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Static and dynamic stability of an elastically restrained Beck column with an attached end mass

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Abstract

The static and dynamic stability of an elastically restrained Beck column with an attached end mass subjected simultaneously to gravity and follower axial forces is presented. The analytical solution takes into account the simultaneous effects of: (1) translational and rotational elastic restraints at the base support; (2) the uniform mass per unit length of the column (including any additional uniformly distributed mass) and rotary inertia of the column; (3) the rotary and translational inertias of the attached end mass; (4) the distance from the centroid of the attached end mass to the column free end; and (5) the shear deformation. The analytical eigenvalue solution can be used to study these effects on the static and dynamic stability of the Beck column. The analytical method and eigenvalue equation capture the static buckling (or divergence) as well as the dynamic (flutter) instability of cantilever columns elastically restrained at the base and subjected to any combinations of gravity and follower compressive axial forces applied at the free end. A parametric study is carried out on the effects of the translational and rotational inertias of the end mass, the distance from the centroid of the attached end mass to the column free end, and the rotational restraint on the static and dynamic stability of a perfectly clamped Beck column. Four comprehensive examples are presented to show the simplicity and effectiveness of the analytical method, and the obtained results compared with those obtained analytically and experimentally by others researchers.
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1. Introduction

The static and dynamic stability of beams and columns subjected to gravity and follower forces are of great importance in engineering, particularly in aerospace rocket structures. The stability of elastic systems subjected to nonconservative forces is studied by Bolotin [1]. Pedersen [2] analyzed a cantilever follower force problem with a concentrated end mass, a linear elastic spring, and a partial follower force at the free end, but did not take into account the distance from the column's free end to the end mass and the rotary inertia of the end mass. Kounadis [3] studied the effects of translational and rotary inertias on the dynamic stability behavior of a cantilever Timoshenko column with concentrated masses attached along its span and elastically restrained at the support and subjected to a follower force at the free end. Ryu and Sugiyama [4] studied the

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Nomenclature

a distance from centroid of the attached end mass to the column free end
A area of the column cross section
A_s effective shear area
b^2 frequency parameter of the column
c damping coefficient
E Young's modulus of the material
F^2 axial load parameter
G shear modulus of the material
I principal moment of inertia about its plane of bending of the column
j rotary inertia of the mass attached at the top end of the column
K stiffness of the rotational restraint at the column base
L column span
m mass attached at the top end of the column
$\bar{m}$ uniform mass per unit length of the column (including any additional uniformly distributed mass)
M bending moment along the column
P_f non-conservative tangential follower force applied at the free end
P_o gravity axial load applied at the centroid of the attached end mass $= mg$
$P = P_f + P_o$ total axial force
r radius of gyration of the column cross section
R slenderness parameter
s^2 ratio of bending to shear stiffness
S stiffness of the lateral bracing provided at the column base
V shear force
y column's lateral deflection
Y shape function
η non-conservativeness parameter

effects of the size of the attached end mass and its rotary inertia on the dynamic stability behavior of a perfectly clamped cantilever Timoshenko column subjected to a follower force using the finite element method (FEM). Sugiyama et al. [5] investigated experimentally the dynamic behavior of a cantilever column with end and intermediate attached rigid masses and the experimental results were compared with those calculated using the FEM neglecting the effects of shear deformations. Sugiyama et al. [6] studied experimentally and analytically the dynamic behavior of a viscoelastic clamped column, taking into account its internal damping. Ryu et al. [7] also studied the dynamic behavior of a cantilever Timoshenko column with an end mass subjected to combined actions of a conservative force (the self weight of the column) and a non-conservative sub-tangential force (follower force), including the effects of the rotary and translational inertias, the size of the end mass, and the shear deformations along the column. Later, Sugiyama et al. [8] verified the above study experimentally. The exact determination of free vibration frequencies and critical load levels of a beam with elastic supports and concentrated masses subjected to both, conservative and nonconservative forces, is investigated by Glabisz [9]. Langthjem and Sugiyama [10] presented a complete survey on the dynamic stability of columns subjected to follower loads. Wang [11] studied the static and dynamic stability of an Euler–Bernoulli column elastically restrained at both ends subjected to a follower force.

The main objective of this paper is to present an analytical solution to the static (divergence) and dynamic (flutter) stability of Beck's column subjected to any combination of a follower force and gravity vertical load including the simultaneous effects of: (1) the restrained base support with rotational and translational restoring elastic springs; (2) rotary and translational inertias of the column mass; (3) the size of the end mass and its rotary and translational inertias; and (4) the shear deformation.

The proposed solution does not consider the effect of damping and assumes that the end mass is much larger than the mass of the column. These two assumptions allow the problem to be solved analytically. When damping is omitted, the obtained buckling load is higher than the buckling load calculated when damping is taken into account and vanished [8]. However, the solution presented in this paper can be used to predict results of flutter tests where a rocket motor with a short burnout time is used [4,5,8,12] and also to predict the upper bound of the critical load for Beck's column with lumped damping [13].

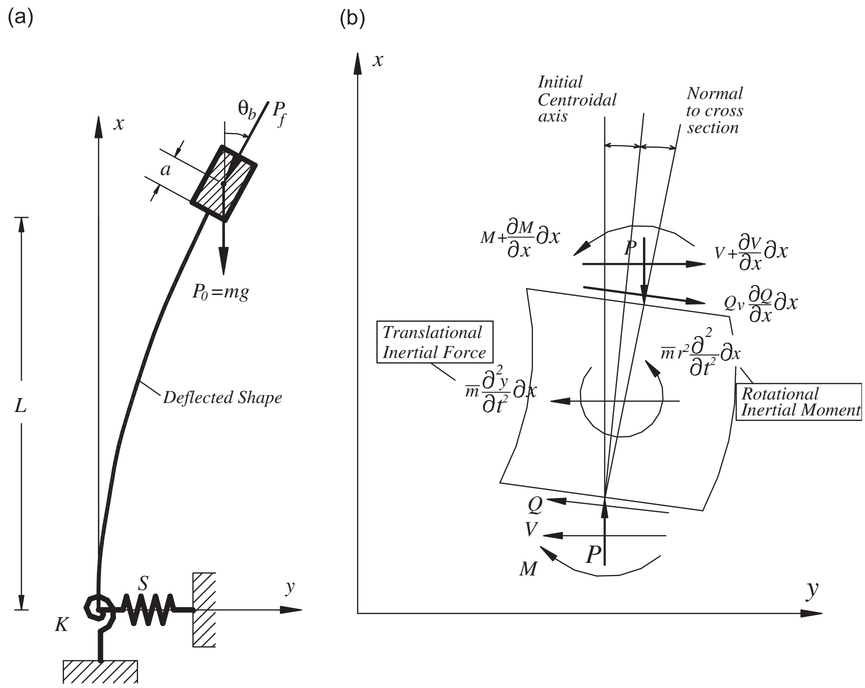

Fig. 1. Beck column with semi-rigid connections at the base: (a) structural model of; and (b) differential element.

