], \tag{5}$$

with $[A_0] = [\lambda]$ and $[A_1] = [\phi]^{\mathrm{T}}[\Delta k][\phi]$.

Now, introducing a 'book-keeping parameter', ε, to keep track of the orders of the different quantities involved in the eigenvalue problem, Eq. (4) may be expressed as

$$[[A_0] + \varepsilon[A_1]]\{\hat{\psi}^{(j)}\} = \hat{\lambda}_j\{\hat{\psi}^{(j)}\}, \tag{6}$$

with $\varepsilon = 1$. It may be noted that the equations obtained by setting $\varepsilon = 0$ in Eq. (6) represent the unperturbed eigenvalue problem, which can be solved using the initial (linear) system properties. Considering a straightforward perturbation expansion for the approximate solutions of $\hat{\lambda}_j$ and $\hat{\psi}^{(j)}$, one can write

$$\hat{v}_j = \sum_{i=0}^{N} \varepsilon^i v_{ij} + O(\varepsilon^{N+1}), \tag{7}$$

where $\hat{v}_j$ takes the value of either $\hat{\lambda}_j$ or $\hat{\psi}^{(j)}$. Here, $O()$ represents the order of magnitude of a quantity and the expansion is said to be valid up to $O(\varepsilon^N)$, with an error of $O(\varepsilon^{N+1})$. Considering an expansion uniformly valid up to $O(\varepsilon)$ (setting $N = 1$ in Eq. (7)), the perturbed (post-yield) eigenvalues and eigenvectors may be expressed as

$$\hat{\lambda}_i = \lambda_{0i} + \varepsilon\lambda_{1i}, \quad i = 1, 2, \ldots, n \tag{8}$$

and

$$\{\hat{\psi}^{(i)}\} = \{u_{0i}\} + \varepsilon\{u_{1i}\}, \quad i = 1, 2, \ldots, n, \tag{9}$$

where λ_{0i} and $\{u_{0i}\}$ are of $O(\varepsilon^0)$; λ_{1i} and $\{u_{1i}\}$ are of $O(\varepsilon)$. The assumed expansion in Eqs. (8) and (9) will satisfy the following orthogonality condition:

$$\{\hat{\psi}^{(i)}\}^{\mathrm{T}}\{\hat{\psi}^{(j)}\} = \delta_{ij}, \quad i, j = 1, 2, 3, \ldots, n, \tag{10}$$

where δ_{ij} is the Kronecker delta.

On substituting the above expansions of $\hat{\lambda}_i$ and $\{\hat{\psi}^{(i)}\}$ in Eq. (6) and equating the coefficients of equal powers of ε up to $O(\varepsilon)$, the following hierarchy of equations is obtained:

Order ε^0:

$$[A_0]\{u_{0i}\} = \lambda_{0i}\{u_{0i}\}, \tag{11}$$

Order ε^1:

$$[A_0]\{u_{1i}\} + [A_1]\{u_{0i}\} = \lambda_{0i}\{u_{1i}\} + \lambda_{1i}\{u_{0i}\}, \tag{12}$$

where $i = 1, 2, \ldots, n$. Similarly, on utilizing the orthogonality condition given by Eq. (10) and considering terms up to $O(\varepsilon)$, the following hierarchy of equations is obtained:

Order ε^0:

$$\{u_{0j}\}^{\mathrm{T}}\{u_{0i}\} = \delta_{ij}, \tag{13}$$

Order ε^1:

$$\{u_{0j}\}^{\mathrm{T}}\{u_{1i}\} + \{u_{1j}\}^{\mathrm{T}}\{u_{0i}\} = 0, \tag{14}$$

where $i, j = 1, 2, \ldots, n$.

Since $[A_0]$ is a diagonal matrix with its ith diagonal element representing the square of the linear system frequency λ_i, from Eqs. (11) and (13), we can observe that λ_{0i} and $\{u_{0i}\}$ are the ith eigenvalue and eigenvector of the linear system in normal coordinates, i.e., $\lambda_{0i} = \lambda_i$ and $\{u_{0i}\}$ is a vector with its ith element equal to unity and all other elements equal to zero.

Now, by expanding $\{u_{1i}\}$ in terms of base vectors $\{u_{0j}\}, j = 1, 2, \ldots, n$ (see Meirovitch and Ryland [22]), i.e.,

$$\{u_{1i}\} = \sum_{j=1}^{n} a_{ij}\{u_{0j}\} \tag{15}$$

and utilizing Eqs. (11)–(14), the coefficients of expansions a_{ij} in Eq. (15) and correction terms λ_{1i} in Eq. (8) can be determined. Accordingly, on substituting in Eq. (12) the expansion of $\{u_{1i}\}$, given by Eq. (15), pre-multiplying by $\{u_{0j}\}^{\mathrm{T}}$, and on using Eq. (13), the following equation is obtained:

$$\lambda_{1i}\delta_{ij} = (\lambda_{0j} - \lambda_{0i})a_{ij} + \{u_{0j}\}^{\mathrm{T}}[A_1]\{u_{0i}\}, \quad i,\ j = 1, 2, \ldots, n. \tag{16}$$

This equation further leads to

$$\lambda_{1i} = \{u_{0i}\}^{\mathrm{T}}[A_1]\{u_{0i}\} = A_1(i,i), \quad i = 1, 2, \ldots, n \tag{17}$$

and

$$a_{ij} = \frac{\{u_{0j}\}^{\mathrm{T}}[A_1]\{u_{0i}\}}{\lambda_{0i} - \lambda_{0j}} = \frac{A_1(j,i)}{\lambda_{0i} - \lambda_{0j}}, \quad i \neq j,\ \ i,\ j = 1, 2, \ldots, n, \tag{18}$$

where $A_1(j,i)$ denotes the element corresponding to jth row and ith column of $[A_1]$.

