Available online at www.sciencedirect.com
ScienceDirect
Journal of Sound and Vibration 313 (2008) 268–289
JOURNAL OF SOUND AND VIBRATION
www.elsevier.com/locate/jsvi

Strength of modes in rotating machinery

Chong-Won Lee[a,*], Dong-Ju Han[b]

[a]Department of Mechanical Engineering, Center for Noise and Vibration Control (NOVIC), KAIST, Science Town, Daejeon 305-701, Republic of Korea
[b]Korea Institute of Aerospace Technology, Korean Air Limited, Yuseong-gu, Jeonmin-dong, 461-1, Daejeon 305-811, Republic of Korea

Received 15 February 2006; received in revised form 20 August 2007; accepted 11 November 2007
Available online 2 January 2008

Abstract

A new method for the analysis of modal strength of the rotor systems with periodically time-varying parameters due to the presence of either rotating or stationary asymmetry is proposed. The method is based on the complete development of the modal analysis by introducing the modulated coordinates to derive the equivalent infinite-order time-invariant Hill's matrix equation from the finite-order time-varying matrix equation. Depending upon the level of possible contribution of each mode to forced response, the modal strength of a mode is rigorously derived from the norm order analysis of the associated eigenvector; thereby, the modes are classified as strong or weak modes. It is shown that the directional frequency response functions are useful in identifying the strength of modes in detail, or equivalently, the modes of symmetry, anisotropy, asymmetry and coupled asymmetry. Two illustrative examples with a simple, yet general, analysis rotor model and a practical flexible asymmetric rotor finite element (FE) model with an open transverse crack are treated to demonstrate the theoretical findings and effectiveness of the proposed method.

1. Introduction

Use of isotropic rotor model is often found to be sufficient in representing the dynamic behavior of many practical rotors. For isotropic rotors with both stationary and rotating symmetry, the forward and backward modes play a significant role in the prediction of forced response of rotors, whereas their complex conjugate modes may be treated as insignificant modes, because they appear as a pure mathematical consequence of formulating the equations of motion in the real domain [1]. On the other hand, when the stationary (rotating) asymmetry is not negligibly small in rotors, it requires use of anisotropic (asymmetric) rotor model for the rotordynamic analysis. As the rotor anisotropy (asymmetry) increases, the complex conjugate (modulated) modes as well as the original forward and backward modes become increasingly important in the stability analysis and forced response predictions. For general rotors with both anisotropy and asymmetry, the critical speed chart becomes over-crowded with an infinite number of complex conjugate and modulated modes in addition to the original modes, leading to the difficulty in ordering the importance of modes from the chart [2].

*Corresponding author. Tel.: +82 42 869 3016.
E-mail address: cwlee@kaist.ac.kr (C.-W. Lee).

0022-460X/$ - see front matter
doi:10.1016/j.jsv.2007.11.038

The present study utilizes the complex modal analysis method, of which the complete solution was developed newly in Refs. [3,4], for asymmetric rotor systems with anisotropic stators. The method introduces modulated complex stationary coordinates to derive an equivalent, infinite-order time-invariant equation of motion. The characteristics of eigenvalues and eigenvectors are theoretically investigated thoroughly by using the equivalent time-invariant equation of motion. Based on the new modal analysis, the concept for investigating the modal behavior as the modal strength of the system will be developed here in a rigorously way.

In this paper, a new method of ordering the importance of modes is proposed, based on the norm of modal vectors defined in the complex coordinate system. The complex modal vectors associated with each mode are analytically derived from the results of the complex modal analysis of the periodically time-varying linear rotor system [4]. Then the modal vector norm is defined, from which the norm order is analytically derived for weakly anisotropic and asymmetric rotors. According to the modal vector norm order, the modal strength is defined and its effect on magnitude of the directional frequency response functions (dFRFs) is examined. The proposed method can be used to improve the whirl speed chart (Campbell diagram [5,6]) by indicating the modal strength based on the norm analysis of the system eigenvectors of interest. The method is particularly useful for design and operation of general rotor systems with either stationary or rotating asymmetry. It is shown that, for the general rotor systems, modes can be classified into strong and weak modes. The strong modes are the modes that are likely to contribute significantly to the response of rotor to all possible excitation sources, whereas, the weak modes are less significant in response contribution than strong modes. It is also shown that the dFRFs are very useful in identifying the modes of symmetry, anisotropy, asymmetry and coupled asymmetry, which are dependent upon the norm magnitude of the associated eigenvectors.

Finally, two illustrative examples are treated with a simple general analysis rotor model and a practical finite element (FE) model to demonstrate the theoretical findings and effectiveness of the proposed method.

2. Modal analysis of general rotors [3,4]

Rotors in general possess both stationary and rotating asymmetry, leading to a complicated periodically time-varying equation of motion with the period of π/Ω, Ω being the rotational speed of rotor [3,4]. The modal analysis of such systems has been recently developed [3,4]; therefore, the detailed procedure will not be treated here. Instead, the main feature of the modal analysis will be briefly described in this section.

The equation of motion for the periodically time-varying linear rotor system with the period of π/Ω, referring to Fig. 1, can be written as

$$\mathbf{M}_f\ddot{\mathbf{p}}(t) + \mathbf{C}_f\dot{\mathbf{p}}(t) + \mathbf{K}_f\mathbf{p}(t) + \{\mathbf{M}_b\ddot{\bar{\mathbf{p}}}(t) + \mathbf{C}_b\dot{\bar{\mathbf{p}}}(t) + \mathbf{K}_b\bar{\mathbf{p}}(t)\} + e^{j2\Omega t}\{\mathbf{M}_r\ddot{\bar{\mathbf{p}}}(t) + \mathbf{C}_r\dot{\bar{\mathbf{p}}}(t) + \mathbf{K}_r\bar{\mathbf{p}}(t)\} = \mathbf{g}(\mathrm{t}). \tag{1}$$

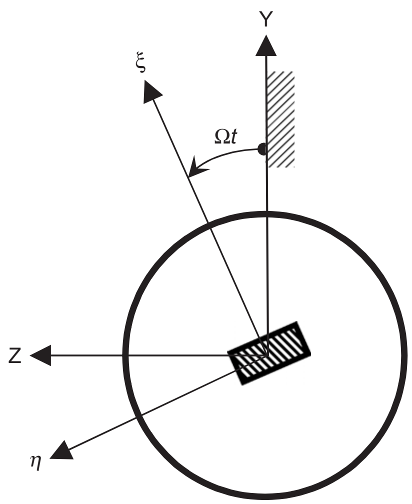

Fig. 1. Stationary and rotating coordinate systems for a rotor with stationary and rotating asymmetry.

Here, the $N \times 1$ complex response and force vectors, $\mathbf{p}(t)$ and $\mathbf{g}(t)$, defined by the real response vectors, $\mathbf{y}(t)$ and $\mathbf{z}(t)$, and the real input vectors, $\mathbf{f}_y(t)$ and $\mathbf{f}_z(t)$, respectively, are

$$\mathbf{p}(t) = \mathbf{y}(t) + \mathrm{j}\mathbf{z}(t), \quad \mathbf{g}(t) = \mathbf{f}_y(t) + \mathrm{j}\mathbf{f}_z(t), \tag{2}$$

where j means the imaginary number; $\mathbf{g}(t)$ includes the force and moment; '$-$' denotes the complex conjugate; $\mathbf{M}$, $\mathbf{C}$ and $\mathbf{K}$ are the $2N \times 2N$ complex matrices representing the generalized mass, damping and stiffness, respectively; and the subscripts f, b and r refer to the symmetric, anisotropic and asymmetric properties, respectively. Note that the system matrices, including the effect of the gyroscopic moment, internal damping and fluid-film-bearing characteristics, may be dependent upon the rotational speed. However, they become constant for given rotational speed. For an *isotropic rotor*; $\mathbf{C}_b = \mathbf{K}_b = \mathbf{M}_r = \mathbf{C}_r = \mathbf{K}_r = \mathbf{0}$; for an *anisotropic rotor*; $\mathbf{M}_r = \mathbf{C}_r = \mathbf{K}_r = \mathbf{0}$; and, for an *asymmetric rotor*; $\mathbf{C}_b = \mathbf{K}_b = \mathbf{0}$. Note here that the periodically time-varying terms, which are preceded by $\mathrm{e}^{\mathrm{j}2\Omega t}$ in Eq. (1), inherently appear, as both rotating and stationary asymmetries exist in the system. When neither the rotating nor the stationary asymmetry exists, the equation of motion becomes, or it can be transformed to, a time-invariant linear differential equation.

Introducing a set of modulated complex coordinate and force vectors, $\mathbf{p}_{;n}$ and $\mathbf{g}_{;n}$, where the modulation index, n, is an arbitrary integer, defined as [3,4]

$$\mathbf{p}_{;n}(t) \equiv \mathbf{p}(t)\mathrm{e}^{\mathrm{j}2n\Omega t}, \quad \mathbf{g}_{;n}(t) \equiv \mathbf{g}(t)\mathrm{e}^{\mathrm{j}2n\Omega t}, \tag{3}$$

we can easily transform Eq. (1) to an infinite order matrix equation given by

$$\underset{\sim}{\mathbf{M}}\,\underset{\sim}{\ddot{\mathbf{p}}}(t) + \underset{\sim}{\mathbf{C}}\,\underset{\sim}{\dot{\mathbf{p}}}(t) + \underset{\sim}{\mathbf{K}}\,\underset{\sim}{\mathbf{p}}(t) = \underset{\sim}{\mathbf{g}}(t), \tag{4}$$

where

$$\underset{\sim}{\mathbf{M}} = \begin{bmatrix} \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \iddots \\ \cdots & \bar{\mathbf{M}}_f & \bar{\mathbf{M}}_b & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots \\ \cdots & \mathbf{M}_b & \mathbf{M}_f & \mathbf{M}_r & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots \\ \cdots & \mathbf{0} & \bar{\mathbf{M}}_r & \bar{\mathbf{M}}_f & \bar{\mathbf{M}}_b & \mathbf{0} & \mathbf{0} & \cdots \\ \cdots & \mathbf{0} & \mathbf{0} & \mathbf{M}_b & \mathbf{M}_f & \mathbf{M}_r & \mathbf{0} & \cdots \\ \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \bar{\mathbf{M}}_r & \bar{\mathbf{M}}_f & \bar{\mathbf{M}}_b & \cdots \\ \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{M}_b & \mathbf{M}_f & \cdots \\ \iddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix},$$

$$\underset{\sim}{\mathbf{C}} = \begin{bmatrix} \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \iddots \\ \cdots & \bar{\mathbf{C}}_{f;1} & \bar{\mathbf{C}}_{b;1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots \\ \cdots & \mathbf{C}_{b;-1} & \mathbf{C}_{f;-1} & \mathbf{C}_{r;0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots \\ \cdots & \mathbf{0} & \bar{\mathbf{C}}_{r;1} & \bar{\mathbf{C}}_{f;0} & \bar{\mathbf{C}}_{b;0} & \mathbf{0} & \mathbf{0} & \cdots \\ \cdots & \mathbf{0} & \mathbf{0} & \mathbf{C}_{b;0} & \mathbf{C}_{f;0} & \mathbf{C}_{r;1} & \mathbf{0} & \cdots \\ \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \bar{\mathbf{C}}_{r;0} & \bar{\mathbf{C}}_{f;-1} & \bar{\mathbf{C}}_{b;-1} & \cdots \\ \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{C}_{b;1} & \mathbf{C}_{f;1} & \cdots \\ \iddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix},$$

$$\underset{\sim}{\mathbf{K}} = \begin{bmatrix} \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \iddots \\ \cdots & \bar{\mathbf{K}}_{f;1} & \bar{\mathbf{K}}_{b;1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots \\ \cdots & \mathbf{K}_{b;-1} & \mathbf{K}_{f;-1} & \mathbf{K}_{r;0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots \\ \cdots & \mathbf{0} & \bar{\mathbf{K}}_{r;1} & \bar{\mathbf{K}}_{f;0} & \bar{\mathbf{K}}_{b;0} & \mathbf{0} & \mathbf{0} & \cdots \\ \cdots & \mathbf{0} & \mathbf{0} & \mathbf{K}_{b;0} & \mathbf{K}_{f;0} & \mathbf{K}_{r;1} & \mathbf{0} & \cdots \\ \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \bar{\mathbf{K}}_{r;0} & \bar{\mathbf{K}}_{f;-1} & \bar{\mathbf{K}}_{b;-1} & \cdots \\ \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{K}_{b;1} & \mathbf{K}_{f;1} & \cdots \\ \iddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix},$$

$$\underset{\sim}{\mathbf{p}}(t) = \begin{Bmatrix} \vdots \\ \bar{\mathbf{p}}_{;1}(t) \\ \mathbf{p}_{;-1}(t) \\ \bar{\mathbf{p}}_{;0}(t) \\ \mathbf{p}_{;0}(t) \\ \bar{\mathbf{p}}_{;-1}(t) \\ \mathbf{p}_{;1}(t) \\ \vdots \end{Bmatrix}, \quad \underset{\sim}{\mathbf{g}}(t) = \begin{Bmatrix} \vdots \\ \bar{\mathbf{g}}_{;1}(t) \\ \mathbf{g}_{;-1}(t) \\ \bar{\mathbf{g}}_{;0}(t) \\ \mathbf{g}_{;0}(t) \\ \bar{\mathbf{g}}_{;-1}(t) \\ \mathbf{g}_{;1}(t) \\ \vdots \end{Bmatrix}$$

and

$$\mathbf{C}_{i;n} = \mathbf{C}_i - \mathrm{j}4n\Omega\mathbf{M}_i, \quad \mathbf{K}_{i;n} = \mathbf{K}_i - \mathrm{j}2n\Omega\mathbf{C}_i - 4n^2\Omega^2\mathbf{M}_i, \quad i = \mathbf{r}, \mathbf{b}, \mathbf{f}, \quad n = 0, \pm 1, \pm 2, \ldots .$$

Note that the differential equation (1) with periodically time-varying parameters is transformed into Eq. (4) with time-invariant parameters, at the expense of introducing the coordinate vector of infinite dimension.

Assuming the solution form of $\underset{\sim}{\mathbf{p}}(t) = \underset{\sim c}{\mathbf{u}}\,\mathrm{e}^{\lambda t}$ for the homogeneous part of Eq. (4), we obtain the latent value problem given as [1]

$$\underset{\sim}{\mathbf{D}}(\lambda^i_{r(m)})\underset{\sim cr(m)}{\mathbf{u}^i} = \underset{\sim}{\mathbf{0}} \quad \text{and} \quad \underset{\sim cr(m)}{\bar{\mathbf{v}}^{i\mathrm{T}}}\underset{\sim}{\mathbf{D}}(\lambda^i_{r(m)}) = \underset{\sim}{\mathbf{0}}^{\mathrm{T}}, \quad r = \pm 1, \pm 2, \ldots, N, \quad m = 0, \pm 1, \pm 2, \ldots, \quad i = B, F, \tag{5}$$

where the lambda matrix of degree two is given by

$$\underset{\sim}{\mathbf{D}}(\lambda) = \lambda^2\underset{\sim}{\mathbf{M}} + \lambda\underset{\sim}{\mathbf{C}} + \underset{\sim}{\mathbf{K}}$$

and the latent roots (eigenvalues) are obtained from the characteristic equation

$$|\underset{\sim}{\mathbf{D}}(\lambda)| = 0$$

and the corresponding right and left latent vectors take the form of

$$\underset{\sim c}{\mathbf{u}} = \left\{ \cdots \quad \hat{\mathbf{u}}^{\mathrm{T}}_{;1} \quad \mathbf{u}^{\mathrm{T}}_{;-1} \quad \hat{\mathbf{u}}^{\mathrm{T}}_{;0} \quad \mathbf{u}^{\mathrm{T}}_{;0} \quad \hat{\mathbf{u}}^{\mathrm{T}}_{;-1} \quad \mathbf{u}^{\mathrm{T}}_{;1} \quad \cdots \right\}^{\mathrm{T}},$$

$$\underset{\sim c}{\mathbf{v}} = \left\{ \cdots \quad \hat{\mathbf{v}}^{\mathrm{T}}_{;1} \quad \mathbf{v}^{\mathrm{T}}_{;-1} \quad \hat{\mathbf{v}}^{\mathrm{T}}_{;0} \quad \mathbf{v}^{\mathrm{T}}_{;0} \quad \hat{\mathbf{v}}^{\mathrm{T}}_{;-1} \quad \mathbf{v}^{\mathrm{T}}_{;1} \quad \cdots \right\}^{\mathrm{T}}.$$

Here, each pair of eigenvalues, equal in value but different in sign of subscript, forms a complex conjugate pair. The subscript $r(m)$ refers to the rth eigen (latent) solution in cluster m. Cluster m consists of only the set of eigensolutions associated with the modulation index m, or equivalently, with the shifted eigenvalues by $\mathrm{j}2m\Omega$.