2. Structural model

Consider a column, as shown in Fig. 1(a), elastically restrained at the base end and free at the other end. It is assumed that: (1) the stiffness of the rotational and translational springs at the base are K and S, respectively; (2) the column is made of a homogenous linear elastic material with modulus of elasticity E and shear modulus G; (3) the column's centroidal axis is a straight line; (4) the column is subjected to a combination of a gravity axial force $P_o = mg$ and a tangential follower force P_f, both applied at a distance a from the free end; (5) the column's transverse cross section is doubly symmetric (i.e., its centroid and shear center coincide with each other) with a total cross-sectional area A, shear area A_s, principal moment of inertia I about its plane of bending, and uniform mass per unit length $\bar{m}$ with a radius of gyration r (that includes any additional uniformly distributed mass along the member besides its own); (6) a mass of magnitude m with rotary inertia j is attached at the column free end (with its mass center located at a distance a from the column free end); and (7) all transverse deflections, rotations, and strains along the column are relatively small when compared to its dimensions.

3. Formulation of the problem

The transverse and bending equilibrium equations of the column's differential element as shown in Fig. 1(b), assuming that the column's weight is not as important as the end mass weight, are:

$$\frac{\partial V}{\partial x} = \bar{m}\frac{\partial^2 y}{\partial x^2} + c\frac{\partial y}{\partial x}, \tag{1}$$

$$\frac{\partial M}{\partial x} = V + (P_f + P_o)\frac{\partial y}{\partial x} - \bar{m}r^2\frac{\partial^2 \theta}{\partial t^2}. \tag{2}$$

Knowing that

$$V = GA_s\gamma - P\theta, \quad EI\frac{\partial\theta}{\partial x} = -M, \quad \frac{\partial y}{\partial x} = \theta + \gamma, \quad P = P_f + P_o, \tag{3}$$

and deriving Eq. (2) with respect to x, for $(\partial y/\partial x)^2 \ll 1$:

$$(P + GA_s)\frac{\partial\theta}{\partial x} - GA_s\frac{\partial^2 y}{\partial x^2} + \bar{m}\frac{\partial^2 y}{\partial t^2} + c\frac{\partial y}{\partial x} = 0, \tag{4}$$

$$EI\frac{\partial^3\theta}{\partial x^3} = (GA_s + P)\frac{\partial\theta}{\partial x} - (GA_s + P)\frac{\partial^2 y}{\partial x^2} + \bar{m}r^2\frac{\partial^3\theta}{\partial x\,\partial t^2}. \tag{5}$$

Solving the system formed by Eqs. (4) and (5):

$$\frac{EI}{P + GA_s}\left(GA_s\frac{\partial^4 y}{\partial x^4} - \bar{m}\frac{\partial^4 y}{\partial x^2\partial t^2} - c\frac{\partial^3 y}{\partial x^2\partial t}\right) = \left(GA_s\frac{\partial^2 y}{\partial x^2} - \bar{m}\frac{\partial^2 y}{\partial t^2} - c\frac{\partial y}{\partial t}\right) - (GA_s + P)\frac{\partial^2 y}{\partial x^2} + \frac{\bar{m}r^2}{P + A_sG}\left(GA_s\frac{\partial^4 y}{\partial x^2\partial t^2} - \bar{m}\frac{\partial^4 y}{\partial t^4} - c\frac{\partial^3 y}{\partial t^3}\right). \tag{6}$$

Note that Eq. (6) cannot be solved analytically because the three damping terms do not allow any decoupling (or separation of variables) as shown next. Currently, it is not feasible to obtain an analytical solution to Eq. (6) with all parameters presented herein in addition to viscous damping. However, by ignoring the effects of damping (i.e., $c = 0$), Eq. (6) is reduced to

$$\frac{EI}{P + GA_s}\left(GA_s\frac{\partial^4 y}{\partial x^4} - \bar{m}\frac{\partial^4 y}{\partial x^2\partial t^2}\right) + GA_s\frac{\partial^2 y}{\partial x^2} - \bar{m}\frac{\partial^2 y}{\partial t^2} + (GA_s + P)\frac{\partial^2 y}{\partial x^2} - \frac{\bar{m}r^2}{P + A_sG}\left(GA_s\frac{\partial^4 y}{\partial x^2\partial t^2} - \bar{m}\frac{\partial^4 y}{\partial t^4}\right) = 0. \tag{7}$$

Assuming harmonic variations of the bending deflection and using separation of variables [i.e., $y(x,t) = Y(x)e^{i\omega t}$ with $Y(x)$ representing the shape function associated with the lateral deflection along the member] and substituting it into the governing differential Eq. (7) becomes

$$\frac{d^4\bar{y}}{d\bar{x}^4} + (b^2s^2 + F^2 + F^4s^2 + b^2R^2)\frac{d^2\bar{y}}{d\bar{x}^2} + (b^4R^2s^2 - b^2 - b^2F^2s^2)\bar{y} = 0, \tag{8}$$

where

$$b^2 = \frac{\bar{m}\omega^2L^4}{EI}, \quad F^2 = \frac{PL^2}{EI}, \quad R = \frac{r}{L}, \quad s^2 = \frac{EI}{GA_sL^2}, \quad \bar{x} = \frac{x}{L}, \text{ and } \quad \bar{y} = \frac{Y}{L} \tag{9}$$

are the frequency parameter, axial load parameter, slenderness parameter, ratio of bending to shear stiffness, dimensionless distance from the restrained end, and dimensionless shape function, respectively.