On substituting Eq. (15) in Eq. (14) and after simplifying, one can obtain $a_{ii} = 0$ and $a_{ij} = -a_{ji}$ for $i,\ j = 1, 2, \ldots, n$ and thus, $\{u_{1i}\}$ becomes

$$\{u_{1i}\} = \sum_{j=1;\, j\neq i}^{n} \frac{A_1(j,i)}{\lambda_i - \lambda_j}\{u_{0j}\}, \quad i,\ j = 1, 2, \ldots, n. \tag{19}$$

Finally, the results of the above derivation can be summarized as

$$\hat{\lambda}_i = \lambda_i + A_1(i,i), \quad i = 1, 2, \ldots, n \tag{20}$$

and

$$\hat{\psi}_i^{(i)} = 1.0, \quad \hat{\psi}_j^{(i)} = -\frac{A_1(j,i)}{\lambda_i - \lambda_j}, \quad i \neq j = 1, 2, \ldots, n \tag{21}$$

as $\{u_{0i}\}$ is a vector with its ith element equal to unity and all other elements equal to zero. The perturbed ith eigenvector (i.e., ith mode shape) in general coordinates becomes

$$\{\hat{\phi}^{(i)}\} = [\phi]\{\hat{\psi}^{(i)}\}. \tag{22}$$

Pre-multiplying both sides of Eq. (22) by $[\phi]^{\mathrm{T}}[m]$, one can write

$$[\hat{\psi}] = [\phi]^{\mathrm{T}}[m][\hat{\phi}], \tag{23}$$

which implies that $[\hat{\psi}]$ is essentially a coupling matrix between pre-yield and post-yield modal matrices with respect to the mass matrix.

3. Vibration of shear building

Let us consider a n-dof shear building as shown in Fig. 1. Let k_i, y_i and m_i be its ith story stiffness, the height of its ith story from its base and the mass of its ith floor, respectively, where $i = 1, 2, \ldots, n$. Let us also consider that n is large and the eigenproperties of the building can be approximated as that of an equivalent uniform cantilever shear beam of length L. Thus the ith frequency of the building, ω_i and the jth component of the ith orthonormal mode shape of the building, $\phi_j^{(i)}$ can be approximated as follows:

$$\omega_i = (2i - 1)\frac{\pi c}{2L} \tag{24}$$

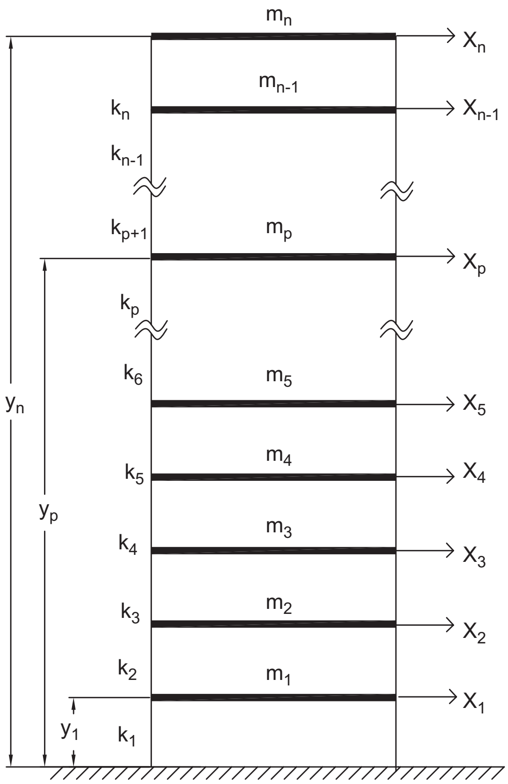

Fig. 1. Schematic diagram of a n degree-of-freedom shear building.

and

$$\phi_j^{(i)} = b \sin(2i-1)\frac{\pi y_j}{2L}, \tag{25}$$

where $i,\ j = 1, 2, \ldots, n$, c is the shear wave velocity of the equivalent shear beam and $b = \sqrt{2/(\rho L)}$, ρ being the mass per unit length of the equivalent shear beam. Note that this assumption of eigenproperties of a lumped mass model in terms of the eigenproperties of a uniform shear beam has been proven to be reasonable for lower modes. Based on these assumptions, the results obtained from the perturbation formulation, can be interpreted for the following three cases.