Eq. (5) can be rewritten as

$$\underset{\sim}{\mathbf{D}}(\lambda)\underset{\sim c}{\mathbf{u}} = \begin{bmatrix} \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \iddots \\ \cdots & \bar{\mathbf{D}}_{f;1} & \bar{\mathbf{D}}_{b;1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots \\ \cdots & \mathbf{D}_{b;-1} & \mathbf{D}_{f;-1} & \mathbf{D}_{r;0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots \\ \cdots & \mathbf{0} & \bar{\mathbf{D}}_{r;1} & \bar{\mathbf{D}}_{f;0} & \bar{\mathbf{D}}_{b;0} & \mathbf{0} & \mathbf{0} & \cdots \\ \cdots & \mathbf{0} & \mathbf{0} & \mathbf{D}_{b;0} & \mathbf{D}_{f;0} & \mathbf{D}_{r;1} & \mathbf{0} & \cdots \\ \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \bar{\mathbf{D}}_{r;0} & \bar{\mathbf{D}}_{f;-1} & \bar{\mathbf{D}}_{b;-1} & \cdots \\ \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{D}_{b;1} & \mathbf{D}_{f;1} & \cdots \\ \iddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \begin{Bmatrix} \vdots \\ \hat{\mathbf{u}}_{;1} \\ \mathbf{u}_{;-1} \\ \hat{\mathbf{u}}_{;0} \\ \mathbf{u}_{;0} \\ \hat{\mathbf{u}}_{;-1} \\ \mathbf{u}_{;1} \\ \vdots \end{Bmatrix} = \underset{\sim}{\mathbf{0}}, \tag{6a}$$

$$\underset{\sim c}{\bar{\mathbf{v}}^{\mathrm{T}}}\,\underset{\sim}{\mathbf{D}}(\lambda) = \left\{ \cdots \quad \hat{\bar{\mathbf{v}}}^{\mathrm{T}}_{;1} \quad \bar{\mathbf{v}}^{\mathrm{T}}_{;-1} \quad \hat{\bar{\mathbf{v}}}^{\mathrm{T}}_{;0} \quad \bar{\mathbf{v}}^{\mathrm{T}}_{;0} \quad \hat{\bar{\mathbf{v}}}^{\mathrm{T}}_{;-1} \quad \bar{\mathbf{v}}^{\mathrm{T}}_{;1} \quad \cdots \right\}$$
$$\times \begin{bmatrix} \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \iddots \\ \cdots & \bar{\mathbf{D}}_{f;1} & \bar{\mathbf{D}}_{b;1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots \\ \cdots & \mathbf{D}_{b;-1} & \mathbf{D}_{f;-1} & \mathbf{D}_{r;0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots \\ \cdots & \mathbf{0} & \bar{\mathbf{D}}_{r;1} & \bar{\mathbf{D}}_{f;0} & \bar{\mathbf{D}}_{b;0} & \mathbf{0} & \mathbf{0} & \cdots \\ \cdots & \mathbf{0} & \mathbf{0} & \mathbf{D}_{b;0} & \mathbf{D}_{f;0} & \mathbf{D}_{r;1} & \mathbf{0} & \cdots \\ \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \bar{\mathbf{D}}_{r;0} & \bar{\mathbf{D}}_{f;-1} & \bar{\mathbf{D}}_{b;-1} & \cdots \\ \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{D}_{b;1} & \mathbf{D}_{f;1} & \cdots \\ \iddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} = \underset{\sim}{\mathbf{0}}^{\mathrm{T}}, \tag{6b}$$

where the $N \times N$ block matrices of the Hill's infinite-order matrix $\underset{\sim}{\mathbf{D}}(\lambda)$ with $3N$ bandwidth are given as

$$\mathbf{D}_{i;n}(\lambda) = \lambda^2\mathbf{M}_i + \lambda\mathbf{C}_{i;n} + \mathbf{K}_{i;n}; \quad i = \mathbf{r}, \mathbf{b}, \mathbf{f}; \quad n = 0, \pm 1, \pm 2, \ldots . \tag{6c}$$

And the bi-orthonormality condition becomes [1]

$$\underset{\sim c}{\bar{\mathbf{v}}^{\mathrm{T}}}\left[\frac{\mathrm{d}}{\mathrm{d}\lambda}\underset{\sim}{\mathbf{D}}(\lambda)\right]\underset{\sim c}{\mathbf{u}} = 1. \tag{7}$$

In the above expressions (5)–(7), the sub- and superscripts are omitted for notational simplicity.

The relations for every pair of latent vectors $(\hat{\mathbf{u}}^{\mathrm{T}}_{;n}, \mathbf{u}^{\mathrm{T}}_{;n})^{\mathrm{T}}$ associated with the eigenvalue $\lambda^i_{r(m)}$ become

$$\bar{\mathbf{u}}^i_{-r(-m);n} = \hat{\mathbf{u}}^i_{r(m);n}, \quad (\bar{\mathbf{v}}^i_{-r(-m);n} = \hat{\mathbf{v}}^i_{r(m);n}), \quad \lambda^i_{r(m)} = \bar{\lambda}^i_{-r(-m)} \tag{8a}$$

and the structure of the eigenvalues and the corresponding latent vectors can be arranged as follows:

eigenvalues:

$$\begin{aligned} \{\lambda^i_{r(m)}\} &= \{\lambda^i_{r(0)} + \mathrm{j}2m\Omega\} \\ &= \{\ldots, (\text{cluster } -m), \ldots, (\text{cluster } 0), \ldots, (\text{cluster } m), \ldots\} \\ &= \{\ldots, (\ldots, \bar{\lambda}^F_{N(0)} - \mathrm{j}2m\Omega, \bar{\lambda}^B_{N(0)} - \mathrm{j}2m\Omega, \ldots, \bar{\lambda}^F_{1(0)} - \mathrm{j}2m\Omega, \bar{\lambda}^B_{1(0)} - \mathrm{j}2m\Omega, \\ &\qquad \lambda^F_{1(0)} - \mathrm{j}2m\Omega, \lambda^B_{1(0)} - \mathrm{j}2m\Omega, \ldots, \lambda^F_{N(0)} - \mathrm{j}2m\Omega, \lambda^B_{N(0)} - \mathrm{j}2m\Omega, \ldots), \ldots, \\ &\qquad (\ldots, \bar{\lambda}^F_{N(0)}, \bar{\lambda}^B_{N(0)}, \ldots, \bar{\lambda}^F_{1(0)}, \bar{\lambda}^B_{1(0)}, \lambda^F_{1(0)}, \lambda^B_{1(0)}, \ldots \lambda^F_{N(0)}, \lambda^B_{N(0)}, \ldots), \ldots, \\ &\qquad (\ldots, \bar{\lambda}^F_{N(0)} + \mathrm{j}2m\Omega, \bar{\lambda}^B_{N(0)} + \mathrm{j}2m\Omega, \ldots, \bar{\lambda}^F_{1(0)} + \mathrm{j}2m\Omega, \bar{\lambda}^B_{1(0)} + \mathrm{j}2m\Omega, \\ &\qquad \lambda^F_{1(0)} + \mathrm{j}2m\Omega, \lambda^B_{1(0)} + \mathrm{j}2m\Omega, \ldots, \lambda^F_{N(0)} + \mathrm{j}2m\Omega, \lambda^B_{N(0)} + \mathrm{j}2m\Omega, \ldots), \ldots\ldots\}. \end{aligned} \tag{8b}$$

latent vectors:

$$\underset{\sim c}{\mathbf{u}} = \left\{ \cdots \quad \bar{\mathbf{u}}^{iT}_{-r(-m);1} \quad \mathbf{u}^{iT}_{r(m);-1} \quad \bar{\mathbf{u}}^{iT}_{-r(-m);0} \quad \mathbf{u}^{iT}_{r(m);0} \quad \bar{\mathbf{u}}^{iT}_{-r(-m);-1} \quad \mathbf{u}^{iT}_{r(m);1} \quad \cdots \right\}^{T},$$
$$\underset{\sim c}{\mathbf{v}} = \left\{ \cdots \quad \bar{\mathbf{v}}^{iT}_{-r(-m);1} \quad \mathbf{v}^{iT}_{r(m);-1} \quad \bar{\mathbf{v}}^{iT}_{-r(-m);0} \quad \mathbf{v}^{iT}_{r(m);0} \quad \bar{\mathbf{v}}^{iT}_{-r(-m);-1} \quad \mathbf{v}^{iT}_{r(m);1} \quad \cdots \right\}^{T}. \tag{8c}$$

Here, $\lambda^i_{r(0)}$ and $\lambda^i_{r(m)} = \lambda^i_{r(0)} + \mathrm{j}2m\Omega$ are the eigenvalues of the periodically time-varying system (1) and the transformed time-invariant system (4), respectively. Since it holds $Re\{\lambda^i_{r(m)}\} = Re\{\lambda^i_{r(0)}\}$, the stability of the periodically time-varying system (1) is preserved irrespective of introduction of the modulated complex coordinates. The time-periodic latent vectors associated with the periodically time-varying system (1), which is normally obtained by using Floquet theory, can be expanded in terms of the complex harmonic function $\mathrm{e}^{\mathrm{j}2m\Omega t}$. It is shown in Ref. [4] that the latent vector elements in Eq. (8c) correspond to the Fourier coefficient vectors of the time-periodic latent vectors.

Assuming the solution form of $\mathbf{p}(t) = \underset{\sim}{\mathbf{u}}\,\underset{\sim}{\eta}(t)$ for Eq. (4) and using the bi-orthonormality condition (7), we can obtain the infinite set of complex modal equations of motion as

$$\dot{\eta}^i_{r(m)} = \lambda^i_{r(m)}\eta^i_{r(m)} + \underset{\sim cr(m)}{\bar{\mathbf{v}}^{iT}}\,\underset{\sim}{\mathbf{g}}(t), \quad r = \pm1, \pm2, \ldots, \pm N, \quad i = B, F, \quad m = 0, \pm1, \pm2, \ldots \tag{9a}$$

and the forced response vector $\mathbf{p}(t)$ of the general rotor system as [4,7]

$$\mathbf{p}(t) = \sum_{i=B,F}\sum_{r=-N}^{N}{}' \sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}\left\{\int_0^t \mathrm{e}^{\lambda^i_{r(m)}(t-\tau)}\left[\mathbf{u}^i_{r(m);0}\bar{\mathbf{v}}^{iT}_{r(m);n}\mathbf{g}_{;n}(\tau) + \mathbf{u}^i_{r(m);0}\bar{\hat{\mathbf{v}}}^{iT}_{r(m);n}\bar{\mathbf{g}}_{;-n}(\tau)\right]\mathrm{d}\tau\right\}. \tag{9b}$$

Fourier transforming equation (9b), we obtain

$$\begin{aligned}\mathbf{P}(\mathrm{j}\omega) &= \sum_{n=-\infty}^{\infty}\left\{\left[\sum_{i=B,F}\sum_{r=-N}^{N}{}'\sum_{m=-\infty}^{\infty}\frac{\mathbf{u}^i_{r(m);0}\bar{\mathbf{v}}^{iT}_{r(m);n}}{\mathrm{j}\omega-\lambda^i_{r(m)}}\right]\mathbf{G}_{;n}(\mathrm{j}\omega) + \left[\sum_{i=B,F}\sum_{r=-N}^{N}{}'\sum_{m=-\infty}^{\infty}\frac{\mathbf{u}^i_{r(m);0}\bar{\hat{\mathbf{v}}}^{iT}_{r(m);n}}{\mathrm{j}\omega-\lambda^i_{r(m)}}\right]\hat{\mathbf{G}}_{;-n}(\mathrm{j}\omega)\right\}\\ &= \sum_{n=-\infty}^{\infty}\left\{\mathbf{H}_{\mathbf{g}_{;-n}\mathbf{p}}(\mathrm{j}\omega)\mathbf{G}_{;n}(\mathrm{j}\omega) + \mathbf{H}_{\hat{\mathbf{g}}_{;-n}\mathbf{p}}(\mathrm{j}\omega)\hat{\mathbf{G}}_{;-n}(\mathrm{j}\omega)\right\},\end{aligned} \tag{10}$$

where the Fourier transforms of the modulated excitation vectors are given by

$$\mathbf{G}_{;n}(\mathrm{j}\omega) = \mathbf{G}\{\mathrm{j}(\omega - 2n\Omega)\}, \quad \hat{\mathbf{G}}_{;n}(\mathrm{j}\omega) = \hat{\mathbf{G}}\{\mathrm{j}(\omega + 2n\Omega)\}. \tag{11}$$

Here, $\mathbf{G}(\mathrm{j}\omega)$ and $\hat{\mathbf{G}}(\mathrm{j}\omega)$ are the Fourier transforms of $\mathbf{g}(t)$ and $\bar{\mathbf{g}}(t)$, respectively. Although there are still an infinite number of dFRMs in Eq. (10), we introduce four dFRMs that are important in characterizing the system asymmetry and anisotropy, as

$$\begin{aligned}\mathbf{H}_{\mathbf{g}_{;0}\mathbf{p}_{;0}}(\mathrm{j}\omega) &= \sum_{m=-\infty}^{\infty}\sum_{i=B,F}\sum_{r=-N}^{N}{}'\left[\frac{\mathbf{u}^i_{r(m);0}\bar{\mathbf{v}}^{iT}_{r(m);0}}{\mathrm{j}\omega-\lambda^i_{r(m)}}\right], \quad \mathbf{H}_{\hat{\mathbf{g}}_{;0}\mathbf{p}_{;0}}(\mathrm{j}\omega) = \sum_{m=-\infty}^{\infty}\sum_{i=B,F}\sum_{r=-N}^{N}{}'\left[\frac{\mathbf{u}^i_{r(m);0}\mathbf{v}^{iT}_{-r(-m);0}}{\mathrm{j}\omega-\lambda^i_{r(m)}}\right],\\ \mathbf{H}_{\hat{\mathbf{g}}_{;-1}\mathbf{p}_{;0}}(\mathrm{j}\omega) &= \sum_{m=-\infty}^{\infty}\sum_{i=B,F}\sum_{r=-N}^{N}{}'\left[\frac{\mathbf{u}^i_{r(m);0}\mathbf{v}^{iT}_{-r(-m);-1}}{\mathrm{j}\omega-\lambda^i_{r(m)}}\right], \quad \mathbf{H}_{\mathbf{g}_{;-1}\mathbf{p}_{;0}}(\mathrm{j}\omega) = \sum_{m=-\infty}^{\infty}\sum_{i=B,F}\sum_{r=-N}^{N}{}'\left[\frac{\mathbf{u}^i_{r(m);0}\bar{\mathbf{v}}^{iT}_{r(m);1}}{\mathrm{j}\omega-\lambda^i_{r(m)}}\right].\end{aligned} \tag{12}$$

Here, $\mathbf{H}_{\mathbf{g}_{;0}\mathbf{p}_{;0}}(\mathrm{j}\omega)$ is referred to as the normal dFRM that represents the system symmetry, $\mathbf{H}_{\hat{\mathbf{g}}_{;0}\mathbf{p}_{;0}}(\mathrm{j}\omega)$ is referred to as the reverse dFRM that represents the effect of system anisotropy, and, $\mathbf{H}_{\hat{\mathbf{g}}_{;-1}\mathbf{p}_{;0}}(\mathrm{j}\omega)$ and $\mathbf{H}_{\mathbf{g}_{;-1}\mathbf{p}_{;0}}(\mathrm{j}\omega)$ are referred to as the modulated dFRMs that represent the effect of system asymmetry and the coupled effect of system anisotropy and asymmetry, respectively.