The solution to Eq. (8) is

$$Y(\bar{x}) = C_1 \sin \beta\bar{x} + C_2 \cos \beta\bar{x} + C_3 \sinh \alpha\bar{x} + C_4 \cosh \alpha\bar{x}, \tag{10}$$

where

$$\alpha = \sqrt{-\Omega + \sqrt{\Omega^2 - \varepsilon}}, \quad \beta = \sqrt{\Omega + \sqrt{\Omega^2 - \varepsilon}}, \quad \Omega = \frac{b^2s^2 + F^2 + F^4s^2 + b^2R^2}{2},$$
$$\varepsilon = b^4R^2s^2 - b^2 - b^2F^2s^2. \tag{11}$$

Once the solution for the shape function $Y(x)$ is obtained, $\theta(x,t)$ can be found integrating Eq. (4):

$$\Theta(\bar{x}) = \lambda C_1 \cos \beta\bar{x} - \lambda C_2 \sin \beta\bar{x} + \delta C_3 \cosh \alpha\bar{x} + \delta C_4 \sinh \alpha\bar{x}, \tag{12}$$

where

$$\lambda = \frac{\beta^2 - b^2s^2}{(1 + F^2s^2)\beta}, \quad \delta = \frac{\alpha^2 + b^2s^2}{(1 + F^2s^2)\alpha}, \quad EI\frac{\partial\theta}{\partial x} = -M, \quad V = GA_s\gamma - P\theta. \tag{13}$$

Expressions for Y, θ, M, and V are expressed in terms of constants C_1, C_2, C_3, and C_4, respectively, which can be determined using the following boundary conditions:

$$V(0,t) = Sy(0,t), \tag{14}$$

$$M(0,t) = -K\theta(0,t), \tag{15}$$

$$V(L,t) = -m\frac{\partial^2}{\partial t^2}\left(y + a\frac{\partial y}{\partial x}\right)\bigg|_{x=L} - P_f\theta(L,t), \tag{16}$$

$$M(L,t) = -aP_o\theta(L,t) + am\frac{\partial^2}{\partial t^2}\left(y + a\frac{\partial y}{\partial x}\right)\bigg|_{x=L} + j\frac{\partial^2\theta}{\partial t^2}\bigg|_{x=L}. \tag{17}$$

Substituting the corresponding terms into Eq. (14)–(17), the following matrix system is obtained:

$$\begin{bmatrix} A_{11} & A_{12} & A_{13} & A_{14} \\ A_{21} & A_{22} & A_{23} & A_{24} \\ A_{31} & A_{32} & A_{33} & A_{34} \\ A_{41} & A_{42} & A_{43} & A_{44} \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \tag{18}$$

where

$$\begin{aligned}
&A_{11} = \frac{b^2s^2}{\bar{S}\beta}, \quad A_{12} = -1, \quad A_{13} = -\frac{b^2s^2}{\bar{S}\alpha}, \quad A_{14} = -1, \quad A_{21} = \lambda, \quad A_{22} = \frac{\lambda\beta}{\bar{K}},\\
&A_{23} = \delta, \quad A_{24} = -\frac{\alpha\delta}{\bar{K}},\\
&A_{31} = \left(-\frac{b^2}{\beta} + \mu b^2\bar{a}\beta - \lambda F^2\eta\right)\cos\beta + \mu b^2 \sin\beta,\\
&A_{32} = \left(\frac{b^2}{\beta} - \mu b^2\bar{a}\beta + \lambda F^2\eta\right)\sin\beta + \mu b^2 \cos\beta,\\
&A_{33} = \left(\frac{b^2}{\alpha} + \mu b^2\bar{a}\alpha - \delta F^2\eta\right)\cosh\alpha + \mu b^2 \sinh\alpha,\\
&A_{34} = \left(\frac{b^2}{\alpha} + \mu b^2\bar{a}\alpha - \delta F^2\eta\right)\sinh\alpha + \mu b^2 \cosh\alpha,\\
&A_{41} = (\bar{a}F_o^2\lambda + \bar{a}^2\mu b^2\beta + \bar{j}b^2\lambda)\cos\beta + (\lambda\beta + \bar{a}\mu b^2)\sin\beta,\\
&A_{42} = -(\bar{a}F_o^2\lambda + \bar{a}^2\mu b^2\beta + \bar{j}b^2\lambda)\sin\beta + (\lambda\beta + \bar{a}\mu b^2)\cos\beta,\\
&A_{43} = (\bar{a}F_o^2\delta + \bar{a}^2\mu b^2\alpha + \bar{j}b^2\delta)\cosh\alpha - (\alpha\delta - \bar{a}\mu b^2)\sinh\alpha,\\
&A_{44} = (\bar{a}F_o^2\delta + \bar{a}^2\mu b^2\alpha + \bar{j}b^2\delta)\sinh\alpha - (\alpha\delta - \bar{a}\mu b^2)\cosh\alpha,
\end{aligned} \tag{19}$$

and

$$\bar{S} = \frac{SL}{GA_s}, \quad \bar{K} = \frac{KL}{EI}, \quad \mu = \frac{m}{\bar{m}L}, \quad \bar{j} = \frac{j}{\bar{m}L^3}, \quad \bar{a} = \frac{a}{L}, \quad F_o^2 = \frac{P_oL^2}{EI}, \quad F_f^2 = \frac{P_fL^2}{EI}, \quad \eta = \frac{F_f^2}{F^2}. \tag{20}$$