3.1. Case 1: yielding of a member of the bottom story

Let us assume that yielding occurs in a member of the bottom story. This case may represent the scenario of yielding of the shear wall structures or even some frame structures during a strong earthquake. Let the reduction in stiffness of this member due to yielding be δk_1, where $0 < \delta < 1$. Hence, only the (1,1)th element of $[\Delta k]$ is $-\delta k_1$ and all other elements are equal to zero. Therefore, one can express $A_1(i,j) = -\phi_1^{(j)}\phi_1^{(i)}\delta k_1$ and $\hat{\lambda}_i$ as

$$\hat{\lambda}_i = \lambda_i\left[1 - (\phi_1^{(i)})^2\frac{\delta k_1}{\lambda_i}\right], \quad i = 1, 2, \ldots, n. \tag{26}$$

Let e_i represents the change in ith eigenvalue (square of the ith frequency) due to the reduction in stiffness resulting from the yielding and normalized with respect to λ_i, i.e., $e_i = (\hat{\lambda}_i - \lambda_i)/\lambda_i$. Therefore, one can write

$$e_i = -(\phi_1^{(i)})^2 \frac{\delta k_1}{\lambda_i}, \quad i = 1, 2, \ldots, n. \tag{27}$$

Now, replacing $\phi_1^{(i)}$ with $b\sin(2i-1)\pi y_1/2L$, and substituting $[(2i-1)\pi c/2L]^2$ for λ_i, one can write

$$e_i = -\frac{[b\sin((2i-1)\pi y_1/2L)]^2 \delta k_1}{[(2i-1)\pi c/2L]^2}, \quad i = 1, 2, \ldots, n. \tag{28}$$

Simplifying above expression as

$$e_i = -\frac{[1-\cos((2i-1)\pi y_1/L)]b^2 \delta k_1}{2[(2i-1)\pi c/2L]^2}, \quad i = 1, 2, \ldots, n, \tag{29}$$

assuming $y_1 \to 0$ due to large value of n, expanding $\cos((2i-1)\pi y_1/L)$ in cosine series for i not being too large, and neglecting higher order terms, Eq. (29) can be written as

$$e_i = -\left[1 - (2i-1)^2 \frac{\pi^2}{12} \frac{y_1^2}{L^2}\right] \frac{y_1^2}{c^2} b^2 \delta k_1, \quad i = 1, 2, \ldots, n. \tag{30}$$

Similarly, for $i,\ j = 1, 2, \ldots, n$ and $i \neq j$, $\hat{\psi}_j^{(i)}$ can be expressed as

$$\begin{aligned} \hat{\psi}_j^{(i)} &= -\frac{\phi_1^{(j)}\phi_1^{(i)}\delta k_1}{\lambda_i - \lambda_j} \\ &= -\frac{b^2 \sin((2j-1)\pi y_1/2L)\sin((2i-1)\pi y_1/2L)}{[(2i-1)^2-(2j-1)^2](\pi c/2L)^2}. \end{aligned} \tag{31}$$

Expanding sine terms in series for small value of $(2j-1)\pi y_1/2L$ and $(2i-1)\pi y_1/2L$, and ignoring higher order terms, one can write

$$\hat{\psi}_j^{(i)} = -\frac{(2j-1)(2i-1)(by_1/c)^2}{(2i-1)^2-(2j-1)^2}. \tag{32}$$

Observation 1: From Eq. (30), one can observe that starting from $i = 1$, as i increases, $|e_i|$ reduces. This implies that due to yielding of any member in the bottom story, absolute percentage change in eigenvalues of a lower mode is significantly greater than that of a higher mode.

Observation 2: From Eq. (32) and keeping in mind that $\hat{\psi}_i^{(i)} = 1$, one can say that for a given i, $(i \geqslant 1)$, as $|i-j|$ increases, $|[(2j-1)(2i-1)]/[(2i-1)^2-(2j-1)^2]|$ reduces and thus $|\hat{\psi}_j^{(i)}|$ reduces. This implies that the coupling between pre-yield (linear) and post-yield modal matrices reduces with the separation of modes. In other words, any lower linear mode (say the fundamental mode) will still be approximately orthogonal with any higher instantaneous modes (say the fifth mode) of post-yield structure.

3.2. *Case 2: yielding of a member of the top story*

Let us consider that the yielding occurs in a member of the top story (i.e., farthest from building's base). Let the reduction in stiffness of this member be δk_n, where $0 < \delta < 1$. Therefore, one can observe that both the $(n-1, n-1)$th and (n, n)th terms of $[\Delta k]$ are equal to $-\delta k_n$, and both the $(n-1, n)$th and $(n, n-1)$th terms of $[\Delta k]$ are equal to δk_n. Thus, after simplification, one can write,

$$A(i,j) = -\delta k_n(\phi_n^{(i)} - \phi_{n-1}^{(i)})(\phi_n^{(j)} - \phi_{n-1}^{(j)}), \quad i,\ j = 1, 2, \ldots, n. \tag{33}$$

Therefore, $\hat{\lambda}_i$ can be expressed as

$$\hat{\lambda}_i = \lambda_i\left(1 - \frac{\delta k_n(\phi_n^{(i)} - \phi_{n-1}^{(i)})^2}{\lambda_i}\right), \quad i = 1, 2, \ldots, n. \tag{34}$$