3. Strength of modes for general rotors based on reduced-order Hill's matrix

The latent value problem (6) can be reduced, as a good approximation for practical use, to a finite-order matrix–vector equation of motion. Without loss of generality, focusing on the modulated coordinates, for every r and i in the mth cluster, the homogeneous matrix equation (6a) for general rotors with weak asymmetry δ and anisotropy Δ, can be reduced to the equation of motion, based on the $6N \times 6N$ reduced order Hill's matrix, given as

$$\tilde{\mathbf{D}}_6(\lambda) \cdot \tilde{\mathbf{u}}_c = \begin{bmatrix} \bar{\mathbf{D}}_{f;1} & \Delta\bar{\mathbf{D}}_{b;1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \Delta\mathbf{D}_{b'';-1} & \mathbf{D}_{f;-1} & \delta\mathbf{D}_{r;0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \delta\bar{\mathbf{D}}_{r;1} & \bar{\mathbf{D}}_{f;0} & \Delta\bar{\mathbf{D}}_{b;0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \Delta\mathbf{D}_{b;0} & \mathbf{D}_{f;0} & \delta\mathbf{D}_{r;1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \delta\bar{\mathbf{D}}_{r;0} & \bar{\mathbf{D}}_{f;-1} & \Delta\bar{\mathbf{D}}_{b;-1} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \Delta\mathbf{D}_{b;1} & \mathbf{D}_{f;1} \end{bmatrix} \begin{Bmatrix} \hat{\mathbf{u}}^i_{r(m);1} \\ \mathbf{u}^i_{r(m);-1} \\ \hat{\mathbf{u}}^i_{r(m);0} \\ \mathbf{u}^i_{r(m);0} \\ \hat{\mathbf{u}}^i_{r(m);-1} \\ \mathbf{u}^i_{r(m);1} \end{Bmatrix} = \begin{Bmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \end{Bmatrix}, \tag{13a}$$

$$\tilde{\bar{\mathbf{v}}}_c^{\mathrm{T}} \cdot \tilde{\mathbf{D}}_6(\lambda) = \begin{Bmatrix} \tilde{\bar{\mathbf{v}}}^i_{r(m);1} \\ \bar{\mathbf{v}}^i_{r(m);-1} \\ \tilde{\bar{\mathbf{v}}}^i_{r(m);0} \\ \bar{\mathbf{v}}^i_{r(m);0} \\ \tilde{\bar{\mathbf{v}}}^i_{r(m);-1} \\ \bar{\mathbf{v}}^i_{r(m);1} \end{Bmatrix}^{\mathrm{T}} \begin{bmatrix} \bar{\mathbf{D}}_{f;1} & \Delta\bar{\mathbf{D}}_{b;1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \Delta\mathbf{D}_{b;-1} & \mathbf{D}_{f;-1} & \delta\mathbf{D}_{r;0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \delta\bar{\mathbf{D}}_{r;1} & \bar{\mathbf{D}}_{f;0} & \Delta\bar{\mathbf{D}}_{b;0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \Delta\mathbf{D}_{b;0} & \mathbf{D}_{f;0} & \delta\mathbf{D}_{r;1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \delta\bar{\mathbf{D}}_{r;0} & \bar{\mathbf{D}}_{f;-1} & \Delta\bar{\mathbf{D}}_{b;-1} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \Delta\mathbf{D}_{b;1} & \mathbf{D}_{f;1} \end{bmatrix} = \begin{Bmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \end{Bmatrix}^{\mathrm{T}}, \tag{13b}$$

where the lambda matrices of degree two associated with the weak dynamic stiffness matrices are given by

$$\Delta\mathbf{D}_{b;n} = \lambda^2\Delta\mathbf{M}_{b;n} + \lambda\Delta\mathbf{C}_{b;n} + \Delta\mathbf{K}_{b;n}, \quad \delta\mathbf{D}_{r;n} = \lambda^2\delta\mathbf{M}_{r;n} + \lambda\delta\mathbf{C}_{r;n} + \delta\mathbf{K}_{r;n}, \quad n = 0, \pm 1, \pm 2, \ldots .$$

Here, the bi-orthonormality condition (7) reduces to

$$\tilde{\bar{\mathbf{v}}}_c^{\mathrm{T}}\left[\frac{\mathrm{d}}{\mathrm{d}\lambda}\tilde{\mathbf{D}}_6(\lambda)\right]\tilde{\mathbf{u}}_c = \left\{\tilde{\bar{\mathbf{v}}}_1^{\mathrm{T}} \quad \bar{\mathbf{v}}_{-1}^{\mathrm{T}} \quad \tilde{\bar{\mathbf{v}}}_0^{\mathrm{T}} \quad \bar{\mathbf{v}}_0^{\mathrm{T}} \quad \tilde{\bar{\mathbf{v}}}_{-1}^{\mathrm{T}} \quad \bar{\mathbf{v}}_1^{\mathrm{T}}\right\}\left[\frac{\mathrm{d}}{\mathrm{d}\lambda}\tilde{\mathbf{D}}_6(\lambda)\right]\left\{\hat{\mathbf{u}}_1^{\mathrm{T}} \quad \mathbf{u}_{-1}^{\mathrm{T}} \quad \hat{\mathbf{u}}_0^{\mathrm{T}} \quad \mathbf{u}_0^{\mathrm{T}} \quad \hat{\mathbf{u}}_{-1}^{\mathrm{T}} \quad \mathbf{u}_1^{\mathrm{T}}\right\}^{\mathrm{T}} = 1. \tag{14}$$

The determinants of the dynamic stiffness matrices are configured as

$$\begin{aligned} |\mathbf{D}_{f;0}(\lambda)| &= \prod_{\substack{r=1\\m=0,\pm1}}^{N}(\lambda-\lambda^{oF}_{r(m)})(\lambda-\lambda^{oB}_{r(m)}), \quad |\bar{\mathbf{D}}_{f;0}(\lambda)| = \prod_{\substack{r=1\\m=0,\pm1}}^{N}(\lambda-\lambda^{oF}_{-r(m)})(\lambda-\lambda^{oB}_{-r(m)}),\\ |\mathbf{D}_{f;-1}(\lambda)| &= \prod_{\substack{r=1\\m=0,\pm1}}^{N}(\lambda-\lambda^{oF}_{r(m)})(\lambda-\lambda^{oB}_{r(m)}), \quad |\bar{\mathbf{D}}_{f;-1}(\lambda)| = \prod_{\substack{r=1\\m=0,\pm1}}^{N}(\lambda-\lambda^{oF}_{-r(m)})(\lambda-\lambda^{oB}_{-r(m)}),\\ |\mathbf{D}_{f;1}(\lambda)| &= \prod_{\substack{r=1\\m=0,\pm1}}^{N}(\lambda-\lambda^{oF}_{r(m)})(\lambda-\lambda^{oB}_{r(m)}), \quad |\bar{\mathbf{D}}_{f;1}(\lambda)| = \prod_{\substack{r=1\\m=0,\pm1}}^{N}(\lambda-\lambda^{oF}_{-r(m)})(\lambda-\lambda^{oB}_{-r(m)}), \end{aligned} \tag{15}$$

where $\lambda^{oi}_{r(m)}$, $i = B, F$; $r = 1, 2, \ldots, N$; $m = 0, \pm 1$, are the eigenvalues of the associated isotropic system, i.e. the rotor with $\delta = \Delta = 0$. The characteristic equation becomes

$$\begin{aligned} |\tilde{\mathbf{D}}_6(\lambda)| &= \prod_{\substack{r=1\\m=0,\pm1\\i=B,F}}^{N}(\lambda-\lambda^i_{r(m)})(\lambda-\lambda^i_{-r(-m)}) = \prod_{\substack{r=1\\m=0,\pm1\\i=B,F}}^{N}(\lambda-\lambda^i_{r(m)})(\lambda-\bar{\lambda}^i_{r(m)})\\ &= \prod_{r=1}^{N}(\lambda-\bar{\lambda}^F_{r(0)})(\lambda-\lambda^F_{r(0)})(\lambda-\bar{\lambda}^F_{r(-1)})(\lambda-\lambda^F_{r(-1)})(\lambda-\bar{\lambda}^F_{r(1)})(\lambda-\lambda^F_{r(1)})\\ &\quad\times(\lambda-\bar{\lambda}^B_{r(0)})(\lambda-\lambda^B_{r(0)})(\lambda-\bar{\lambda}^B_{r(-1)})(\lambda-\lambda^B_{r(-1)})(\lambda-\bar{\lambda}^B_{r(1)})(\lambda-\lambda^B_{r(1)}) = 0. \end{aligned} \tag{16}$$

The eigenvalues and the corresponding latent vectors are related to each other, i.e. for $i = F, B$ and $r = 1, 2, \ldots, N$

$$\begin{array}{lll}
\underline{\text{eigenvalue}} & \underline{\text{left latent vector}} & \underline{\text{right latent vector}} \\
\lambda^i_{r(0)} & \begin{bmatrix}\bar{\hat{\mathbf{v}}}^{\mathrm{T}}_{;1} & \bar{\mathbf{v}}^{\mathrm{T}}_{;-1} & \bar{\hat{\mathbf{v}}}^{\mathrm{T}}_{;0} & \bar{\mathbf{v}}^{\mathrm{T}}_{0} & \bar{\hat{\mathbf{v}}}^{\mathrm{T}}_{;-1} & \bar{\mathbf{v}}^{\mathrm{T}}_{;1}\end{bmatrix}^{i\mathrm{T}}_{r(0)} & \begin{bmatrix}\hat{\mathbf{u}}^{\mathrm{T}}_{;1} & \mathbf{u}^{\mathrm{T}}_{;-1} & \hat{\mathbf{u}}^{\mathrm{T}}_{;0} & \mathbf{u}^{\mathrm{T}}_{;0} & \hat{\mathbf{u}}^{\mathrm{T}}_{;-1} & \mathbf{u}^{\mathrm{T}}_{;1}\end{bmatrix}^{i\mathrm{T}}_{r(0)}, \\
\lambda^i_{-r(0)} = \bar{\lambda}^i_{r(0)} & \begin{bmatrix}\bar{\hat{\mathbf{v}}}^{\mathrm{T}}_{;1} & \bar{\mathbf{v}}^{\mathrm{T}}_{;-1} & \bar{\hat{\mathbf{v}}}^{\mathrm{T}}_{;0} & \bar{\mathbf{v}}^{\mathrm{T}}_{0} & \bar{\hat{\mathbf{v}}}^{\mathrm{T}}_{;-1} & \bar{\mathbf{v}}^{\mathrm{T}}_{;1}\end{bmatrix}^{i\mathrm{T}}_{-r(0)} & \begin{bmatrix}\hat{\mathbf{u}}^{\mathrm{T}}_{;1} & \mathbf{u}^{\mathrm{T}}_{;-1} & \hat{\mathbf{u}}^{\mathrm{T}}_{;0} & \mathbf{u}^{\mathrm{T}}_{;0} & \hat{\mathbf{u}}^{\mathrm{T}}_{;-1} & \mathbf{u}^{\mathrm{T}}_{;1}\end{bmatrix}^{i\mathrm{T}}_{-r(0)}, \\
\lambda^i_{r(1)} \to \lambda^i_{r(0)} + \mathrm{j}2\Omega & \begin{bmatrix}\bar{\hat{\mathbf{v}}}^{\mathrm{T}}_{;1} & \bar{\mathbf{v}}^{\mathrm{T}}_{;-1} & \bar{\hat{\mathbf{v}}}^{\mathrm{T}}_{;0} & \bar{\mathbf{v}}^{\mathrm{T}}_{0} & \bar{\hat{\mathbf{v}}}^{\mathrm{T}}_{;-1} & \bar{\mathbf{v}}^{\mathrm{T}}_{;1}\end{bmatrix}^{i\mathrm{T}}_{r(1)} & \begin{bmatrix}\hat{\mathbf{u}}^{\mathrm{T}}_{;1} & \mathbf{u}^{\mathrm{T}}_{;-1} & \hat{\mathbf{u}}^{\mathrm{T}}_{;0} & \mathbf{u}^{\mathrm{T}}_{;0} & \hat{\mathbf{u}}^{\mathrm{T}}_{;-1} & \mathbf{u}^{\mathrm{T}}_{;1}\end{bmatrix}^{i\mathrm{T}}_{r(-1)}, \\
\lambda^i_{-r(1)} = \bar{\lambda}^i_{r(-1)} \to \bar{\lambda}^i_{r(0)} + \mathrm{j}2\Omega & \begin{bmatrix}\bar{\hat{\mathbf{v}}}^{\mathrm{T}}_{;1} & \bar{\mathbf{v}}^{\mathrm{T}}_{;-1} & \bar{\hat{\mathbf{v}}}^{\mathrm{T}}_{;0} & \bar{\mathbf{v}}^{\mathrm{T}}_{0} & \bar{\hat{\mathbf{v}}}^{\mathrm{T}}_{;-1} & \bar{\mathbf{v}}^{\mathrm{T}}_{;1}\end{bmatrix}^{i\mathrm{T}}_{-r(1)} & \begin{bmatrix}\hat{\mathbf{u}}^{\mathrm{T}}_{;1} & \mathbf{u}^{\mathrm{T}}_{;-1} & \hat{\mathbf{u}}^{\mathrm{T}}_{;0} & \mathbf{u}^{\mathrm{T}}_{;0} & \hat{\mathbf{u}}^{\mathrm{T}}_{;-1} & \mathbf{u}^{\mathrm{T}}_{;1}\end{bmatrix}^{i\mathrm{T}}_{-r(1)}, \\
\lambda^i_{r(-1)} \to \lambda^i_{r(0)} - \mathrm{j}2\Omega & \begin{bmatrix}\bar{\hat{\mathbf{v}}}^{\mathrm{T}}_{;1} & \bar{\mathbf{v}}^{\mathrm{T}}_{;-1} & \bar{\hat{\mathbf{v}}}^{\mathrm{T}}_{;0} & \bar{\mathbf{v}}^{\mathrm{T}}_{0} & \bar{\hat{\mathbf{v}}}^{\mathrm{T}}_{;-1} & \bar{\mathbf{v}}^{\mathrm{T}}_{;1}\end{bmatrix}^{i\mathrm{T}}_{r(-1)} & \begin{bmatrix}\hat{\mathbf{u}}^{\mathrm{T}}_{;1} & \mathbf{u}^{\mathrm{T}}_{;-1} & \hat{\mathbf{u}}^{\mathrm{T}}_{;0} & \mathbf{u}^{\mathrm{T}}_{;0} & \hat{\mathbf{u}}^{\mathrm{T}}_{;-1} & \mathbf{u}^{\mathrm{T}}_{;1}\end{bmatrix}^{i\mathrm{T}}_{r(-1)}, \\
\lambda^i_{-r(-1)} = \bar{\lambda}^i_{r(1)} \to \bar{\lambda}^i_{r(0)} - \mathrm{j}2\Omega & \begin{bmatrix}\bar{\hat{\mathbf{v}}}^{\mathrm{T}}_{;1} & \bar{\mathbf{v}}^{\mathrm{T}}_{;-1} & \bar{\hat{\mathbf{v}}}^{\mathrm{T}}_{;0} & \bar{\mathbf{v}}^{\mathrm{T}}_{0} & \bar{\hat{\mathbf{v}}}^{\mathrm{T}}_{;-1} & \bar{\mathbf{v}}^{\mathrm{T}}_{;1}\end{bmatrix}^{i\mathrm{T}}_{-r(-1)} & \begin{bmatrix}\hat{\mathbf{u}}^{\mathrm{T}}_{;1} & \mathbf{u}^{\mathrm{T}}_{;-1} & \hat{\mathbf{u}}^{\mathrm{T}}_{;0} & \mathbf{u}^{\mathrm{T}}_{;0} & \hat{\mathbf{u}}^{\mathrm{T}}_{;-1} & \mathbf{u}^{\mathrm{T}}_{;1}\end{bmatrix}^{i\mathrm{T}}_{-r(-1)}.
\end{array} \tag{17}$$