The algebraic characteristic equation of the 4×4 matrix in Eq. (18) is as follows:

$$\begin{aligned}
&A_c\cos\beta\cosh\alpha + B_c\sin\beta\sinh\alpha + (\alpha\delta + \beta\lambda)[C_c\cos\beta\sinh\alpha + D_c\sin\beta\cosh\alpha]\\
&\quad + E_sF^2\{F_c\cos\beta\cosh\alpha + G_c\sin\beta\sinh\alpha\\
&\quad + (\alpha\delta + \beta\lambda)[H_c\cos\beta\sinh\alpha + I_c\sin\beta\cosh\alpha] + J_c\} = 0,
\end{aligned} \tag{21}$$

where

$$A_c = -\alpha^2\delta^2 - \beta^2\lambda^2 + \frac{\mu b^2 s^2}{\bar{S}}(\alpha\delta+\beta\lambda)^2 + \mu\{\bar{a}\alpha^2\beta^2(\delta^2-\lambda^2) + b^2[2\bar{j}\alpha\beta\delta\lambda + \bar{a}(\alpha\delta-\beta\lambda)]\}$$
$$+\frac{1}{\bar{K}}\{b^2\bar{j}(\alpha\delta+\beta\lambda)^2 + \mu b^2[\bar{a}^2(\alpha^2+\beta^2)(\alpha\delta+\beta\lambda)] - \bar{a}\bar{j}\alpha\beta[\alpha\beta(\delta^2-\lambda^2)-\delta\lambda(\alpha^2-\beta^2)]\}$$
$$+\frac{b^2s^2}{\bar{K}\bar{S}}|2\alpha\beta\delta\lambda + \mu\{\bar{a}\alpha\beta\delta\lambda(\alpha^2-\beta^2) + b^2[\bar{a}(\alpha\delta-\beta\lambda) - \bar{j}(\alpha^2\delta^2+\beta^2\lambda^2)]\}|,$$

$$B_c = -\delta\lambda(\alpha^2-\beta^2) + \frac{\alpha\beta\mu}{\bar{K}}(\alpha\delta+\beta\lambda)^2 + \mu\left\{-\bar{j}\alpha\beta b^2(\delta^2-\lambda^2) + \bar{a}\left[2\alpha^2\beta^2\delta\lambda + b^2(\beta\delta+\alpha\lambda)\right]\right\}$$
$$+\frac{b^4s^2}{\bar{S}}\left|-\frac{\bar{j}}{\alpha\beta}(\alpha\delta+\beta\lambda)^2 + \mu\left\{-\bar{a}^2(\alpha^2+\beta^2)\left(\frac{\lambda}{\alpha}+\frac{\delta}{\beta}\right) + \bar{a}\bar{j}[-\delta\lambda(\alpha^2-\beta^2)+\alpha\beta(\delta^2-\lambda^2)]\right\}\right|$$
$$+\frac{b^2s^2}{\bar{K}\bar{S}}\left\{b^2\mu\left[-\bar{j}\delta\lambda(\alpha^2-\beta^2) - \frac{\bar{a}}{\alpha\beta}(\alpha^3\delta+\beta^3\lambda)\right] + \frac{1}{\alpha\beta}(\alpha^4\delta^2-\beta^4\delta^2) - \bar{a}\mu\alpha\beta(\alpha^2\delta^2+\beta^2\lambda^2)\right\},$$

$$C_c = -\alpha\beta\lambda\mu + b^2\left\{\bar{a}^2\delta\mu\frac{\alpha^2+\beta^2}{\alpha\delta+\beta\lambda} + \bar{j}\left[\delta - \bar{a}\mu\alpha\beta\delta\frac{\beta\delta-\alpha\lambda}{\alpha\delta+\beta\lambda}\right]\right\} + \frac{b^2s^2}{\bar{S}}\left[\mu b^2\left(\frac{\bar{a}}{\alpha}-\bar{j}\delta\right) + \frac{\beta\lambda}{\alpha} + \bar{a}\mu\alpha\beta\lambda\right]$$
$$+\frac{1}{\bar{K}}\left[-\alpha^2\delta + \bar{a}\mu\alpha^2\beta^2\delta + b^2\mu(\alpha+\bar{j}\alpha\beta\lambda)\right]$$
$$+\frac{b^2s^2}{\bar{K}\bar{S}}\left\{\alpha^2\delta\mu + b^2\left[-\frac{\bar{a}^2}{\alpha}\mu\beta\lambda\frac{\alpha^2+\beta^2}{\alpha\delta+\beta\lambda} + \bar{j}\left(-\frac{\beta\lambda}{\alpha} + \bar{a}\mu\beta^2\lambda\right)\right]\right\},$$

$$D_c = \mu\alpha\beta\delta + b^2\left[\bar{j}\lambda + \bar{a}^2\lambda\mu\frac{\alpha^2+\beta^2}{\alpha\delta+\beta\lambda} - \bar{a}\mu\bar{j}\alpha\beta\lambda\frac{\beta\delta-\alpha\lambda}{\alpha\delta+\beta\lambda}\right] + \frac{b^2s^2}{\bar{S}}\left[\frac{\alpha\delta}{\beta} - \bar{a}\mu\alpha\beta\delta - \beta^2\mu\left(\frac{\bar{a}}{\beta}+\bar{j}\lambda\right)\right]$$
$$+\frac{1}{\bar{K}}\left[\beta^2\lambda + \bar{a}\mu\alpha^2\beta^2\lambda + b^2\mu(\bar{a}\beta - \bar{j}\alpha\beta\delta)\right]$$
$$+\frac{b^2s^2}{\bar{K}\bar{S}}\left[-\beta^2\lambda\mu + b^2\left(-\bar{j}\frac{\alpha\delta}{\beta} - \bar{a}^2\mu\frac{\alpha\delta}{\beta}\frac{\alpha^2+\beta^2}{\alpha\delta+\beta\lambda} + \bar{a}\bar{j}\mu\alpha^2\delta\frac{\beta\delta-\alpha\lambda}{\alpha\delta+\beta\lambda}\right)\right],$$

$$E_c = \alpha\delta\left(1+\frac{b^2s^2}{\bar{K}\bar{S}}\right)[-\bar{a}b^2\mu + \bar{a}\beta^3\lambda\mu - 2\beta\lambda(1+\bar{j}b^2\mu)] - \bar{a}\beta\lambda\mu\left(\alpha^3\delta + \frac{b^4s^2}{\bar{K}\bar{S}}\right) + \bar{a}b^2\beta\lambda\mu\left(1-\frac{s^2\alpha^3\delta}{\bar{K}\bar{S}}\right),$$