Now, from the eigenvalue problem of the linear system, one can write

$$k_n(\phi_n^{(i)} - \phi_{n-1}^{(i)}) = \lambda_i m_n \phi_n^{(i)}, \quad i = 1, 2, \ldots, n. \tag{35}$$

Using Eq. (35) in Eq. (34), one can express $\hat{\lambda}_i$ as

$$\hat{\lambda}_i = \lambda_i \left(1 - \frac{\delta (m_n \phi_n^{(i)})^2}{k_n} \lambda_i \right), \quad i = 1, 2, \ldots, n. \tag{36}$$

Replacing e_i with $(\hat{\lambda}_i - \lambda_i)/\lambda_i$, Eq. (36) can be written as

$$e_i = -\lambda_i \delta \frac{m_n^2 (\phi_n^{(i)})^2}{k_n}, \quad i = 1, 2, \ldots, n. \tag{37}$$

Substituting $\phi_n^{(i)}$ with $b \sin((2i-1)\pi y_n / 2L)$ and λ_i with $[(2i-1)\pi c/2L]^2$, one can write

$$e_i = -\frac{b^2 \delta m_n^2}{k_n} \left((2i-1) \frac{\pi c}{2L} \right)^2 \left[\sin \left((2i-1) \frac{\pi y_n}{2L} \right) \right]^2, \quad i = 1, 2, \ldots, n. \tag{38}$$

For a large value of n, $y_n \to L$ and thus, Eq. (38) can be approximated as

$$e_i = -\frac{b^2 \delta m_n^2}{k_n} \left(\frac{(2i-1)\pi c}{2L} \right)^2, \quad i = 1, 2, \ldots, n. \tag{39}$$

Similarly, on using Eq. (35) in Eq. (33) and then substituting the result in Eq. (21), one can express $\hat{\psi}_j^{(i)}$ as

$$\hat{\psi}_j^{(i)} = -\frac{\delta \phi_n^{(j)} \phi_n^{(i)} \lambda_i \lambda_j m_n^2}{(\lambda_i - \lambda_j) k_n}, \quad i \neq j, \ i, \ j = 1, 2, \ldots, n. \tag{40}$$

Replacing $\phi_n^{(i)}$, $\phi_n^{(j)}$ λ_i and λ_j with those obtained by Eqs. (24)–(25), and then approximating y_n as L and simplifying, one can write

$$\hat{\psi}_j^{(i)} = -\frac{\delta m_n^2 b^2}{k_n} \frac{\left[(2i-1)(2j-1)\pi c/2L \right]^2}{(2i-1)^2 - (2j-1)^2}, \quad i \neq j, \ i, \ j = 1, 2, \ldots, n. \tag{41}$$

Observation 3: From Eq. (39), it can be observed that as i increases starting from $i = 1$, $|e_i|$ also increases. This implies that due to yielding of any member in the top story, absolute percentage change in eigenvalues of a lower mode will be less than that of a higher mode.

Observation 4: For a given i, $(i \geqslant 1)$, as $|i - j|$ increases, $|[(2j-1)(2i-1)]^2/(2i-1)^2 - (2j-1)^2|$ reduces and thus $|\hat{\psi}_j^{(i)}|$ reduces. This implies that the coupling between the initial linear and post-yield modal matrices reduces with the separation of modes, i.e., any two well-separated modes (one from the linear and the other from the post-yield range of behavior) still remain approximately orthonormal with respect to the mass matrix.

3.3. Case 3: yielding occurs in a member of the pth story

Let the reduction in stiffness due to yielding of any member of the pth story from the base of the building be δk_p, where $0 < \delta < 1$. Therefore, both the $(p-1, p-1)$th and (p, p)th terms of $[\Delta k]$ become $-\delta k_p$, and both the $(p-1, p)$th and $(p, p-1)$th terms of $[\Delta k]$ become δk_p. Thus, one can express $A_1(i,j) = -(\phi_i^{(p-1)} - \phi_i^{(p)})(\phi_j^{(p-1)} - \phi_j^{(p)})\delta k_p$. Therefore, $\hat{\lambda}_i$ becomes

$$\hat{\lambda}_i = \lambda_i \left(1 - (\phi_i^{(p-1)} - \phi_i^{(p)})^2 \frac{\delta k_p}{\lambda_i} \right), \quad i = 1, 2, \ldots, n. \tag{42}$$

Replacing $(\hat{\lambda}_i - \lambda_i)/\lambda_i$ with e_i, Eq. (42) can be expressed as

$$e_i = -(\phi_i^{(p-1)} - \phi_i^{(p)})^2 \frac{\delta k_p}{\lambda_i}, \quad i = 1, 2, \ldots, n. \tag{43}$$

Substituting $\phi_i^{(p)}$, $\phi_i^{(p-1)}$ and λ_i with $b\sin((2i-1)\pi y_p/2L)$, $b\sin((2i-1)(\pi y_{p-1})/2L)$ and $(2i-1)\pi c/2L$, respectively, and then by simplifying for $i = 1, 2, \ldots, n$, one can write

$$e_i = -4b^2\left(\sin\left[(2i-1)(y_{p-1}-y_p)\frac{\pi}{4L}\right]\cos\left[(2i-1)(y_{p-1}+y_p)\frac{\pi}{4L}\right]\right)^2 \times \frac{\delta k_p}{[(2i-1)\pi c/2L]^2}. \tag{44}$$