By introducing the norm defined in the $6N$ dimensional vector space, we can derive the relations given by

$$\begin{aligned}
&||\mathbf{D}_{f;0}(\lambda^i_{r(m)})\mathbf{u}^i_{r(m);0}|| \sim \begin{cases} O(\delta, \Delta) & \text{for} \quad r(m) = r(0), \\ O(1) & \text{otherwise}, \end{cases} \\
&||\bar{\mathbf{D}}_{f;0}(\lambda^i_{r(m)})\hat{\mathbf{u}}^i_{r(m);0}|| \sim \begin{cases} O(\delta, \Delta) & \text{for} \quad r(m) = -r(0), \\ O(1) & \text{otherwise}, \end{cases} \\
&||\mathbf{D}_{f;-1}(\lambda^i_{r(m)})\mathbf{u}^i_{r(m);-1}|| \sim \begin{cases} O(\delta, \Delta) & \text{for} \quad r(m) = r(-1), \\ O(1) & \text{otherwise}, \end{cases} \\
&||\bar{\mathbf{D}}_{f;-1}(\lambda^i_{r(m)})\hat{\mathbf{u}}^i_{r(m);-1}|| \sim \begin{cases} O(\delta, \Delta) & \text{for} \quad r(m) = -r(+1) \\ O(1) & \text{otherwise}, \end{cases} \\
&||\mathbf{D}_{f;1}(\lambda^i_{r(m)})\mathbf{u}^i_{r(m);1}|| \sim \begin{cases} O(\delta, \Delta) & \text{for} \quad r(m) = r(1) \\ O(1) & \text{otherwise}, \end{cases} \\
&||\bar{\mathbf{D}}_{f;1}(\lambda^i_{r(m)})\hat{\mathbf{u}}^i_{r(m);1}|| \sim \begin{cases} O(\delta, \Delta) & \text{for} \quad r(m) = -r(-1) \\ O(1) & \text{otherwise}. \end{cases}
\end{aligned} \tag{18}$$

Here, $O(1)$ means that the norm order is independent of the perturbation δ and Δ, and $O(\delta, \Delta) = O(\delta) \oplus O(\Delta)$, where $\oplus$ denotes the direct sum. The first relation in Eqs. (18) can be derived from the fourth block matrix equation in Eq. (13a), as given in detail in Appendix A, and the rest of the relations can be similarly derived without difficulty.

For $r(m) = r(0)$ where $||\mathbf{D}_{f;0}(\lambda^i_{r(m)})\mathbf{u}^i_{r(m);0}|| \sim O(\delta, \Delta)$, we can derive, by removing the fourth block matrix equation in Eq. (13a) and using $||\mathbf{u}^i_{r(0);0}|| \sim O(1)$, the norm order of right latent (modal) vectors as

$$\begin{aligned}
||\hat{\mathbf{u}}^i_{r(0);1}|| &= ||\delta\Delta^2\mathbf{D}^{-1}_{f;1}\bar{\mathbf{D}}_{b;1}[\delta^2\mathbf{D}_{r;0}\bar{\mathbf{D}}^{-1}_{f;0}\bar{\mathbf{D}}_{r;1} - \mathbf{E}_{f;-1}]^{-1}\mathbf{D}_{r;0}\bar{\mathbf{D}}^{-1}_{f;0}\bar{\mathbf{D}}_{b;0}\mathbf{u}^i_{r(0);0}|| \\
&\approx || - \delta\Delta^2\mathbf{D}^{-1}_{f;1}\bar{\mathbf{D}}_{b;1}\mathbf{D}^{-1}_{f;-1}\mathbf{D}_{r;0}\bar{\mathbf{D}}^{-1}_{f;0}\bar{\mathbf{D}}_{b;0}\mathbf{u}^i_{r(0);0}|| \sim O(\delta\Delta^2)||\mathbf{u}^i_{r(0);0}|| \sim O(\delta\Delta^2),
\end{aligned}$$

$$||\mathbf{u}^i_{r(0);-1}|| = || - \delta\Delta[\delta^2\mathbf{D}_{r;0}\bar{\mathbf{D}}^{-1}_{f;0}\bar{\mathbf{D}}_{r;1} - \mathbf{E}_{f;-1}]^{-1}\mathbf{D}_{r;0}\bar{\mathbf{D}}^{-1}_{f;0}\bar{\mathbf{D}}_{b;0}\mathbf{u}^i_{r(0);0}|| \approx ||\delta\Delta\mathbf{D}^{-1}_{f;-1}\mathbf{D}_{r;0}\bar{\mathbf{D}}^{-1}_{f;0}\bar{\mathbf{D}}_{b;0}\mathbf{u}^i_{r(0);0}|| \sim O(\delta\Delta),$$

$$||\hat{\mathbf{u}}^i_{r(0);0}|| = ||\Delta[\delta^2\bar{\mathbf{D}}_{r;1}\mathbf{E}^{-1}_{f;-1}\mathbf{D}_{r;0} - \bar{\mathbf{D}}_{f;0}]^{-1}\bar{\mathbf{D}}_{b;0}\mathbf{u}^i_{r(0);0}|| \approx || - \Delta\bar{\mathbf{D}}^{-1}_{f;0}\bar{\mathbf{D}}_{b;0}\mathbf{u}^i_{r(0);0}|| \sim O(\Delta),$$

$$||\mathbf{u}^{i}_{r(0);1}|| = ||\delta\Delta \mathbf{D}^{-1}_{f;1}\mathbf{D}_{b;1}\bar{\mathbf{E}}^{-1}_{f;-1}\bar{\mathbf{D}}_{r;0}\mathbf{u}^{i}_{r(0);0}|| \approx \delta\Delta||\mathbf{D}^{-1}_{f;1}\mathbf{D}_{b;1}\bar{\mathbf{D}}^{-1}_{f;-1}\bar{\mathbf{D}}_{r;0}\mathbf{u}^{i}_{r(0);0}|| \sim O(\delta\Delta),$$

$$||\hat{\mathbf{u}}^{i}_{r(0);-1}|| = ||-\delta\bar{\mathbf{E}}^{-1}_{f;-1}\bar{\mathbf{D}}_{r;0}\mathbf{u}^{i}_{r(0);0}|| \approx ||-\delta\bar{\mathbf{D}}^{-1}_{f;-1}\bar{\mathbf{D}}_{r;0}\mathbf{u}^{i}_{r(0);0}|| \sim O(\delta), \tag{19}$$

where

$$\mathbf{E}_{f;-1}(\lambda^{i}_{r(0)}) = \mathbf{D}_{f;-1}(\lambda^{i}_{r(0)}) - \Delta^{2}\mathbf{D}_{b;-1}(\lambda^{i}_{r(0)})\bar{\mathbf{D}}^{-1}_{f;1}(\lambda^{i}_{r(0)})\bar{\mathbf{D}}_{b;1}(\lambda^{i}_{r(0)}) \approx \mathbf{D}_{f;-1}(\lambda^{i}_{r(0)}).$$

For the rest of the modes with $r(m) = -r(0)$, $r(-1)$, $-r(1)$, $r(1)$ and $-r(-1)$, we can derive the similar relations to Eqs. (19), which are given in Appendix B. Similarly, the norm orders of the corresponding left latent (complex conjugate adjoint) vectors can be derived, which is not repeated here.

Table 1 summarizes the norm order for the right and left latent vectors. In practice, it is convenient to normalize the norms of the right and left latent vectors so that they become identical. The shaded area indicates the norm order for the right and left latent vectors obtained based on the $4N \times 4N$ reduced-order Hill's matrix. Note that the norm order analysis is consistent, irrespective of the order of approximation with Hill's infinite matrix. The thick-framed area represents the reference norm order associated with the strength of modes. The modes with reference norm order of 1, $O(1)$, (less than 1) may be referred to as the 'strong (weak) modes,' because the contribution of a modal response to the total response will be proportional to the corresponding modal norm. The modes with reference norm order of 1, Δ and δ are associated with symmetry, anisotropy and asymmetry of the system, respectively, and the rest of the modes are associated with the coupling of asymmetry and anisotropy. The weak modes tend to vanish as the degree of anisotropy and asymmetry diminishes. In particular, the modes of the coupled anisotropy and asymmetry are vulnerable to the degree of both asymmetry and anisotropy, so that they are not likely to be easily captured in practice.

Using the results in Table 1 and the matrix norm properties in Appendix C, we can express the norm of the dFRFs, in terms of norm order of residue matrices, as

$$\begin{aligned}
&||\mathbf{H}_{\mathbf{g};0\mathbf{p};0}(\mathrm{j}\omega)|| \\
&\leqslant \sum_{r=1}^{N}\sum_{i=B,F}\left\{\frac{||\mathbf{u}^{i}_{r(0);0}\bar{\mathbf{v}}^{i\mathrm{T}}_{r(0);0}||}{|\mathrm{j}\omega-\lambda^{i}_{r(0)}|}+\frac{||\mathbf{u}^{i}_{r(-1);0}\bar{\mathbf{v}}^{i\mathrm{T}}_{r(-1);0}||}{|\mathrm{j}\omega-\lambda^{i}_{r(-1)}|}+\frac{||\mathbf{u}^{i}_{r(1);0}\bar{\mathbf{v}}^{i\mathrm{T}}_{r(1);0}||}{|\mathrm{j}\omega-\lambda^{i}_{r(1)}|}+\frac{||\mathbf{u}^{i}_{-r(0);0}\bar{\mathbf{v}}^{i\mathrm{T}}_{-r(0);0}||}{|\mathrm{j}\omega-\lambda^{i}_{-r(0)}|}+\frac{||\mathbf{u}^{i}_{-r(-1);0}\bar{\mathbf{v}}^{i\mathrm{T}}_{-r(-1);0}||}{|\mathrm{j}\omega-\lambda^{i}_{-r(-1)}|}\right.\\
&\qquad\left.+\frac{||\mathbf{u}^{i}_{-r(1);0}\bar{\mathbf{v}}^{i\mathrm{T}}_{-r(1);0}||}{|\mathrm{j}\omega-\lambda^{i}_{-r(1)}|}\right\}\\
&\sim \sum_{r=1}^{N}\sum_{i=B,F}\left\{\frac{\boldsymbol{O}(\mathbf{1})}{|\mathrm{j}\omega-\lambda^{i}_{r(0)}|}+\frac{O(\delta^{2}\Delta^{2})}{|\mathrm{j}\omega-\lambda^{i}_{r(-1)}|}+\frac{O(\delta^{2}\Delta^{2})}{|\mathrm{j}\omega-\lambda^{i}_{r(1)}|}+\frac{O(\Delta^{2})}{|\mathrm{j}\omega-\lambda^{i}_{-r(0)}|}+\frac{O(\delta^{2}\Delta^{4})}{|\mathrm{j}\omega-\lambda^{i}_{-r(-1)}|}+\frac{O(\delta^{2})}{|\mathrm{j}\omega-\lambda^{i}_{-r(1)}|}\right\},
\end{aligned} \tag{20a}$$

Table 1
Modal strength in terms of vector norm order: use of $6N \times 6N$ reduced order Hill's matrix

$r(m)$, $r>0$ for $\lambda^{B,F}_{r(m)}$	$\lVert\hat{\mathbf{u}}^{i}_{r(m);1}\rVert$ $\lVert\hat{\mathbf{v}}^{i}_{r(m);1}\rVert$	$\lVert\mathbf{u}^{i}_{r(m);-1}\rVert$ $\lVert\mathbf{v}^{i}_{r(m);-1}\rVert$	$\lVert\hat{\mathbf{u}}^{i}_{r(m);0}\rVert$ $\lVert\hat{\mathbf{v}}^{i}_{r(m);0}\rVert$	$\lVert\mathbf{u}^{i}_{r(m);0}\rVert$ $\lVert\mathbf{v}^{i}_{r(m);0}\rVert$	$\lVert\hat{\mathbf{u}}^{i}_{r(m);-1}\rVert$ $\lVert\hat{\mathbf{v}}^{i}_{r(m);-1}\rVert$	$\lVert\mathbf{u}^{i}_{r(m);1}\rVert$ $\lVert\mathbf{v}^{i}_{r(m);1}\rVert$
$r(0)$	$O(\delta^{2}\Delta)$	$\boldsymbol{O}(\delta\boldsymbol{\Delta})$	$\boldsymbol{O}(\boldsymbol{\Delta})$	$\underline{\boldsymbol{O}(1)}$	$\boldsymbol{O}(\boldsymbol{\delta})$	$O(\delta\Delta)$
$-r(0)$	$O(\delta\Delta)$	$\boldsymbol{O}(\boldsymbol{\delta})$	$\boldsymbol{O}(1)$	$\underline{\boldsymbol{O}(\boldsymbol{\Delta})}$	$\boldsymbol{O}(\boldsymbol{\delta\Delta})$	$O(\delta\Delta^{2})$
$r(-1)$	$O(\Delta)$	$\boldsymbol{O}(1)$	$\boldsymbol{O}(\boldsymbol{\delta})$	$\underline{\boldsymbol{O}(\boldsymbol{\delta\Delta})}$	$\boldsymbol{O}(\boldsymbol{\delta}^{2}\boldsymbol{\Delta})$	$O(\delta^{2}\Delta^{2})$
$-r(1)$	$O(\delta^{2}\Delta^{2})$	$\boldsymbol{O}(\boldsymbol{\delta}^{2}\boldsymbol{\Delta})$	$\boldsymbol{O}(\boldsymbol{\delta\Delta})$	$\underline{\boldsymbol{O}(\boldsymbol{\delta})}$	$\boldsymbol{O}(1)$	$O(\Delta)$
$r(1)$	$O(\delta^{2}\Delta^{3})$	$O(\delta^{2}\Delta^{2})$	$O(\delta\Delta^{2})$	$\underline{\boldsymbol{O}(\boldsymbol{\delta\Delta})}$	$O(\Delta)$	$O(1)$
$-r(-1)$	$O(1)$	$O(\Delta)$	$O(\delta\Delta)$	$\underline{\boldsymbol{O}(\boldsymbol{\delta\Delta}^{2})}$	$O(\delta^{2}\Delta^{2})$	$O(\delta^{2}\Delta^{3})$

Bold italic indicates the results from $4N \times 4N$ reduced-order Hill's matrix and the underlined area indicates the representative modal strengths.