$$F_c = -\eta\left[\alpha\beta\delta\lambda(\alpha\delta-\beta\lambda)\left(\frac{1}{b^2}+\frac{s^2}{\bar{K}\bar{S}}\right) + \frac{\bar{a}}{\bar{K}}(\alpha\delta+\beta\lambda)^2\right] + 2\bar{a}\alpha\beta\delta\lambda\mu - \frac{\bar{a}b^2s^2\mu}{\bar{K}\bar{S}}(\alpha^2\delta^2+\beta^2\lambda^2)$$
$$+\frac{\bar{a}}{\bar{K}}\{(\alpha\delta+\beta\lambda)^2 - \bar{a}\mu\alpha\beta[\alpha\beta(\delta^2-\lambda^2)-\delta\lambda(\alpha^2-\beta^2)]\},$$

$$G_c = \eta\left[\frac{\bar{a}b^2s^2}{\alpha\beta\bar{S}}(\alpha\delta+\beta\lambda)^2 - \frac{\alpha\delta\beta\lambda}{b^2}(\beta\delta+\alpha\lambda) + \frac{s^2\delta\lambda}{\bar{K}\bar{S}}(\alpha^3\delta+\beta^3\lambda)\right] - \frac{\bar{a}b^2s^2\mu\delta\lambda}{\bar{K}\bar{S}}(\alpha^2-\beta^2) - \bar{a}\mu\alpha\beta(\delta^2-\lambda^2)$$
$$+\frac{b^2s^2}{\bar{S}}\left\{-\frac{\bar{a}}{\alpha\beta}(\alpha\delta+\beta\lambda)^2 + \bar{a}^2\mu[\alpha\beta(\delta^2-\lambda^2)-\delta\lambda(\alpha^2-\beta^2)]\right\},$$

$$H_c = \bar{a}\delta + \eta\left(-\bar{a}\delta + \frac{\bar{a}b^2s^2}{\alpha\bar{K}\bar{S}}\beta\lambda - \frac{s^2\beta\delta\lambda}{\bar{S}} - \frac{\alpha^2\beta\delta\lambda}{b^2K}\right) - \frac{\bar{a}b^2s^2\delta\mu}{\bar{S}} - \bar{a}^2\alpha\beta\delta\mu\frac{\beta\delta-\alpha\lambda}{\alpha\delta+\beta\lambda}$$
$$+\frac{\bar{a}\alpha gb\lambda\mu}{\bar{K}} + \frac{b^2s^2}{\bar{K}\bar{S}}\left(-\frac{\bar{a}\beta\lambda}{\alpha} + \bar{a}^2\beta^2\lambda\mu\frac{\beta\delta-\alpha\lambda}{\alpha\delta+\beta\lambda}\right),$$

$$I_c = \bar{a}\lambda + \eta\left[\frac{\bar{a}b^2s^2\alpha\delta}{\beta\bar{K}\bar{S}} - \bar{a}\lambda + \frac{s^2\alpha\delta\lambda}{\bar{S}} - \frac{\alpha\beta^2\delta\lambda}{b^2\bar{K}}\right] - \frac{\bar{a}\mu\alpha\beta\delta}{\bar{K}} - \frac{\bar{a}b^2s^2\mu\lambda}{\bar{S}}$$
$$- \bar{a}^2\mu\alpha\beta\lambda\frac{\beta\delta-\alpha\lambda}{\alpha\delta+\beta\lambda} + \frac{\bar{a}}{\bar{K}\bar{S}}\left(-\frac{b^2s^2\alpha\delta}{\beta} + \bar{a}\mu\alpha^2\delta\frac{\beta\delta-\alpha\lambda}{\alpha\delta+\beta\lambda}\right),$$
$$J_c = \alpha\beta\delta\lambda\eta(\alpha\delta-\beta\lambda)\left(\frac{1}{b^2} + \frac{s^2}{\bar{S}\bar{K}}\right) - 2\bar{a}\alpha\beta\delta\lambda\mu\left(1 + \frac{b^2s^2}{\bar{K}\bar{S}}\right). \quad (22)$$

4. Parametric studies

A series of parametric studies was carried out to determine the effects of the attached mass m, its rotary inertia j, and the distance a on the static and dynamic stability of Beck's column with various values of rotational restrain (i.e., with $K = 1$, 3, 9 and ∞) and fixed to lateral displacement at the base (i.e., $S = \infty$).

The frequencies at which the column vibrates when the applied axial load is $P = 0$ are the natural frequencies. The variations of these natural frequencies with the parameter $\mu = m/\bar{m}L$ are shown in Fig. 2 for an Euler–Bernoulli column (i.e. $GA_s = \infty$) with $\bar{a} = 0.2$, $\bar{j} = 0.1$ and four different values of $\bar{K}$. Note that the second-mode natural frequency is not affected by the value of $m/\bar{m}L$ when both β and $j/\bar{m}L^3$ remain constant. However, the first-mode natural frequency is affected significantly. Furthermore, as expected, the higher the rotational restrain, the higher the frequency.

The static buckling load is the load at which Eq. (21) is satisfied when $\omega = 0$. Fig. 3 shows the effect that parameter β has on these static buckling loads for $\mu = 10$ and $\bar{j} = 0.1$. As expected, these results indicate that the static buckling load is reduced as the parameter β increases. It is interesting to note that the curve shown in Fig. 3(a) decrease faster than the curve in Fig. 3(b). This means that the influence of parameter β is greater on the second buckling load than on the first buckling load.