For $y_{p-1} - y_p = -\varepsilon$, with ε being very small for large value of n, Eq. (44) can be approximated as

$$e_i = -b^2\delta k_p\left(\frac{\varepsilon}{c}\right)^2\cos^2\left[(2i-1)\frac{\pi y_p}{2L}\right], \quad i = 1, 2, \ldots, n. \tag{45}$$

$|e_i|$ will be maximum when $\cos^2((2i-1)\pi y_p/2L)$ will be maximum, i.e., when $(2i-1) = 2L/y_p$ or mode shape, $b\sin((2i-1)\pi y_p/2L) = 0$. In the same way, $|e_i|$ will be minimum when $b\sin((2i-1)\pi y_p/2L) = 0$.

Now,

$$\hat{\psi}_j^{(i)} = -\frac{(\phi_i^{(p-1)} - \phi_i^{(p)})(\phi_j^{(p-1)} - \phi_j^{(p)})\delta k_p}{\lambda_i - \lambda_j}, \quad i \neq j, \ i, \ j = 1, 2, \ldots, n. \tag{46}$$

Substituting mode shapes and eigenvectors of Eq. (46) and simplifying, for $i, \ j = 1, 2, \ldots, n$ and $i \neq j$, one can write

$$\hat{\psi}_j^{(i)} = \frac{(2i-1)(2j-1)\cos((2i-1)\pi y_p/L)\cos((2j-1)\pi y_p/L)}{(2j-1)^2 - (2i-1)^2}\left(\frac{2b\varepsilon}{c}\right)^2\delta k_p. \tag{47}$$

Observation 5: From Eq. (45) and subsequent discussion, one can notice that for the ith mode shape, if y_p is such that the value of this mode shape at the pth dof is close to zero then, $|e_i|$ will be maximum for the ith mode. For any other mode, say jth mode, $|e_j|$ will be less compared to $|e_i|$. The value of $|e_j|$ reduces as the value of mode shape at the pth dof increases compared to that of the ith mode.

Observation 6: From Eq. (47), it can be observed that for a given i, as $|i-j|$ increases, $|\hat{\psi}_j^{(i)}|$ reduces.

It may be mentioned here that although the results of all three cases discussed here are based on a slight change in stiffness due to the nonlinear behavior of the assumed building, these results give a good insight into the problem under consideration. However, for a large change in stiffness, it may be necessary to consider a higher order expansion and better approximations to obtain better results.

4. Numerical illustration

In order to investigate the trend in change of eigenproperties obtained from the perturbation formulation, which is derived based on the approximation of building behavior as that of a shear building with large number of dofs and a small change in stiffness, a 4 dof shear building is considered (see Fig. 1 and consider $n = 4$). For this system, the inter-story stiffness properties and mass distribution along the height of the building are considered to be uniform, i.e., $k_1 = k_2 = k_3 = k_4 = 2.04 \times 10^8$ N/m and $m_1 = m_2 = m_3 = m_4 = 1.0 \times 10^5$ kg. For this building, two different cases of stiffness reduction due to yielding (Cases 1 and 2), are considered. A 25% reduction in the instantaneous stiffness of the bottom story is assumed for Case 1 and the same percentage reduction of instantaneous stiffness in the top story is considered for Case 2.

Table 1 shows the eigenvalues of the four-story shear building in initial linear and the post-yield ranges, along with the percentage change in eigenvalues for all three cases, e_i (%), $i = 1, 2, \ldots, n$. These values are obtained by solving the associated eigenvalue problems and not using the expressions obtained from the perturbation formulation. Note that the value of e_i is negative for all values of i implying a reduction in eigenvalues due to the yielding considered in Case 1 and Case 2. It can be observed from this table that starting from the fundamental mode, as the mode number increases, the absolute value of percentage change in eigenvalues reduces for Case 1 and increases for Case 2. However, for Case 2, this absolute percentage change in eigenvalues increases with the increase in mode number for the first three modes but not for the fourth mode. This is due to the fact that in the analytical derivation, the frequencies and mode shapes of a lumped-mass shear building is approximated as that of an equivalent uniform shear beam. It has been found that this

Table 1
Eigenvalues of the four-story shear building in initial linear range and different post-yield ranges

Mode no.	Linear	Case 1: 25% stiffness reduction of bottom story		Case 2: 25% stiffness reduction of top story	
i	λ_i (Hz^2)	$\hat{\lambda}_i$ (Hz^2)	e_i (%)	$\hat{\lambda}_i$ (Hz^2)	e_i (%)
1	6.23	5.44	−12.75	6.12	−1.75
2	51.67	47.01	−9.02	46.06	−10.87
3	121.29	115.93	−4.42	107.76	−11.16
4	182.52	180.42	−1.15	175.94	−3.60

Table 2
Element of matrix $[\hat{\psi}] = [\phi]^{\mathrm{T}}[m][\hat{\phi}]$ of the four-story shear building for different yielding scenarios