$$
\begin{aligned}
&||\mathbf{H}_{\hat{\mathbf{g}};0\mathbf{p};0}(\mathrm{j}\omega)|| \\
&\leqslant \sum_{r=1}^{N}\sum_{i=B,F}\left\{ \frac{||\mathbf{u}^{i}_{r(0);0}\bar{\hat{\mathbf{v}}}^{i\mathrm{T}}_{r(0);0}||}{|\mathrm{j}\omega-\lambda^{i}_{r(0)}|} + \frac{||\mathbf{u}^{i}_{r(-1);0}\bar{\hat{\mathbf{v}}}^{i\mathrm{T}}_{r(-1);0}||}{|\mathrm{j}\omega-\lambda^{i}_{r(-1)}|} + \frac{||\mathbf{u}^{i}_{r(1);0}\bar{\hat{\mathbf{v}}}^{i\mathrm{T}}_{r(1);0}||}{|\mathrm{j}\omega-\lambda^{i}_{r(1)}|} + \frac{||\mathbf{u}^{i}_{-r(0);0}\bar{\hat{\mathbf{v}}}^{i\mathrm{T}}_{-r(0);0}||}{|\mathrm{j}\omega-\lambda^{i}_{-r(0)}|} + \frac{||\mathbf{u}^{i}_{-r(-1);0}\bar{\hat{\mathbf{v}}}^{i\mathrm{T}}_{-r(-1);0}||}{|\mathrm{j}\omega-\lambda^{i}_{-r(-1)}|} \right. \\
&\qquad \left. + \frac{||\mathbf{u}^{i}_{-r(1);0}\bar{\hat{\mathbf{v}}}^{i\mathrm{T}}_{-r(1);0}||}{|\mathrm{j}\omega-\lambda^{i}_{-r(1)}|} \right\} \\
&\sim \sum_{r=1}^{N}\sum_{i=B,F}\left\{ \frac{\boldsymbol{O(\Delta)}}{|\mathrm{j}\omega-\lambda^{i}_{r(0)}|} + \frac{O(\delta^2\Delta)}{|\mathrm{j}\omega-\lambda^{i}_{r(-1)}|} + \frac{O(\delta^2\Delta^3)}{|\mathrm{j}\omega-\lambda^{i}_{r(1)}|} + \frac{\boldsymbol{O(\Delta)}}{|\mathrm{j}\omega-\lambda^{i}_{-r(0)}|} + \frac{O(\delta^2\Delta^3)}{|\mathrm{j}\omega-\lambda^{i}_{-r(-1)}|} + \frac{O(\delta^2\Delta)}{|\mathrm{j}\omega-\lambda^{i}_{-r(1)}|} \right\},
\end{aligned} \tag{20b}
$$

$$
\begin{aligned}
&||\mathbf{H}_{\hat{\mathbf{g}};-1\mathbf{p};0}(\mathrm{j}\omega)|| \\
&\leqslant \sum_{r=1}^{N}\sum_{i=B,F}\left\{ \frac{||\mathbf{u}^{i}_{r(0);0}\bar{\hat{\mathbf{v}}}^{i\mathrm{T}}_{r(0);-1}||}{|\mathrm{j}\omega-\lambda^{i}_{r(0)}|} + \frac{||\mathbf{u}^{i}_{r(-1);0}\bar{\hat{\mathbf{v}}}^{i\mathrm{T}}_{r(-1);-1}||}{|\mathrm{j}\omega-\lambda^{i}_{r(-1)}|} + \frac{||\mathbf{u}^{i}_{r(1);0}\bar{\hat{\mathbf{v}}}^{i\mathrm{T}}_{r(1);-1}||}{|\mathrm{j}\omega-\lambda^{i}_{r(1)}|} + \frac{||\mathbf{u}^{i}_{-r(0);0}\bar{\hat{\mathbf{v}}}^{i\mathrm{T}}_{-r(0);-1}||}{|\mathrm{j}\omega-\lambda^{i}_{-r(0)}|} + \frac{||\mathbf{u}^{i}_{-r(-1);0}\bar{\hat{\mathbf{v}}}^{i\mathrm{T}}_{-r(-1);-1}||}{|\mathrm{j}\omega-\lambda^{i}_{-r(-1)}|} \right. \\
&\qquad \left. + \frac{||\mathbf{u}^{i}_{-r(1);0}\bar{\hat{\mathbf{v}}}^{i\mathrm{T}}_{-r(1);-1}||}{|\mathrm{j}\omega-\lambda^{i}_{-r(1)}|} \right\} \\
&\sim \sum_{r=1}^{N}\sum_{i=B,F}\left\{ \frac{\boldsymbol{O(\delta)}}{|\mathrm{j}\omega-\lambda^{i}_{r(0)}|} + \frac{O(\delta^3\Delta^2)}{|\mathrm{j}\omega-\lambda^{i}_{r(-1)}|} + \frac{O(\delta\Delta^2)}{|\mathrm{j}\omega-\lambda^{i}_{r(1)}|} + \frac{O(\delta\Delta^2)}{|\mathrm{j}\omega-\lambda^{i}_{-r(0)}|} + \frac{O(\delta^3\Delta^4)}{|\mathrm{j}\omega-\lambda^{i}_{-r(-1)}|} + \frac{\boldsymbol{O(\delta)}}{|\mathrm{j}\omega-\lambda^{i}_{-r(1)}|} \right\},
\end{aligned} \tag{20c}
$$

Table 2
Eigenvalues of the simple gyroscopic rotor with both stationary and rotating asymmetry: $N=1$, $\Omega=0.5$

Eigenvalues						
Reduced matrix order					Remarks	Mode
2 N	4 N	6 N	8 N	10N		
–	–	–	–	−0.0255−j5.3554	$\sim\bar{\lambda}^F-\mathrm{j}4\Omega$	$\lambda^F_{-1(-2)}$
–	–	–	–	−0.0127−j3.2858	$\sim\bar{\lambda}^B-\mathrm{j}4\Omega$	$\lambda^B_{-1(-2)}$
–	–	–	−0.0291−j2.6861	−0.0269−j2.6712	$\sim\lambda^F-\mathrm{j}4\Omega$	$\lambda^F_{1(-2)}$
–	–	–	−0.0143−j4.7477	−0.0146−j4.7266	$\sim\lambda^B-\mathrm{j}4\Omega$	$\lambda^B_{1(-2)}$
–	–	−0.0255−j3.3554	−0.0254−j3.3359	−0.0273−j3.3281	$\sim\bar{\lambda}^F-\mathrm{j}2\Omega$	$\lambda^F_{-1(-1)}$
–	–	−0.0127−j1.2858	−0.0131−j1.2812	−0.0131−j1.2812	$\sim\bar{\lambda}^B-\mathrm{j}2\Omega$	$\lambda^B_{-1(-1)}$
–	−0.0291−j0.6861	−0.0269−j0.6712	−0.0269−j0.6712	−0.0269−j0.6712	$\sim\lambda^F-\mathrm{j}2\Omega$	$\lambda^F_{1(-1)}$
–	−0.0143−j2.7477	−0.0146−j2.7266	−0.0112−j2.7162	−0.0131−j2.7187	$\sim\lambda^B-\mathrm{j}2\Omega$	$\lambda^B_{1(-1)}$
−0.0255−j1.3553	−0.0254−j1.3359	−0.0273−j1.3281	−0.0269−j1.3287	−0.0269−j1.3287	$\bar{\lambda}^F$	$\lambda^F_{-1(0)}$
−0.0255+j0.7226	−0.0112+j0.7161	−0.0131+j0.7187	−0.0131+j0.7187	−0.0131+j0.7187	$\bar{\lambda}^B$	$\lambda^B_{-1(0)}$
−0.0255−j0.7226	**−0.0112−j0.7161**	**−0.0131−j0.7187**	**−0.0131−j0.7187**	**−0.0131−j0.7187**	$\boldsymbol{\lambda^B}$	$\boldsymbol{\lambda^B_{1(0)}}$
−0.0255+j1.3553	**−0.0254+j1.3359**	**−0.0273+j1.3281**	**−0.0269+j1.3287**	**−0.0269+j1.3287**	$\boldsymbol{\lambda^F}$	$\boldsymbol{\lambda^F_{1(0)}}$
–	−0.0143+j2.7477	−0.0146+j2.7266	−0.0112+j2.7162	−0.0131+j2.7187	$\sim\bar{\lambda}^B+\mathrm{j}2\Omega$	$\lambda^B_{-1(1)}$
–	−0.0291+j0.6861	−0.0269+j0.6712	−0.0269+j0.6712	−0.0269+j0.6712	$\sim\bar{\lambda}^F+\mathrm{j}2\Omega$	$\lambda^F_{-1(1)}$
–	–	−0.0127+j1.2858	−0.0131+j1.2812	−0.0131+j1.2812	$\sim\lambda^B+\mathrm{j}2\Omega$	$\lambda^B_{1(1)}$
–	–	−0.0255+j3.3554	−0.0254+j3.3359	−0.0273+j3.3281	$\sim\lambda^F+\mathrm{j}2\Omega$	$\lambda^F_{1(1)}$
–	–	–	−0.0143+j4.7477	−0.0146+j4.7266	$\sim\bar{\lambda}^B+\mathrm{j}4\Omega$	$\lambda^B_{-1(2)}$
–	–	–	−0.0291+j2.6861	−0.0269+j2.6712	$\sim\bar{\lambda}^F+\mathrm{j}4\Omega$	$\lambda^F_{-1(2)}$
–	–	–	–	−0.0127+j3.2858	$\sim\lambda^B+\mathrm{j}4\Omega$	$\lambda^B_{1(2)}$
–	–	–	–	−0.0255+j5.3554	$\sim\lambda^F+\mathrm{j}4\Omega$	$\lambda^F_{1(2)}$

Bold indicates the basic (strong) modes.

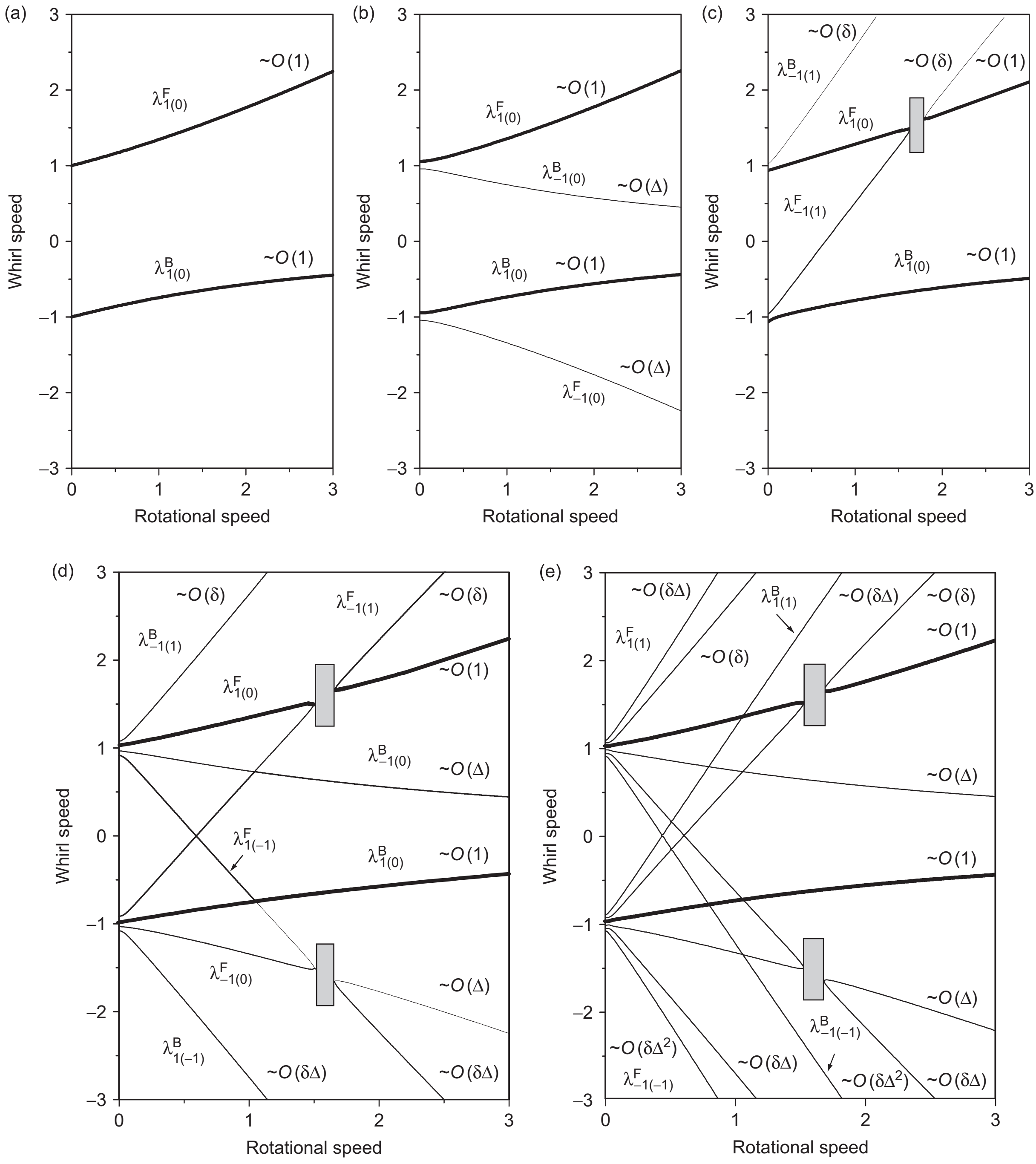

Fig. 2. Whirl speed charts for (a) isotropic rotor ($\Delta = \delta = 0$), (b) anisotropic rotor ($\Delta = 0.1,\ \delta = 0$), (c) asymmetric rotor ($\Delta = 0,\ \delta = 0.1$) and (d, e) simple general rotor ($\Delta = \delta = 0.1$) calculated from 4×4 and 6×6 reduced order Hill's matrix, respectively: thick and thin lines indicate strong and weak modes, respectively; shaded area indicates the unstable speed region.

$$
\begin{aligned}
&||\mathbf{H}_{\mathbf{g};-1\,\mathbf{p};0}(\mathrm{j}\omega)|| \\
&\leqslant \sum_{r=1}^{N}\sum_{i=B,F}\left\{\frac{||\mathbf{u}^{i}_{r(0);0}\bar{\mathbf{v}}^{i\mathrm{T}}_{r(0);1}||}{|\mathrm{j}\omega-\lambda^{i}_{r(0)}|}+\frac{||\mathbf{u}^{i}_{r(-1);0}\bar{\mathbf{v}}^{i\mathrm{T}}_{r(-1);1}||}{|\mathrm{j}\omega-\lambda^{i}_{r(-1)}|}+\frac{||\mathbf{u}^{i}_{r(1);0}\bar{\mathbf{v}}^{i\mathrm{T}}_{r(1);1}||}{|\mathrm{j}\omega-\lambda^{i}_{r(1)}|}+\frac{||\mathbf{u}^{i}_{-r(0);0}\bar{\mathbf{v}}^{i\mathrm{T}}_{-r(0);1}||}{|\mathrm{j}\omega-\lambda^{i}_{-r(0)}|}+\frac{||\mathbf{u}^{i}_{-r(-1);0}\bar{\mathbf{v}}^{i\mathrm{T}}_{-r(-1);1}||}{|\mathrm{j}\omega-\lambda^{i}_{-r(-1)}|}\right. \\
&\qquad\left.+\frac{||\mathbf{u}^{i}_{-r(1);0}\bar{\mathbf{v}}^{i\mathrm{T}}_{-r(1);1}||}{|\mathrm{j}\omega-\lambda^{i}_{-r(1)}|}\right\} \\
&\sim \sum_{r=1}^{N}\sum_{i=B,F}\left\{\frac{O(\boldsymbol{\delta\Delta})}{|\mathrm{j}\omega-\lambda^{i}_{r(0)}|}+\frac{O(\delta^{3}\varDelta^{3})}{|\mathrm{j}\omega-\lambda^{i}_{r(-1)}|}+\frac{O(\boldsymbol{\delta\Delta})}{|\mathrm{j}\omega-\lambda^{i}_{r(1)}|}+\frac{O(\delta\varDelta^{3})}{|\mathrm{j}\omega-\lambda^{i}_{-r(0)}|}+\frac{O(\delta^{3}\varDelta^{5})}{|\mathrm{j}\omega-\lambda^{i}_{-r(-1)}|}+\frac{O(\boldsymbol{\delta\Delta})}{|\mathrm{j}\omega-\lambda^{i}_{-r(1)}|}\right\}, \qquad (20\mathrm{d})
\end{aligned}
$$