The stability map for $\bar{a} = 0.2$, $\mu = 10$ and $\bar{j} = 0.1$ is shown in Fig. 4 for four different values of $\bar{K}$. Note that (1) the value at which the instability type changes from divergence to flutter decreases as the stiffness of the rotational restrain increases; and (2) there is a common point for the values of the rotational spring analyzed, indicating that there is a value for the non-conservativeness parameter at which the instability force is the same

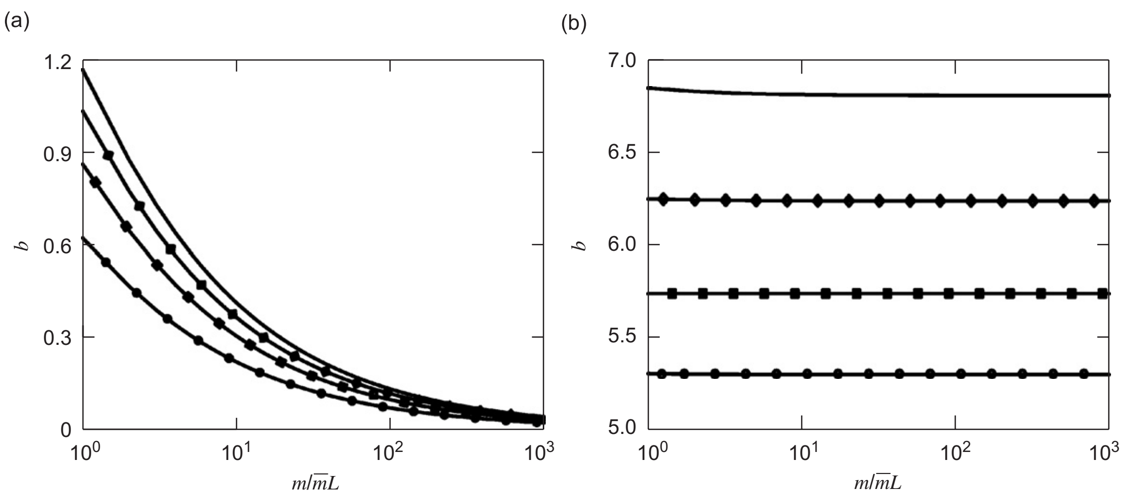

Fig. 2. Variation of the frequency parameter b with μ for $\bar{a} = 0.2$, $\bar{j} = 0.1$: (a) first natural frequency, (b) second natural frequency. —○—○— $\bar{K} = 1$, —□—□— $\bar{K} = 3$, —◇—◇— $\bar{K} = 9$ and ———— $\bar{K} = \infty$.

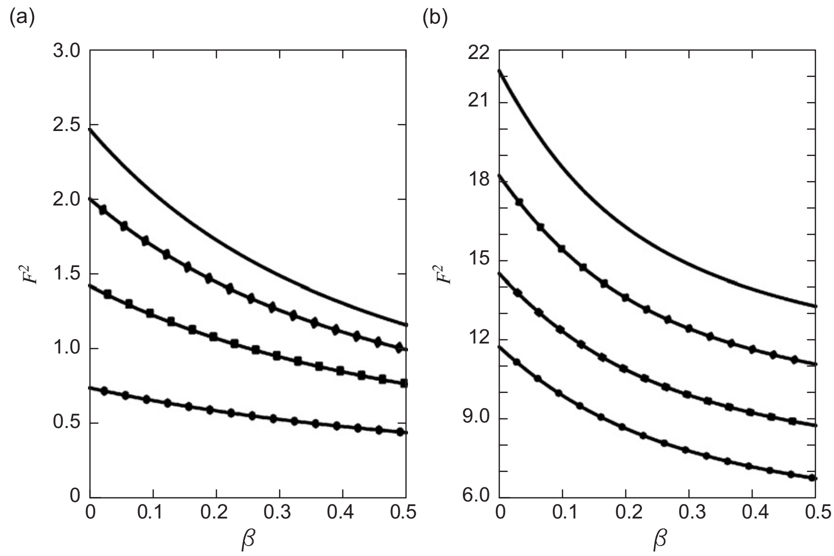

Fig. 3. Variation of the static buckling load with β for $\bar{\mu} = 10$ and $\bar{j} = 0.1$: (a) first buckling load, (b) second buckling load. –○–○– $\bar{K} = 1$, –□–□– $\bar{K} = 3$, –◇–◇– $\bar{K} = 9$ and ——— $\bar{K} = \infty$.

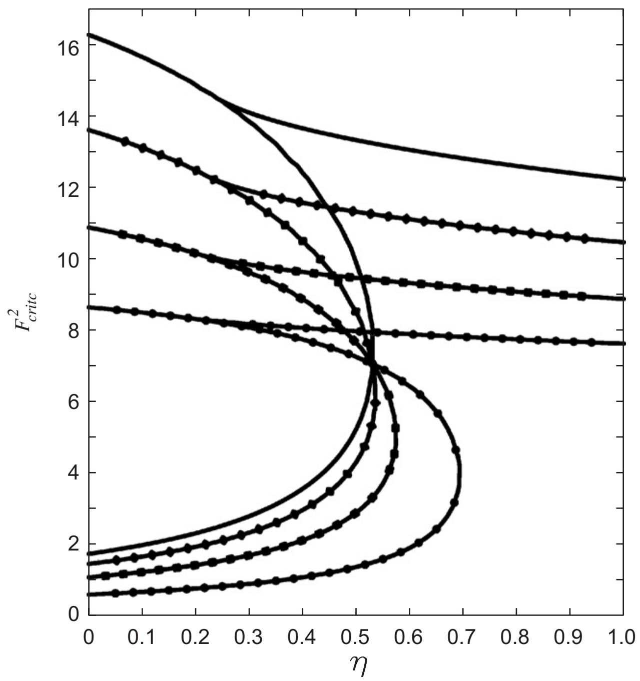

Fig. 4. Stability maps for $\bar{a} = 0.2$, $\mu = 10$ and $\bar{j} = 0.1$. –○–○– $\bar{K} = 1$, –□–□– $\bar{K} = 3$, –◇–◇– $\bar{K} = 9$ and ——— $\bar{K} = \infty$.

regardless of the value of the rotational restraint K. Another feature presented in Fig. 4 is that the first divergence load increases as the non-conservativeness parameter increases, while the second divergence and the flutter loads decrease. This indicates that the follower force has a stabilizing effect on the first static buckling load and a destabilizing effect on the second static buckling load and the flutter load.