$i\backslash j$	Case 1: 25% stiffness reduction of bottom story				Case 2: 25% stiffness reduction of top story			
	1	2	3	4	1	2	3	4
1	0.9988	0.0450	−0.0169	0.0064	0.9998	−0.0191	−0.0103	−0.0038
2	−0.0435	0.9961	0.0731	−0.0219	0.0174	0.9899	−0.1358	−0.0375
3	0.0197	−0.0711	0.9958	0.0542	0.0120	0.1287	0.9804	−0.1485
4	−0.0084	0.0254	−0.0524	0.9983	0.0063	0.0568	0.1421	0.9882

approximation of uniform beam for lumped-mass model leads to significant error in the eigenproperties of higher modes. This is in fact evident by computing mode shapes of this building, where one can observe that the value corresponding to the dof 4 of the third and fourth mode shapes are significantly lower than the same values for the first and second mode which are approximately equal. Note that for a uniform shear beam these values should be same as $\phi_n^{(i)} = b$ for $y_n = L$ and $i = 1, 2, \ldots, n$ from Eq. (25). Nonetheless, keeping in mind that only the first few modes are significant in the seismic response of a building, one can conclude that the trend predicted through the analytical formulation of Case 2 compares well with the numerical results.

Table 2 shows the coupling matrix $[\hat{\psi}] = [\phi]^{\mathrm{T}}[m][\hat{\phi}]$. This coefficient matrix actually shows the strength of the modal coupling terms between the mode shapes of the linear and post-yield ranges. It may be observed that the leading diagonal terms are close to unity. For any column, the value of any term reduces as it goes further away from the leading diagonal term in that column. This means that the coupling of a linear range mode with any post-yield mode is less as the spacing between the linear and the post-yield modes becomes wider. From Tables 1 and 2, one can conclude that the trend obtained by perturbation approach are well applicable even for this four-story shear building.

Now, in order to see how this trend works for short to medium height steel moment-resisting frame buildings, for which the dynamic behavior can be considered to be close to that of an equivalent shear building, two steel moment-resisting frame (SMRF) buildings with four and eight stories are considered. These buildings were previously considered by Santa-Ana and Miranda [23] and were designed using the lateral load distribution specified in the Uniform Building Code, Structural Engineering Design Provisions [25]. The fundamental periods of vibration for these two buildings are representative of those obtained from earthquake records of instrumented existing SMRFs. The buildings have a uniform mass distribution over their height and a non-uniform lateral stiffness distribution. Fig. 2 shows the representative exterior frames of each of the two buildings. The buildings were designed according to strong-column weak-beam criteria of capacity design, i.e., excluding the beam-to-column connections in the top floor, the sum of the plastic moments of the columns framing into each beam–column joint is higher than the sum of plastic moments of the beams framing into the same joint. Lumped mass numerical models of these frames are developed using OpenSees [24], a open source finite element software framework to simulate the performance of structural and geotechnical systems subjected to earthquakes. It is assumed that the buildings are fixed at their base and transverse vibration is considered.

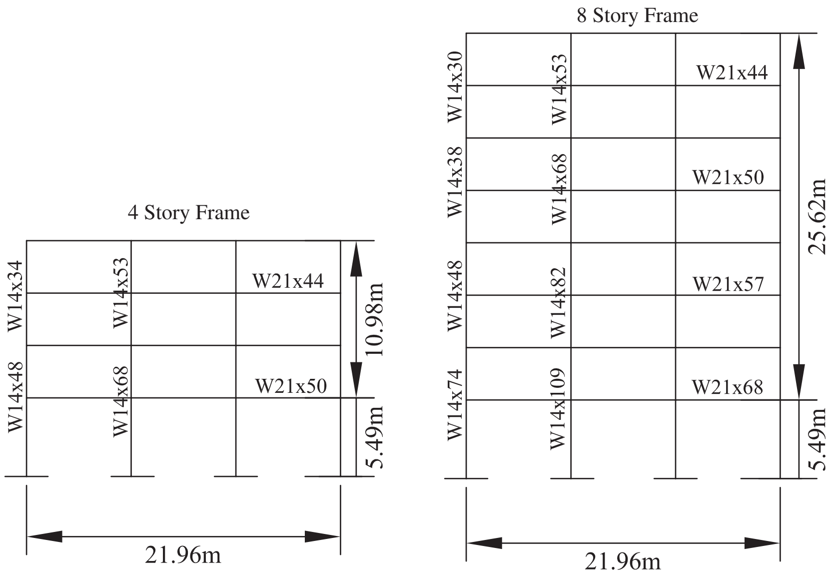

Fig. 2. Two-dimensional frames of the four- and eight-story steel moment-resisting frame buildings considered in this study.