$$
\begin{aligned}
&||\mathbf{H}_{\mathbf{g};1\,\mathbf{p};0}(\mathrm{j}\omega)|| \\
&\leqslant \sum_{r=1}^{N}\sum_{i=B,F}\left\{\frac{||\mathbf{u}^{i}_{r(0);0}\bar{\mathbf{v}}^{i\mathrm{T}}_{r(0);-1}||}{|\mathrm{j}\omega-\lambda^{i}_{r(0)}|}+\frac{||\mathbf{u}^{i}_{r(-1);0}\bar{\mathbf{v}}^{i\mathrm{T}}_{r(-1);-1}||}{|\mathrm{j}\omega-\lambda^{i}_{r(-1)}|}+\frac{||\mathbf{u}^{i}_{r(1);0}\bar{\mathbf{v}}^{i\mathrm{T}}_{r(1);-1}||}{|\mathrm{j}\omega-\lambda^{i}_{r(1)}|}+\frac{||\mathbf{u}^{i}_{-r(0);0}\bar{\mathbf{v}}^{i\mathrm{T}}_{-r(0);-1}||}{|\mathrm{j}\omega-\lambda^{i}_{-r(0)}|}+\frac{||\mathbf{u}^{i}_{-r(-1);0}\bar{\mathbf{v}}^{i\mathrm{T}}_{-r(-1);-1}||}{|\mathrm{j}\omega-\lambda^{i}_{-r(-1)}|}\right. \\
&\qquad\left.+\frac{||\mathbf{u}^{i}_{-r(1);0}\bar{\mathbf{v}}^{i\mathrm{T}}_{-r(1);-1}||}{|\mathrm{j}\omega-\lambda^{i}_{-r(1)}|}\right\} \\
&\sim \sum_{r=1}^{N}\sum_{i=B,F}\left\{\frac{O(\boldsymbol{\delta\,\Delta})}{|\mathrm{j}\omega-\lambda^{i}_{r(0)}|}+\frac{O(\delta\,\boldsymbol{\Delta})}{|\mathrm{j}\omega-\lambda^{i}_{r(-1)}|}+\frac{O(\delta^{3}\varDelta^{3})}{|\mathrm{j}\omega-\lambda^{i}_{r(1)}|}+\frac{O(\boldsymbol{\delta\,\Delta})}{|\mathrm{j}\omega-\lambda^{i}_{-r(0)}|}+\frac{O(\delta\,\varDelta^{3})}{|\mathrm{j}\omega-\lambda^{i}_{-r(-1)}|}+\frac{O(\delta^{3}\varDelta)}{|\mathrm{j}\omega-\lambda^{i}_{-r(1)}|}\right\}, \qquad (20\mathrm{e})
\end{aligned}
$$

$$
\begin{aligned}
&||\mathbf{H}_{\hat{\mathbf{g}};1\,\mathbf{p};0}(\mathrm{j}\omega)|| \\
&\leqslant \sum_{r=1}^{N}\sum_{i=B,F}\left\{\frac{||\mathbf{u}^{i}_{r(0);0}\hat{\bar{\mathbf{v}}}^{i\mathrm{T}}_{r(0);1}||}{|\mathrm{j}\omega-\lambda^{i}_{r(0)}|}+\frac{||\mathbf{u}^{i}_{r(-1);0}\hat{\bar{\mathbf{v}}}^{i\mathrm{T}}_{r(-1);1}||}{|\mathrm{j}\omega-\lambda^{i}_{r(-1)}|}+\frac{||\mathbf{u}^{i}_{r(1);0}\hat{\bar{\mathbf{v}}}^{i\mathrm{T}}_{r(1);1}||}{|\mathrm{j}\omega-\lambda^{i}_{r(1)}|}+\frac{||\mathbf{u}^{i}_{-r(0);0}\hat{\bar{\mathbf{v}}}^{i\mathrm{T}}_{-r(0);1}||}{|\mathrm{j}\omega-\lambda^{i}_{-r(0)}|}+\frac{||\mathbf{u}^{i}_{-r(-1);0}\hat{\bar{\mathbf{v}}}^{i\mathrm{T}}_{-r(-1);1}||}{|\mathrm{j}\omega-\lambda^{i}_{-r(-1)}|}\right. \\
&\qquad\left.+\frac{||\mathbf{u}^{i}_{-r(1);0}\hat{\bar{\mathbf{v}}}^{i\mathrm{T}}_{-r(1);1}||}{|\mathrm{j}\omega-\lambda^{i}_{-r(1)}|}\right\} \\
&\sim \sum_{r=1}^{N}\sum_{i=B,F}\left\{\frac{O(\boldsymbol{\delta\,\Delta}^{2})}{|\mathrm{j}\omega-\lambda^{i}_{r(0)}|}+\frac{O(\boldsymbol{\delta\,\Delta}^{2})}{|\mathrm{j}\omega-\lambda^{i}_{r(-1)}|}+\frac{O(\delta^{3}\,\varDelta^{4})}{|\mathrm{j}\omega-\lambda^{i}_{r(1)}|}+\frac{O(\boldsymbol{\delta\Delta}^{2})}{|\mathrm{j}\omega-\lambda^{i}_{-r(0)}|}+\frac{O(\boldsymbol{\delta\Delta}^{2})}{|\mathrm{j}\omega-\lambda^{i}_{-r(-1)}|}+\frac{O(\delta^{3}\,\varDelta^{2})}{|\mathrm{j}\omega-\lambda^{i}_{-r(1)}|}\right\}, \qquad (20\mathrm{f})
\end{aligned}
$$

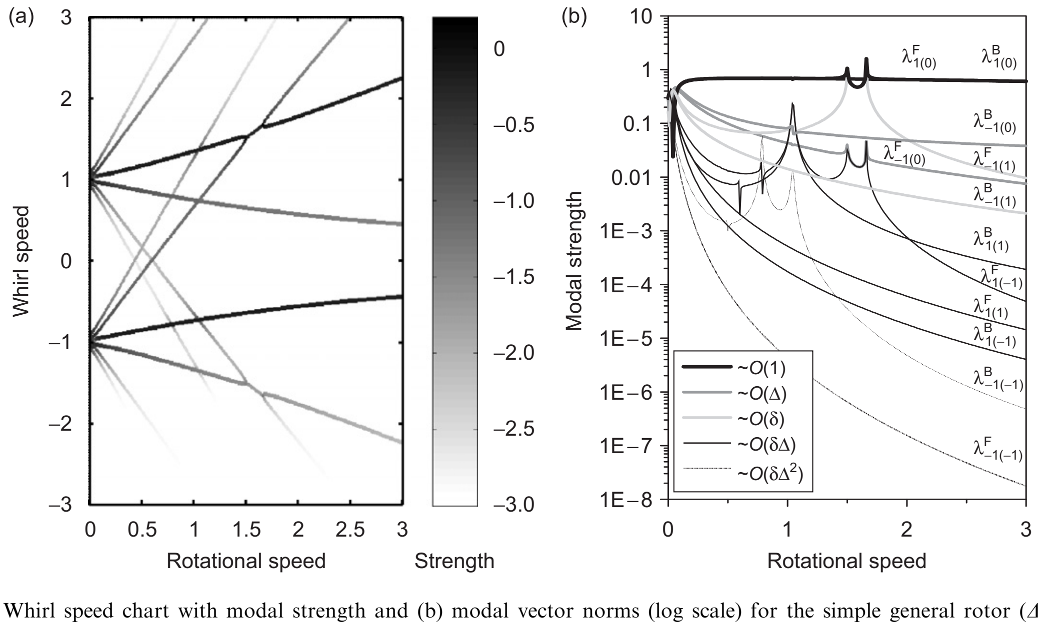

Fig. 3. (a) Whirl speed chart with modal strength and (b) modal vector norms (log scale) for the simple general rotor ($\varDelta=\delta=0.1$) calculated from 6×6 reduced order Hill's matrix.

where the modes with the bold-faced norm order indicate the dominant modes in the associated dFRMs.

From Eqs. (20), we can conclude that

(1) $\mathbf{H}_{\mathbf{g};0\mathbf{p};0}(\mathrm{j}\omega)$ is useful to identify the strong (and thus weak) modes,
(2) $\mathbf{H}_{\hat{\mathbf{g}};0\mathbf{p};0}(\mathrm{j}\omega)$ is a good indicator of degree of anisotropy, irrespective of presence of system asymmetry,
(3) $\mathbf{H}_{\hat{\mathbf{g}};-1\mathbf{p};0}(\mathrm{j}\omega)$ is a good indicator of degree of asymmetry, irrespective of presence of system anisotropy and
(4) $\mathbf{H}_{\mathbf{g};-1\mathbf{p};0}(\mathrm{j}\omega)$ and $\mathbf{H}_{\mathbf{g};1\mathbf{p};0}(\mathrm{j}\omega)$ are very sensitive to the coupled effect of system anisotropy and asymmetry.

So far, for demonstration purpose, the $6N \times 6N$ reduced order matrix equation of motion has been treated, but it can be easily extended to higher order matrix equation of motion. As the reduced order of Hill's infinite matrix increases, the accuracy of eigensolutions certainly improves, as will be demonstrated in Section 2

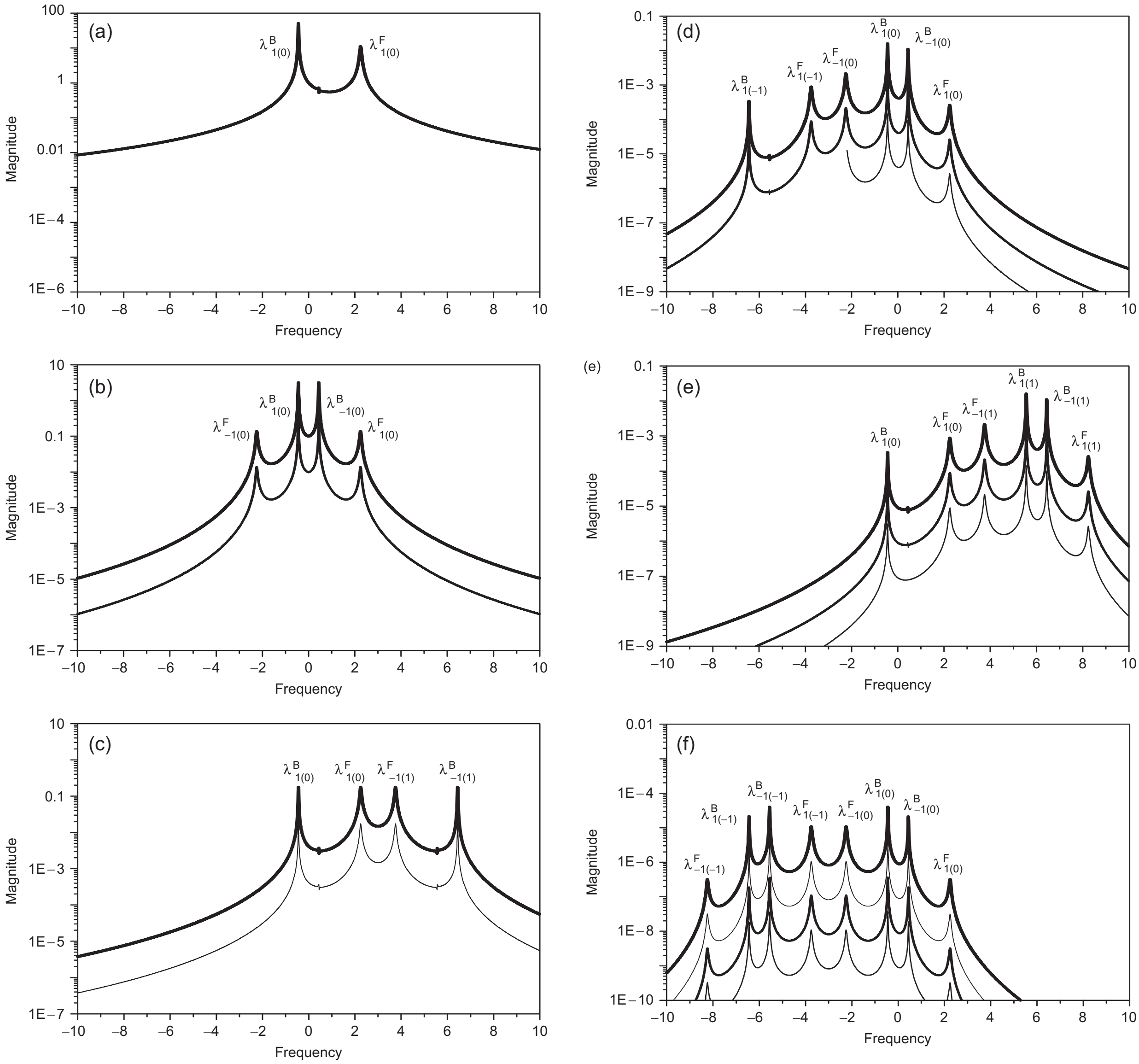

Fig. 4. dFRFs of the simple general gyroscopic rotor with the degree of asymmetry and anisotropy varied (—— $\delta = 0.01,\ \Delta = 0.01$, —— $\delta = 0.01,\ \Delta = 0.1$, —— $\delta = 0.1,\ \Delta = 0.01$, —— $\delta = \Delta = 0.1$): (a) $\left|\mathbf{H}_{\mathbf{g};0\mathbf{p};0}(\omega)\right|$, (b) $\left|\mathbf{H}_{\hat{\mathbf{g}};0\mathbf{p};0}(\omega)\right|$, (c) $\left|\mathbf{H}_{\hat{\mathbf{g}};-1\mathbf{p};0}(\omega)\right|$, (d) $\left|\mathbf{H}_{\mathbf{g};1\mathbf{p};0}(\omega)\right|$, (e) $\left|\mathbf{H}_{\mathbf{g};-1\mathbf{p};0}(\omega)\right|$, (f) $\left|\mathbf{H}_{\hat{\mathbf{g}};1\mathbf{p};0}(\omega)\right|$; $\alpha = 0.6,\ \zeta = 0.02,\ \Omega = 3.0$.

(refer to Table 2). And it has been found that the $6N \times 6N$ reduced order matrix equation of motion gives sufficiently accurate eigensolutions of practical interest [4]. One of the benefits of the suggested norm order analysis is that the norm order of eigenvectors obtained from a lower order matrix equation remains unchanged as the matrix order increases, as shown in Table 1. In conclusion, the modal strength given in Table 1 is always valid, irrespective of the reduced matrix order used for norm order analysis of eigenvectors.

4. Numerical examples

In this section, two rotor models are treated in order to demonstrate the analytical findings in the previous sections.

First, consider the simplest analytical form of general rotor model whose equation of motion is given by

$$\ddot{\mathbf{p}}(t) + (2\zeta - \mathrm{j}\alpha\Omega)\dot{\mathbf{p}}(t) + \mathbf{p}(t) + \delta \mathrm{e}^{\mathrm{j}2\Omega t}\bar{\mathbf{p}}(t) + \Delta\bar{\mathbf{p}}(t) = \mathbf{g}(t), \tag{21}$$

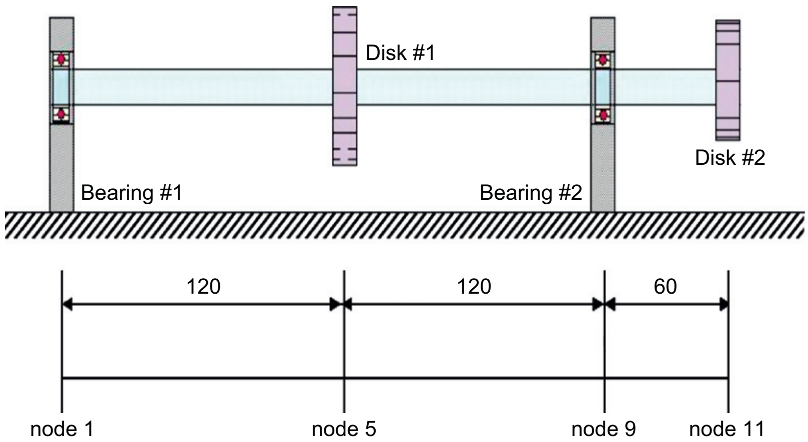

Fig. 5. Configuration of the open cracked flexible shaft-anisotropic-bearing system (FEM model).