5. Comprehensive examples

5.1. Example 1. Elastically restrained cantilever column without end mass

Determine the characteristic equation of the dynamic stability of an elastically retrained cantilever column without end mass subjected to the follower force P_f at its free end. Neglect the effects of shear deformations and those of the rotary inertia of the column. Compare the obtained characteristic equations for the cases with: (i) $K=\infty$; and (ii) $S=\infty$ with those reported by Kounadis [3].

Solution: For a cantilever column without end mass and subjected to the follower force P_f, the parameters given in Eq. (20) become $\mu=\bar{j}=\bar{a}=0$, $\eta=1$. Neglecting the column rotary inertia or $r=0$ and the shear deformation or $GA_s=\infty$:

(i) For $K=\infty$ or $\overline{K}=\infty$. Taking into account these values, the expressions given in Eq. (22) become

$$A_c=-(\alpha^4+\beta^4),\quad B_c=-\alpha\beta(\alpha^2-\beta^2),\quad C_c=\beta^4\alpha\frac{s^2}{\bar{S}},\quad D_c=\alpha^4\beta\frac{s^2}{\bar{S}},\quad E_c=-2\alpha^2\beta^2,$$
$$F_c=-(\alpha^2-\beta^2),\quad G_c=-2\alpha\beta,\quad H_c=-\alpha\beta^2\frac{s^2}{\bar{S}},\quad I_c=\beta\alpha^2\frac{s^2}{\bar{S}},\quad J_c=\alpha^2-\beta^2. \tag{23}$$

Then, the characteristic equation given in Eq. (21) becomes

$$\alpha^2\beta^2(\alpha^2+\beta^2)[\alpha\cos\beta\sinh\alpha+\beta\sin\beta\cosh\alpha]$$
$$-\frac{SL^3}{EI}[\alpha^4+\beta^4+2\alpha^2\beta^2\cos\beta\cosh\alpha+\alpha\lambda(\beta^2-\alpha^2)\sin\beta\sinh\alpha]=0. \tag{24}$$

Note that the expression given in Eq. (24) is identical to that reported by Kounadis [3].

(ii) For $S=\infty$ or $\bar{S}=\infty$. Taking into account these values, expressions given in Eq. (22) become

$$A_c=-(\alpha^4+\beta^4),\quad B_c=-\alpha\beta(\alpha^2-\beta^2),\quad C_c=-\frac{\alpha^3}{\bar{K}},\quad D_c=-\frac{\beta^3}{\bar{K}},\quad E_c=-2\alpha^2\beta^2,$$
$$F_c=-(\alpha^2-\beta^2),\quad G_c=-2\alpha\beta,\quad H_c=-\frac{\alpha}{\bar{K}},\quad I_c=-\frac{\beta}{\bar{K}},\quad J_c=\alpha^2-\beta^2. \tag{25}$$

Then, the characteristic equation given in Eq. (21) becomes

$$\alpha\beta(\alpha^2+\lambda^2)[\beta\cos\beta\sinh\alpha-\alpha\sin\beta\cosh\alpha]+\bar{K}[\alpha^4+\beta^4+2\alpha^2\beta^2\cos\beta\cosh\alpha$$
$$+\alpha\beta(\beta^2-\alpha^2)\sin\beta\sinh\alpha]=0. \tag{26}$$

Note that expression given in Eq. (26) is also identical to that reported by Kounadis [3].

5.2. Example 2. Clamped column with attached top mass

Determine the characteristic equation for the dynamic stability of a perfectly clamped cantilever column (with $S=K=\infty$) with a concentrated end mass (i.e., with $a=j=0$) subjected to the follower force P_f at its free end. Neglect the effects of the rotary inertia of the column and shear deformation. Compare the obtained results with those reported by with those reported by Ryu and Sugiyama [4].

Solution: The parameters in Eq. (20) are as follows: $\bar{a}=\bar{j}=0$; $\eta=1$. Neglecting the effects of the rotary inertia and shear deformations of the clamped column, the parameters given in Eq. (22) are reduced to

$$A_c=-(\alpha^4+\beta^4),\quad B_c=-\alpha\beta(\alpha^2-\beta^2),\quad C_c=-\alpha\beta^2\mu,\quad D_c=\alpha^2\beta\mu,\quad E_c=-2\alpha^2\beta^2,$$
$$F_c=-(\alpha^2-\beta^2),\quad G_c=-2\alpha\beta,\quad H_c=0,\quad I_c=0,\quad J_c=\alpha^2-\beta^2. \tag{27}$$

Substituting these parameters into Eq. (21):

$$-(\alpha^4+\beta^4)\cos\beta\cosh\alpha-\alpha\beta(\alpha^2-\beta^2)\sin\beta\sinh\alpha+\alpha\beta\mu(\alpha^2+\beta^2)[-\beta\cos\beta\sinh\alpha+\alpha\sin\beta\cosh\alpha]$$
$$-2\alpha^2\beta^2F^2\{-(\alpha^2-\beta^2)\cos\beta\cosh\alpha-2\alpha\beta\sin\beta\sinh\alpha+\alpha^2-\beta^2\}=0. \tag{28}$$

Using Eq. (28) it is possible to calculate the critical follower force P for different values of $\mu = m/\bar{m}L$. Those loads are presented in Table 1 and compared with those reported by Ryu and Sugiyama [4]. As it can be seen, the results obtained using Eq. (28) compare well with those reported by Ryu and Sugiyama [4] that used a finite element formulation modeled with 20 column segments.

5.3. Example 3. Perfectly clamped column with end mass

In this example, a perfectly clamped cantilever column subjected to a follower force tested experimentally by Sugiyama et al. [6] is analyzed. The column is a rectangular column with the following properties: $6 \times 30\,\text{mm}^2$ cross section; $\bar{m} = 0.481\,\text{kg/m}$; $EI = 3.43 \times 10^6\,\text{kgf/mm}^2$; $m = 14.18\,\text{kg}$; $j = 0.1196\,\text{kg m}$, $a = 0.2\,\text{m}$; and its length varied from $L = 1.0$ to $1.1\,\text{m}$. Neglect the effects of any external damping in the dynamic stability analysis.