Table 3
Change in eigenvalues of the four-story frame for different yielding scenarios

Mode no.	Linear	Case 1: plastic hinges at the base of middle two columns		Case 2: plastic hinges at middle beam of roof		Case 3: plastic hinges at middle beam of fourth floor	
i	λ_i (Hz^2)	$\hat{\lambda}_i$ (Hz^2)	e_i (%)	$\hat{\lambda}_i$ (Hz^2)	e_i (%)	$\hat{\lambda}_i$ (Hz^2)	e_i (%)
1	0.63	0.42	−32.88	0.63	−0.61	0.62	−2.27
2	6.34	5.44	−14.18	6.08	−4.13	5.77	−9.00
3	22.93	21.77	−5.08	21.73	−5.23	22.01	−4.04
4	48.38	47.36	−2.11	47.74	−1.32	47.74	−1.33

For the four-story frame, three cases of yielding scenarios are considered. They are (i) Case 1: formation of plastic hinges at the base of middle two columns (bottom of interior columns of the first story), (ii) Case 2: formation of plastic hinge at two ends of the middle beam of roof, and (iii) Case 3: formation of plastic hinge at two ends of the middle beam of the fourth floor (dof 3). Table 3 shows the percentage change in eigenvalues in all three cases. It can be observed from this table that as the mode number increases, $|e_i|$ reduces for Case 1 and increases for Case 2. However, similar to the Case 2 of the four-story shear building, this trend does not follow for the fourth mode due to the same reason of mode shapes approximation as discussed for the four-story shear building. For Case 3, $|e_i|$ becomes maximum for $i = 2$ and reduces for any mode with the increase in separation of that mode with respect to the second mode.

The trend found in Case 3 can be explained by observing the mode shapes of this building as given in Fig. 3, in conjunction with *Observation* 5. It can be observed from Fig. 3 that the second mode forms a node near the third dof (fourth floor) and thus $|e_i|$ is maximum for the second mode and reduces for any mode on either side of this mode.

Fig. 4(a) and (b) shows eight mode shapes of the eight-story frame. Note in this figure that the second mode and the fifth mode form node near the seventh floor (dof 6). Similar to the four-story frame, for the eight-story frame three cases of yielding scenarios are considered. They are (i) Case 1: formation of plastic hinge at the base of middle two columns, (ii) Case 2: formation of plastic hinge at two ends of the middle beam of roof, and (iii) Case 3: formation of plastic hinge at two ends of the middle beam of the seventh floor.

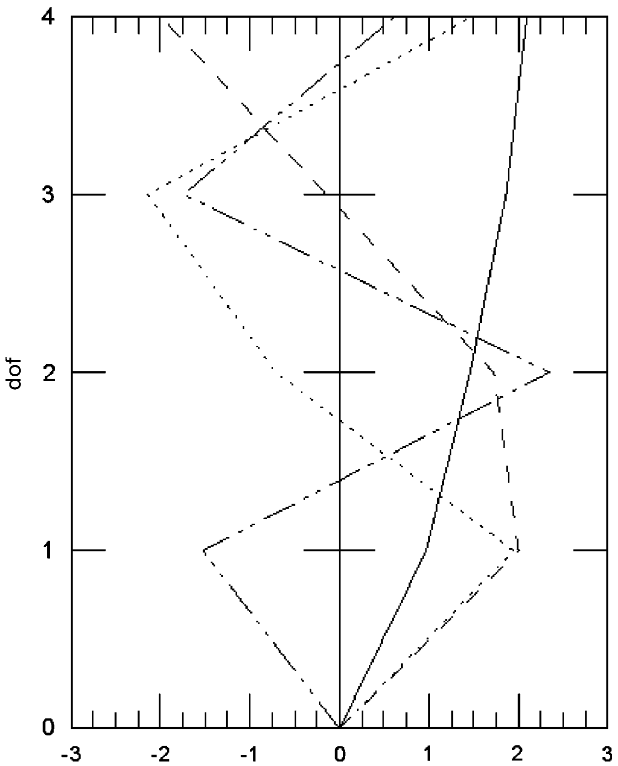

Fig. 3. Orthonormal mode shapes of the four-story steel moment-resisting frame. ——— 1st mode, – – – – 2nd mode, · · · · · · 3rd mode, — · · — 4th mode.

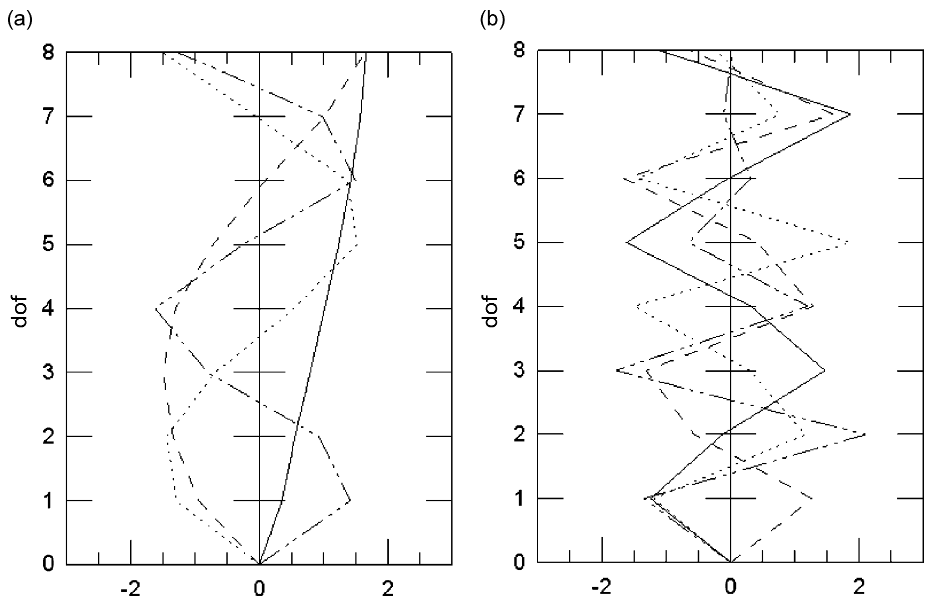

Fig. 4. Orthonormal mode shapes of the eight-story steel moment-resisting frame, (a) first mode to fourth mode: ——— 1st mode, – – – – 2nd mode, · · · · · · 3rd mode, — · · — 4th mode, (b) fifth mode to eighth mode: ——— 5th mode, – – – – 6th mode, · · · · · · 7th mode, — · · — 8th mode.