Table 3
Specifications of the FE model

Mesh data			
# of elements = 10, # of disks = 2, # of bearings = 2			
Shaft			
Length = 30 cm, diameter = 1.0 cm			
Density = 7850 kg/m^3 , Young's modulus = 2.07×10^{11} N/m^2			
Disk			
Node number	Mass (kg)	Pol. inertia (kg m^2)	Dia. inertia (kg m^2)
5,11 (identical)	0.617	7.71×10^{-4}	3.90×10^{-4}
Bearings (degree of anisotropy; $\Delta = 0.13$)			
Node number	Stiffness (N/m)		Damping (N s/m)
1,9 (identical)	$k_{yy} = 3.0 \times 10^8$, $k_{zz} = 2.3 \times 10^8$ $k_{yz} = k_{zy} = 0$		c_{yy}, $c_{zz} = 4.0 \times 10^3$ $c_{yz} = c_{zy} = 0$
Open crack			
Node = 5, Crack depth ratio (a/D) = 0.4			

where ζ is the damping ratio, $\alpha\Omega$ represents the gyroscopic moment, Ω is the rotational speed, and, δ and Δ represent the degree of asymmetry and anisotropy, respectively. The nominal parameters used in the numerical simulations are $\alpha = 0.6$, $\zeta = 0.02$ and $\Delta = \delta = 0.1$.

Table 2 shows the eigenvalues of the simple gyroscopic rotor with both stationary and rotating asymmetry at $\Omega = 0.5$ with the reduced order of Hill's matrix varied from $2N$ ($N = 1$ for this case) to $10N$. In Table 2, the framed area indicates the basic (strong) modes and the rest of the modes are either complex conjugates or modulated values of the basic modes. As the reduced order of Hill's matrix increases, the calculated eigenvalues tend to converge to actual values. From the results given in Table 2, we can conclude that use of $6N \times 6N$ reduced-order Hill's matrix is sufficient to estimate the modes of importance with fair accuracy, including the modes of symmetry, anisotropy, asymmetry and the coupled asymmetry.

Figs. 2(a)–(c) compare the whirl speeds for the isotropic ($\Delta = \delta = 0$) anisotropic ($\Delta = 0.1, \delta = 0$), and asymmetric ($\Delta = 0, \delta = 0.1$) analysis rotor models. Note that the strong modes associated with the rotor symmetry are clearly seen in Fig. 2(a), the weak (complex conjugate) modes associated with the rotor anisotropy appear in Fig. 2(b) and the weak (modulated) modes associated with the rotor asymmetry appear in Fig. 2(c), where there exists an unstable speed region. In Fig. 2, the strong and weak modes are indicated by

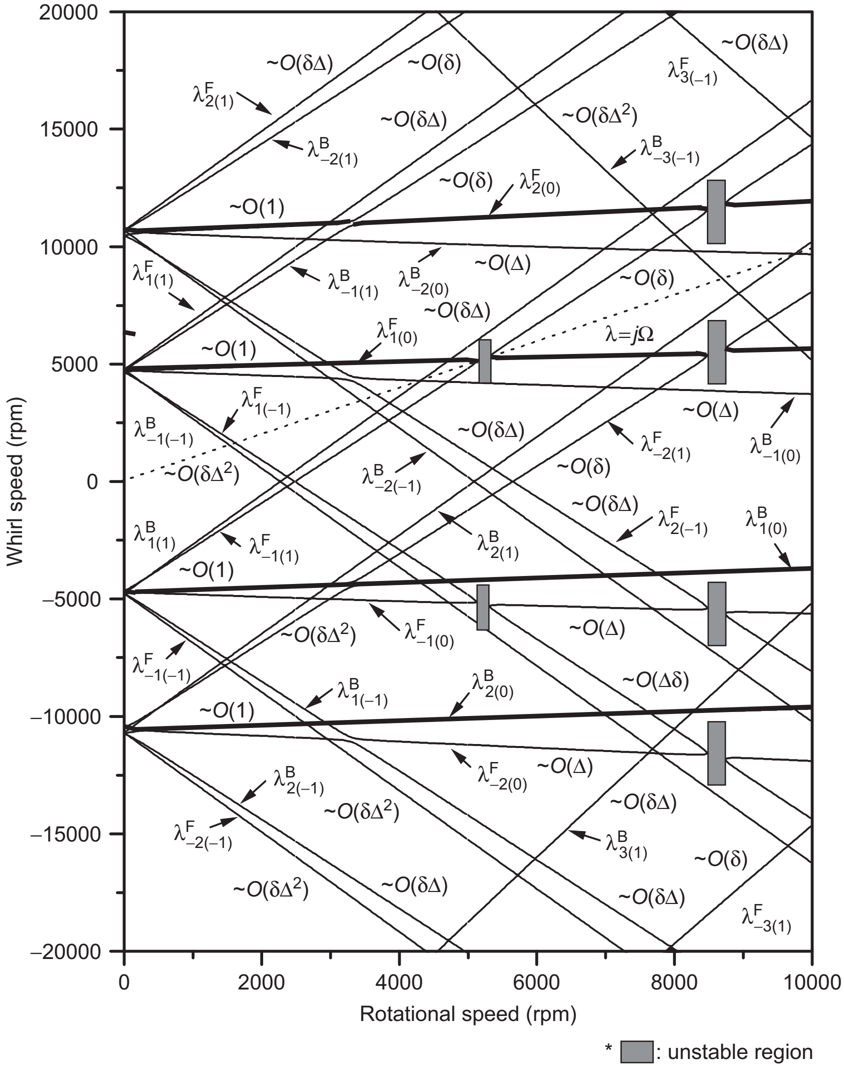

Fig. 6. Whirl speed chart of the flexible asymmetrical rotor with anisotropic stator (FEM model) (▬▬ strong mode —— weak mode).

thick and thin lines, respectively, marked with the norm order of the corresponding modal vectors. Note that, for these three cases, the Hill's matrix reduces to a matrix of finite order, leading to accurate estimation of eigenvalues. Figs. 2(d) and (e) compare the whirl speeds for the general analysis rotor model ($\Delta = \delta = 0.1$) calculated from the reduced Hill's matrix of order 4 and 6. Note that the increase in order of reduced Hill's matrix introduces additional modes of the coupled asymmetry with higher modal norm order of smallness. The weak modes associated with higher modal norm order than $\Delta\delta$ are considered to be relatively less important in potential contributions to the system response. Fig. 3(a) is the whirl speed chart of the simple general rotor ($\Delta = \delta = 0.1$) with the level of modal strength indicated by the gray-scaled lines, according to the norm of the bi-orthonormalized modal vectors calculated from the 6×6 reduced-order Hill's matrix, as shown in Fig. 3(b). Note that the modal norm order analysis is valid over the whole range of rotational speed, except the low-speed region where the gyroscopic moment effect becomes negligibly small and near the unstable speed region.

Fig. 4 shows the six different types of dFRFs, calculated according to Eqs. (20), with the degree of anisotropy and asymmetry varied. Comparing the results in Figs. 3 with the modal norm order analysis given in Eqs. (20), we can conclude that the modal contributions in dFRFs are consistent with the analytical findings given in Eqs. (20), conforming to the observations in the previous section. That is (1) $\mathbf{H}_{\mathbf{g};0\mathbf{p};0}(\mathrm{j}\omega)$ shown in Fig. 4(a) clearly identifies the basic (strong) modes of symmetry, irrespective of the presence of anisotropy and asymmetry; (2) $\mathbf{H}_{\hat{\mathbf{g}};0\mathbf{p};0}(\mathrm{j}\omega)$ shown in Fig. 4(b) clearly identifies the weak modes of anisotropy together with the modes of symmetry, irrespective of the degree of asymmetry; (3) $\mathbf{H}_{\hat{\mathbf{g}};-1\mathbf{p};0}(\mathrm{j}\omega)$ shown in Fig. 4(c) identifies the weak modes of asymmetry together with the modes of symmetry, irrespective of the degree of anisotropy; (4) the magnitudes of $\mathbf{H}_{\mathbf{g};-1\mathbf{p};0}(\mathrm{j}\omega)$, $\mathbf{H}_{\mathbf{g};1\mathbf{p};0}(\mathrm{j}\omega)$ and $\mathbf{H}_{\hat{\mathbf{g}};1\mathbf{p};0}(\mathrm{j}\omega)$ shown in Fig. 4(d)–(f), respectively, which are very sensitive to the coupled effect of system anisotropy and asymmetry, are of higher order of smallness; (5) the degree of asymmetry and anisotropy can be well identified from $\mathbf{H}_{\hat{\mathbf{g}};0\mathbf{p};0}(\mathrm{j}\omega)$ and $\mathbf{H}_{\hat{\mathbf{g}};-1\mathbf{p};0}(\mathrm{j}\omega)$, respectively.

The second numerical example is the FE model of a flexible asymmetric rotor supported by anisotropic bearings with an open crack as shown in Fig. 5 [9]. The material and geometrical properties are listed in

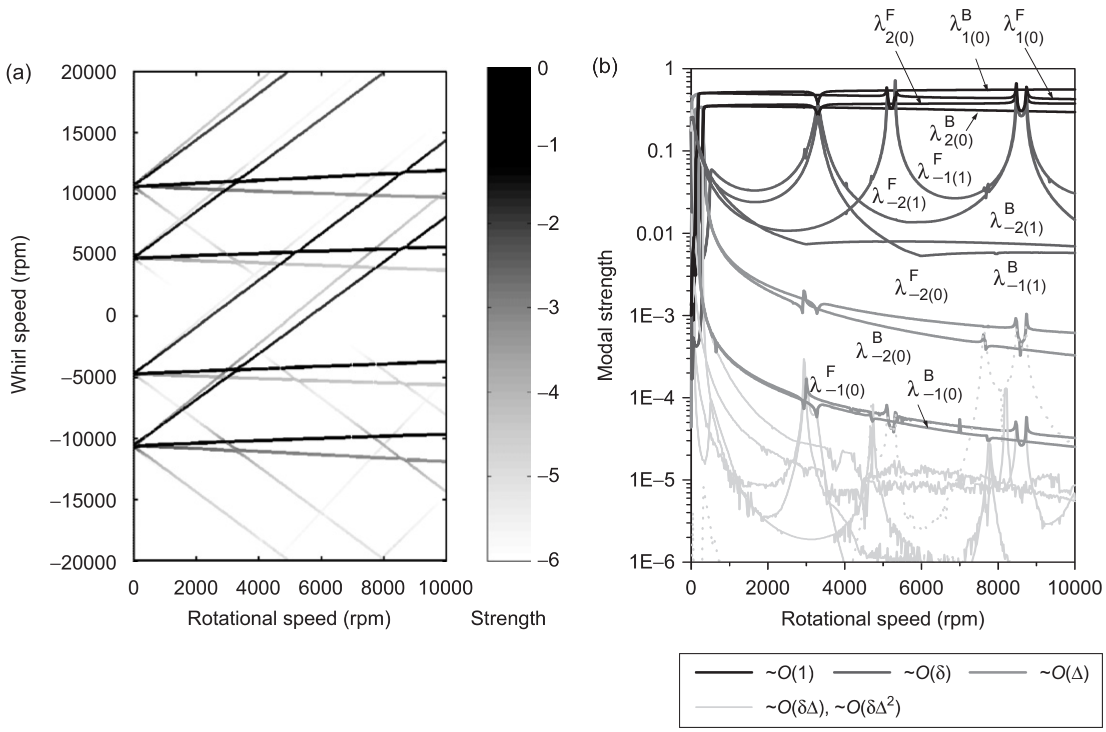

Fig. 7. (a) Whirl speed chart with modal strength and (b) the modal vector norm (log scale) for the flexible asymmetrical rotor with anisotropic stator (FEM model).

Table 3. The model consists of ten Rayleigh beam elements, two rigid disks and two anisotropic bearings. We assume that the shaft stiffness asymmetry is caused by an open transverse crack with the crack depth (a) to shaft diameter (D) ratio $a/D = 0.4$ developed at node #5 [10]. Fig. 6 shows the whirl speeds calculated from the reduced Hill's matrix of order 6. Note that the whirl speed map is rather crowded with the basic, complex conjugate and modulated modes, with multiple unstable speed regions, due to the increased degree of freedom N. It may be difficult to identify the importance of modes in possible modal contributions to forced response of the rotor system from the conventional whirl speed chart, Fig. 6. Fig. 7 checks the modal strengths by the representative modal vector norm value, which is marked by the gray-scaled lines.

Fig. 8 displays the six different types of typical dFRFs for the asymmetric flexible rotor with the anisotropy given in Table 3, as the crack depth ratio a/D takes the values of 0.1, 0.2, 0.3 and 0.4. The dFRFs were calculated at the rotational speed of 4200 rev/min for the case with sensor node #5, exciter node #11 and crack node #5. From Fig. 8, we can conclude, for weakly anisotropic and asymmetric rotors, that

(1) $\mathbf{H}_{\mathbf{g};0\mathbf{p};0}(\mathrm{j}\omega)$ shown in Fig. 8(a) is almost insensitive to the crack growth, revealing the strong, basic modes of symmetry, but hiding the weak modes.
(2) $\mathbf{H}_{\hat{\mathbf{g}};0\mathbf{p};0}(\mathrm{j}\omega)$ shown in Fig. 8(b) is almost insensitive to the crack growth, revealing the weak modes of anisotropy as well as the strong modes of symmetry, but hiding the rest of weak modes.