Solution: The critical axial forces were calculated using Eqs. (21) and (22). Fig. 5 shows that the critical values obtained using these expressions compare well with those experimentally and theoretically reported by Sugiyama et al. [6].

Table 1
Example 2: critical follower force for different values of $m/\bar{m}L$

$m/\bar{m}L$	$P = P_f$ (N)		
	Reported by Ryu and Sugiyama [4]	Calculated using Eq. (28)	Difference (%)
0.0	673.7	673.7	0.00
0.1	590.4	591.2	0.14
0.3	547.0	547.3	0.06
0.5	537.3	539.7	0.45
1.0	544.0	544.7	0.14
2.0	560.4	561.7	0.22
5.0	587.0	590.1	0.53
100	653.2	653.7	0.07

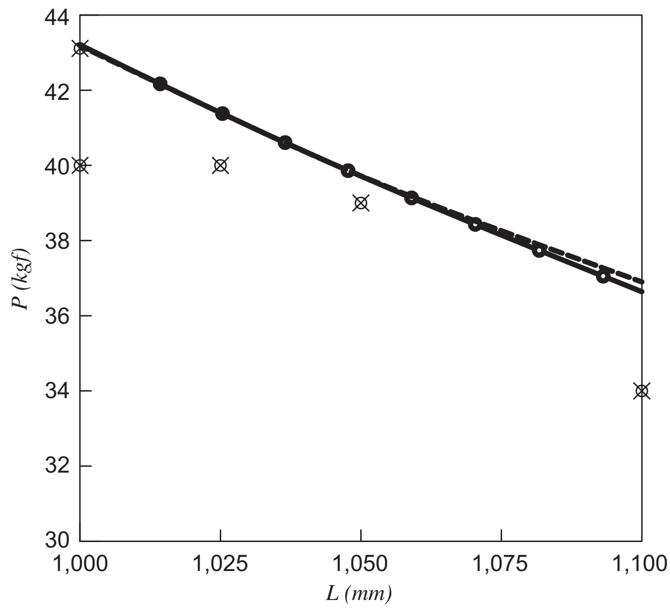

Fig. 5. Example 3: Critical force-versus-column length. —O———O— Eq. (21); — — — — Sugiyama; and ⊠ experimental.

Table 2
Example 4: critical follower force for different values of $\mu = m/\bar{m}L$ and s^2

μ	s^2														
	10^{-2}			10^{-3}			10^{-4}			10^{-5}			10^{-6}		
	Dimensionless critical follower force														
	Ref. [4]	Eq. (21)	Dif. (%)	Ref. [4]	Eq. (21)	Dif. (%)	Ref. [4]	Eq. (21)	Dif. (%)	Ref. [4]	Eq. (21)	Dif. (%)	Ref. [4]	Eq. (21)	Dif. (%)
0.0	15.30	15.88	3.63	19.45	19.46	0.07	20.00	19.99	0.05	20.05	20.05	0.02	20.05	20.05	0.00
0.1	13.92	14.28	2.52	17.07	17.15	0.44	17.47	17.55	0.44	17.57	17.59	0.12	17.57	17.60	0.14
0.3	13.13	13.47	2.52	15.89	15.91	0.15	16.19	16.25	0.37	16.28	16.29	0.03	16.28	16.29	0.06
0.5	13.03	13.38	2.61	15.69	15.71	0.11	15.99	16.03	0.22	15.99	16.06	0.42	15.99	16.06	0.44
1.0	13.32	13.60	2.07	15.79	15.87	0.50	16.09	16.18	0.54	16.19	16.21	0.12	16.19	16.21	0.14
2.0	13.72	14.08	2.54	16.28	16.37	0.56	16.68	16.68	0.01	16.68	16.71	0.20	16.68	16.72	0.22
5.0	14.41	14.82	2.73	17.17	17.02	0.88	17.47	17.53	0.31	17.47	17.56	0.51	17.47	17.56	0.52
100	15.99	16.38	2.36	18.95	18.86	0.50	19.34	19.41	0.37	19.44	19.45	0.05	19.44	19.55	0.54

5.4. Example 4. Effect of shear deformation

Determine the effects of shear deformations for various values of the parameter μ in a clamped column, neglecting the rotary inertia of the end mass and its distance a from the end of the column. Compare the result with those presented by Ryu et al. [4].

Solution: The critical axial forces were calculated using Eqs. (21) and (22). Table 2 shows that the critical values obtained have good agreement with those reported by Ryu et al. [4].

6. Summary and conclusions

The static and dynamic stability analysis of an elastically restrained Beck column with an attached end mass subjected to combined gravity and follower axial forces is presented in a classical manner. The analytical solution takes into account the simultaneous effects of: (1) translational and rotational elastic restraints at the base support; (2) the uniform mass per unit length of the column (including any additional uniformly distributed mass) and rotary inertia of the column; (3) the rotary and translational inertia of the attached end mass; (4) the distance from the centroid of the attached end mass to the column free end; and (5) shear deformations along the column span. The eigenvalue equation (21) can be used to study these effects on Beck's column. The analytical solution captures the static buckling (or divergence) as well as the dynamic (flutter) instability of cantilever columns elastically restrained at the base and subjected to any combinations of gravity and follower compressive axial forces applied at the free end. The analytical method presented and the corresponding equations can be programmed facilitating calculations and efficient computer coding and avoiding cumbersome procedures. A parametric study is carried out on the effects of the rotary and translational inertia of the end mass, the distance from centroid of the attached end mass to the column free end, and the rotary inertia of the column. The analytical method and corresponding equations yield results that agree very well with experimental and analytical studies carried out by other researchers.

Analytical results indicate that for an elastically restrained Beck's column with its base with sideways totally inhibited (i.e., with $S = \infty$): (1) the second-mode natural frequency is not affected significantly by the value of $m/\bar{m}L$ when both β and $j/\bar{m}L^3$ remain constant. However, the first-mode natural frequency is affected significantly; (2) the static buckling load is reduced as the parameter β increases; (3) a follower force P_f has a stabilizing effect on the static stability. However, it has a destabilizing effect on the dynamic stability of the column; and (4) the stability maps have a common point for different values of $\bar{K}$.

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