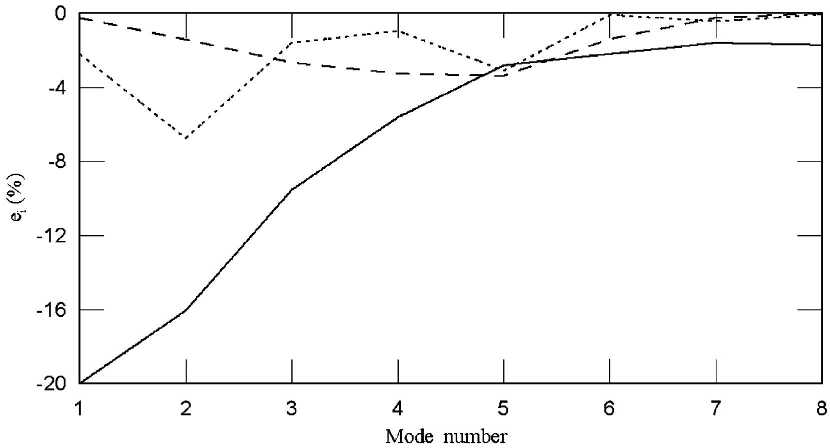

Fig. 5. Percentage change in eigenvalues of the eight-story steel moment-resisting frame for different yielding scenarios. ——— Case 1: plastic hinges at the base of middle two columns, – – – – Case 2: plastic hinges at two ends of middle beam of roof, $\cdots\cdots$ Case 3: plastic hinges at two ends of middle beam of 7th floor.

Fig. 5 shows the percentage change in eigenvalues with mode number in all three cases. It can be observed from this figure that as the mode number increases, $|e_i|$ reduces for Case 1 and increases for Case 2 until the fifth mode after which $|e_i|$ starts to decrease. For Case 3, $|e_i|$ reaches local maxima for the second mode and the fifth mode. Also, note that for Case 3, $|e_i|$ is greater for the second mode than that of the fifth mode.

5. Conclusions

In this paper, a perturbation approach is adopted to obtain an insight into the pattern of change in the instantaneous eigenproperties of a building as a result of reduction in stiffness due to yielding of its different members. For this purpose, analytical approximate relationships between the initial linear range (pre-yield) and post-yield eigenproperties have been established. A numerical study with a four-story shear building and two steel moment-resisting frame buildings (one four-story and the other eight-story) shows that the trends of change in eigenproperties obtained from the analytical study are in good agreement with the numerical study for all the cases of yielding scenarios considered herein. The summary of the trends as obtained for a building are as follows:

(1) A change in stiffness of the bottom story due to yielding will cause a greater percentage change of the eigenvalues in its lower modes than that of its higher modes.
(2) In contrast to above conclusion, for a change in stiffness of the top story, the percentage change in eigenvalues increases with the increase in mode number starting from the fundamental mode. This trend is found to be valid until the first few modes.
(3) In addition, if yielding occurs in such a location of a building that for a mode its mode shape value is close to zero at that location, the mode will have maximum percentage change in its eigenvalue. On the other hand, if yielding occurs in such a location that for a mode, its mode shape value is close to unity at that location, the mode will have minimum percentage change in its eigenvalue. If the above-mentioned condition of mode shape value close to zero is satisfied for multiple modes, the percentage change in eigenvalues will have the local maxima for those modes, with the global maximum occurring in the lowest mode satisfying this condition.
(4) Also, in all above cases, the coupling between a initial linear mode and a post-yield instantaneous mode with respect to the mass matrix reduces with an increase in the spacing between the two modes. This implies that any mode of linear range will still be approximately orthogonal with a post-yield instantaneous mode where the post-yield mode is well separated from the linear mode.

The outcome of this study is based on the first-order perturbation approach and the logical assumptions for shear buildings. Since the vibration characteristics of a short to medium height frame building is close to that of a shear building, the trends of change in eigenproperties predicted through the analytical study are found to be applicable for short to medium height frame buildings too. For very tall buildings, these results may not be applicable as the dynamic behavior of a tall building is more close to bending type than that of shear type and thus predominantly different from that of the shear buildings as considered here.

Acknowledgments

The work reported herein is a part of the Ph.D. dissertation of the author carried out under the guidance of Professor Roberto Villaverde at the University of California, Irvine. Careful review and support of Professor Tara Hutchinson, currently at the University of California, San Diego is greatly appreciated. The author would also like to thank Dr. Debraj Ghosh, currently at Stanford University for his comments regarding this work.

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