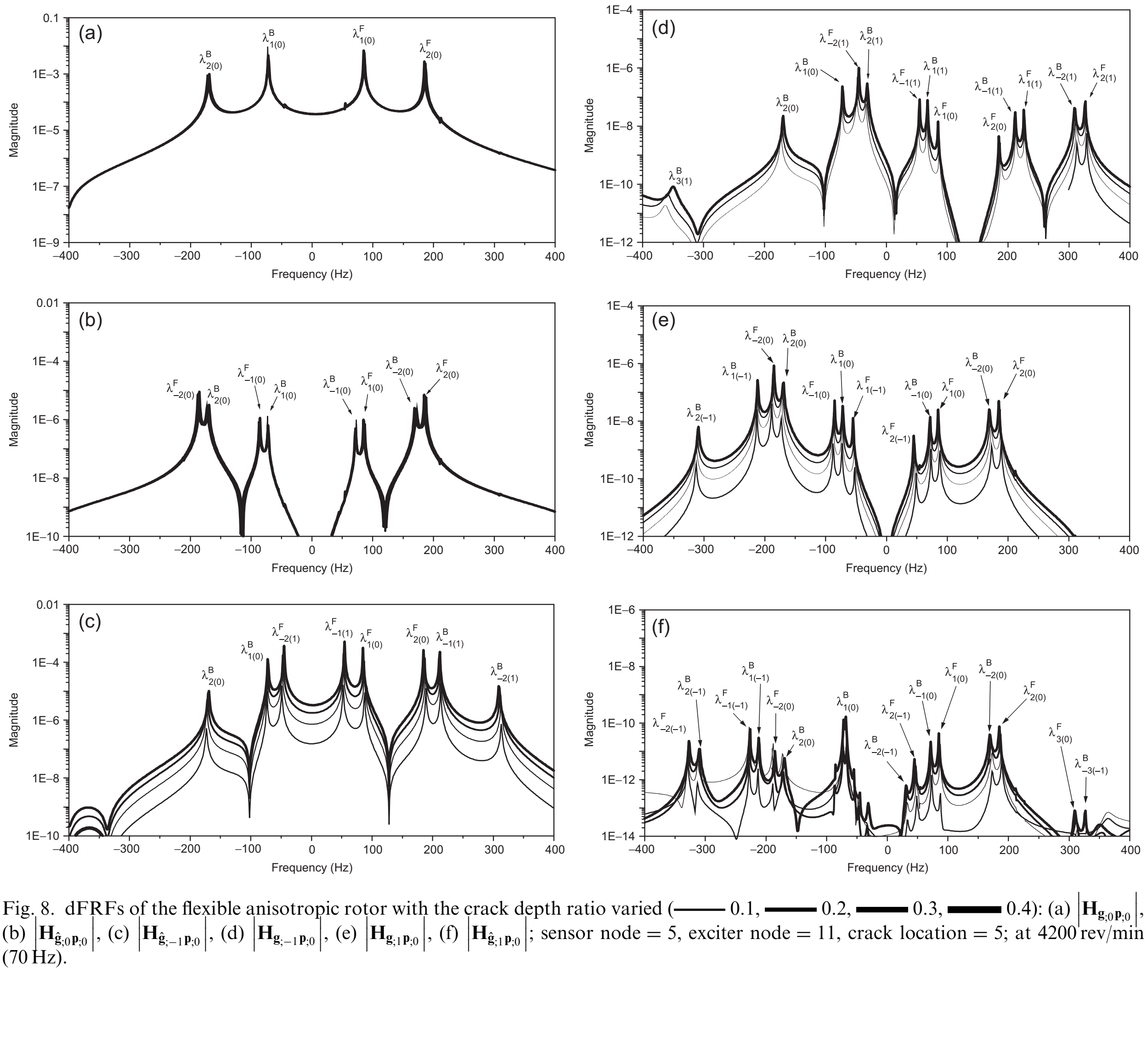

Fig. 8. dFRFs of the flexible anisotropic rotor with the crack depth ratio varied (—— 0.1, —— 0.2, —— 0.3, —— 0.4): (a) $\left|\mathbf{H}_{\mathbf{g};0\mathbf{p};0}\right|$, (b) $\left|\mathbf{H}_{\hat{\mathbf{g}};0\mathbf{p};0}\right|$, (c) $\left|\mathbf{H}_{\hat{\mathbf{g}};-1\mathbf{p};0}\right|$, (d) $\left|\mathbf{H}_{\mathbf{g};-1\mathbf{p};0}\right|$, (e) $\left|\mathbf{H}_{\mathbf{g};1\mathbf{p};0}\right|$, (f) $\left|\mathbf{H}_{\hat{\mathbf{g}};1\mathbf{p};0}\right|$; sensor node = 5, exciter node = 11, crack location = 5; at 4200 rev/min (70 Hz).

(3) $\mathbf{H}_{\hat{\mathbf{g}};-1\mathbf{p};0}(j\omega)$ shown in Fig. 8(c) is very sensitive to the crack growth, revealing the weak modes of asymmetry as well as the strong modes of symmetry, but hiding the rest of weak modes.
(4) $\mathbf{H}_{\mathbf{g};-1\mathbf{p};0}(j\omega)$, $\mathbf{H}_{\mathbf{g};1\mathbf{p};0}(j\omega)$ and $\mathbf{H}_{\hat{\mathbf{g}};1\mathbf{p};0}(j\omega)$ shown in Figs. 8(d)–(f), respectively, is very sensitive to the coupling between the anisotropy and asymmetry, revealing all modes. These plots are so complicated that perhaps their detailed information may not be useful in practice, but they certainly evidence the presence of both asymmetry and anisotropy.

5. Conclusions

A new method of ordering the importance of modes for rotordynamic analysis is proposed, based on the norm of modal vectors defined in the complex coordinate system. And, it is shown that, using the concept of modal norm order, modes can be properly classified according to the modal strength and that the dFRFs are useful in identifying the modal strength more in detail, i.e. the modes of symmetry, anisotropy, asymmetry and coupled asymmetry. The concept of modal strength can be used for improving the whirl speed chart (Campbell diagram) for rotors with anisotropy and/or asymmetry in the sense that the modes with potential contribution to forced response of rotor can be readily identified from the otherwise over-crowded chart.

Appendix A. Stationarity of latent roots

Consider a pair of the typical homogeneous matrix–vector equations in Eq. (13a) given as

$$\Delta\mathbf{D}_{b;0}(\lambda^i_{r(m)})\hat{\mathbf{u}}^i_{r(m);0} + \mathbf{D}_{f;0}(\lambda^i_{r(m)})\mathbf{u}^i_{r(m);0} + \delta\mathbf{D}_{r;1}(\lambda^i_{r(m)})\hat{\mathbf{u}}^i_{r(m);-1} = \mathbf{0}, \tag{A.1}$$

$$\delta\bar{\mathbf{D}}_{r;1}(\lambda^i_{r(m)})\mathbf{u}^i_{r(m);-1} + \bar{\mathbf{D}}_{f;0}(\lambda^i_{r(m)})\hat{\mathbf{u}}^i_{r(m);0} + \Delta\bar{\mathbf{D}}_{b;0}(\lambda^i_{r(m)})\mathbf{u}^i_{r(m);0} = \mathbf{0} \tag{A.2}$$

with

$$\mathbf{D}_{i;n}(\lambda) = \lambda^2\mathbf{M}_i + \lambda\mathbf{C}_{i;n} + \mathbf{K}_{i;n}; \quad i = \mathbf{r}, \mathbf{b}, \mathbf{f}; \quad n = 0, \pm1, \pm2, \ldots . \tag{A.3}$$

Let the perturbed latent roots and corresponding latent vectors be represented by

$$\lambda^i_{r(m)} = \lambda^{0i}_{r(m)} + (\varDelta, \delta)\lambda^{1i}_{r(m)} + (\varDelta, \delta)^2\lambda^{2i}_{r(m)} + O(\varDelta, \delta)^3, \quad r = \pm1, \ldots, \pm N, \tag{A.4}$$

$$\begin{Bmatrix} \mathbf{u}^i_{r(m);-1} \\ \hat{\mathbf{u}}^i_{r(m);0} \\ \mathbf{u}^i_{r(m);0} \\ \hat{\mathbf{u}}^i_{r(m);-1} \end{Bmatrix} = \begin{Bmatrix} \mathbf{u}^{0i}_{r(m);-1} \\ \hat{\mathbf{u}}^{0i}_{r(m);0} \\ \mathbf{u}^{0i}_{r(m);0} \\ \hat{\mathbf{u}}^{0i}_{r(m);-1} \end{Bmatrix} + (\varDelta, \delta)\begin{Bmatrix} \mathbf{u}^{1i}_{r(m);-1} \\ \hat{\mathbf{u}}^{1i}_{r(m);0} \\ \mathbf{u}^{1i}_{r(m);0} \\ \hat{\mathbf{u}}^{1i}_{r(m);-1} \end{Bmatrix} + O(\varDelta, \delta)^2, \tag{A.5}$$

$$\begin{Bmatrix} \hat{\bar{\mathbf{v}}}^i_{r(m):0} \\ \bar{\mathbf{v}}^i_{r(m):0} \end{Bmatrix} = \begin{Bmatrix} \hat{\bar{\mathbf{v}}}^{0i}_{r(m):0} \\ \bar{\mathbf{v}}^{0i}_{r(m):0} \end{Bmatrix} + (\varDelta, \delta)\begin{Bmatrix} \hat{\bar{\mathbf{v}}}^{1i}_{r(m):0} \\ \bar{\mathbf{v}}^{1i}_{r(m):0} \end{Bmatrix} + O(\varDelta, \delta)^2. \tag{A.6}$$

Substituting Eqs. (A.4) and (A.5) into Eq. (A.1), we obtain, using the Taylor-series expansion with respect to the unperturbed states

$$\mathbf{D}_{f;0}(\lambda^i_{r(m)})\mathbf{u}^i_{r(m);0} = \mathbf{D}_{f;0}(\lambda^{0i}_{r(m)})\mathbf{u}^{0i}_{r(m);0} + (\varDelta, \delta)[\lambda^{1i}_{r(m)}\mathbf{D}'_{f;0}(\lambda^{0i}_{r(m)})\mathbf{u}^{0i}_{r(m);0} + \mathbf{D}_{f;0}(\lambda^{0i}_{r(m)})\mathbf{u}^{1i}_{r(m);0}] + O(\varDelta, \delta)^2 \tag{A.7}$$

leading to

$$\begin{aligned} \Delta\mathbf{D}_{b;0}\hat{\mathbf{u}}^i_{r(m);0} + \mathbf{D}_{f;0}\mathbf{u}^i_{r(m);0} + \delta\mathbf{D}_{r;1}\hat{\mathbf{u}}^i_{r(m);-1} = {}& \Delta\mathbf{D}_{b;0}(\lambda^{0i}_{r(m)})[\hat{\mathbf{u}}^{0i}_{r(m);0} + (\varDelta, \delta)\hat{\mathbf{u}}^{1i}_{r(m);0}] + \mathbf{D}_{f;0}(\lambda^{0i}_{r(m)})\mathbf{u}^{0i}_{r(m);0} \\ & + (\varDelta, \delta)[\lambda^{1i}_{r(m)}\mathbf{D}'_{f;0}(\lambda^{0i}_{r(m)})\mathbf{u}^{0i}_{r(m);0} + \mathbf{D}_{f;0}(\lambda^{0i}_{r(m)})\mathbf{u}^{1i}_{r(m);0}] \end{aligned}$$

$$
\begin{aligned}
&+\delta\mathbf{D}_{r;1}(\lambda^{0i}_{r(m)})[\hat{\mathbf{u}}^{0i}_{r(m);-1}+(\Delta,\delta)\hat{\mathbf{u}}^{1i}_{r(m);-1}]+O(\Delta,\delta)^2\\
&=\Delta\mathbf{D}_{b;0}(\lambda^{0i}_{r(m)})\hat{\mathbf{u}}^{0i}_{r(m);0}+\mathbf{D}_{f;0}(\lambda^{0i}_{r(m)})\mathbf{u}^{0i}_{r(m);0}+\delta\mathbf{D}_{r;1}(\lambda^{0i}_{r(m)})\hat{\mathbf{u}}^{0i}_{r(m);-1}\\
&\quad+(\Delta,\delta)\left[\lambda^{1i}_{r(m)}\mathbf{D}'_{f;0}(\lambda^{0i}_{r(m)})\mathbf{u}^{0i}_{r(m);0}+\mathbf{D}_{f;0}(\lambda^{0i}_{r(m)})\mathbf{u}^{1i}_{r(m);0}\right]+O(\Delta,\delta)^2=\mathbf{0}.
\end{aligned}
\tag{A.8}
$$

For $r>0$, since it holds $\mathbf{D}_{f:0}(\lambda^{0i}_{r(m)})\mathbf{u}^{0i}_{r(m);0}=\mathbf{0}$ and $\hat{\mathbf{u}}^{0i}_{r(m);0}=\hat{\mathbf{u}}^{0i}_{r(m);-1}=\mathbf{0}$ for the unperturbed isotropic rotor system, we can derive, from Eqs. (A.8) and (A.6), and using the relation $\bar{\mathbf{v}}^{0i\mathrm{T}}_{r(m);0}\mathbf{D}_{f;0}(\lambda^{0i}_{r(m)})=\mathbf{0}^{\mathrm{T}}$,

$$
\lambda^{1i}_{r(m)}\cong-\frac{\{\bar{\mathbf{v}}^{0i\mathrm{T}}_{r(m);0}+(\Delta,\delta)\bar{\mathbf{v}}^{1i\mathrm{T}}_{r(m);0}\}\mathbf{D}_{f;0}(\lambda^{0i}_{r(m)})\mathbf{u}^{1i}_{r(m);0}}{\{\bar{\mathbf{v}}^{0i\mathrm{T}}_{r(m);0}+(\Delta,\delta)\bar{\mathbf{v}}^{1i\mathrm{T}}_{r(m);0}\}\mathbf{D}'_{f;0}(\lambda^{0i}_{r(m)})\mathbf{u}^{0i}_{r(m);0}}=\frac{(\Delta,\delta)\bar{\mathbf{v}}^{1i\mathrm{T}}_{r(m);0}\mathbf{D}_{f;0}(\lambda^{0i}_{r(m)})\mathbf{u}^{1i}_{r(m);0}}{\bar{\mathbf{v}}^{0i\mathrm{T}}_{r(m);0}\mathbf{D}'_{f;0}(\lambda^{0i}_{r(m)})\mathbf{u}^{0i}_{r(m);0}}\sim O(\Delta,\delta).
\tag{A.9}
$$

And, similarly, for $r<0$, it holds $\bar{\mathbf{D}}_{f:0}(\lambda^{0i}_{r(m)})\hat{\mathbf{u}}^{0i}_{r(m);0}=\mathbf{0}$ and $\mathbf{u}^{0i}_{r(m);0}=\mathbf{0}$ for the unperturbed isotropic rotor system, we can derive, using the relation $\bar{\hat{\mathbf{v}}}^{0i\mathrm{T}}_{r(m);0}\bar{\mathbf{D}}_{f;0}(\lambda^{0i}_{r(m)})=\mathbf{0}^{\mathrm{T}}$,

$$
\lambda^{1i}_{r(m)}\cong-\frac{\{\bar{\hat{\mathbf{v}}}^{0i\mathrm{T}}_{r(m);0}+(\Delta,\delta)\bar{\hat{\mathbf{v}}}^{1i\mathrm{T}}_{r(m);0}\}\bar{\mathbf{D}}_{f;0}(\lambda^{0i}_{r(m)})\hat{\mathbf{u}}^{1i}_{r(m);0}}{\{\bar{\hat{\mathbf{v}}}^{0i\mathrm{T}}_{r(m);0}+(\Delta,\delta)\bar{\hat{\mathbf{v}}}^{1i\mathrm{T}}_{r(m);0}\}\bar{\mathbf{D}}'_{f;0}(\lambda^{0i}_{r(m)})\hat{\mathbf{u}}^{0i}_{r(m);0}}\sim O(\Delta,\delta).
\tag{A.10}
$$

From the above two relations, we can conclude that

$$
\lambda^{i}_{r(m)}=\lambda^{0i}_{r(m)}+(\Delta,\delta)^2\lambda^{2i}_{r(m)}+O(\Delta,\delta)^3,\quad r=\pm1,\dots\pm N
\tag{A.11}
$$

implying that the latent roots are stationary with respect to the perturbations in system parameters. This derivation is mathematically consistent with the previous result derived for the asymmetric rotor with anisotropic stator (refer to Appendix A of Suh et al. [7]).

Substituting Eq. (A.9) into Eq. (A.7), we obtain, for $r>0$,

$$
\mathbf{D}_{f;0}(\lambda^{i}_{r(m)})\mathbf{u}^{i}_{r(m);0}=(\Delta,\delta)\mathbf{D}_{f;0}(\lambda^{0i}_{r(m)})\mathbf{u}^{1i}_{r(m);0}
\tag{A.12}
$$

and similarly, for $r<0$,

$$
\bar{\mathbf{D}}_{f;0}(\lambda^{i}_{r(m)})\hat{\mathbf{u}}^{i}_{r(m);0}=(\Delta,\delta)\bar{\mathbf{D}}_{f;0}(\lambda^{0i}_{r(m)})\hat{\mathbf{u}}^{1i}_{r(m);0}.
\tag{A.13}
$$

Appendix B. Latent vector relations

Here, latent vector relations for general rotor with weak anisotropy and asymmetry for other than the modes associated with $r>0$ and $m=0$ are derived.

For $r(m)=-r(0)$ and $r>0$, where $||\bar{\mathbf{D}}_{f;0}(\lambda^{i}_{-r(0)})\hat{\mathbf{u}}^{i}_{-r(0);0}||\sim O(\delta,\Delta)$, we can derive, by removing the third block matrix equation from Eq. (13a) and using $||\hat{\mathbf{u}}^{i}_{-r(0);0}||\sim O(1)$, the norm order of right latent (modal) vectors as

$$
\begin{aligned}
||\mathbf{u}^{i}_{-r(0);1}||&=||\delta\Delta^2\bar{\mathbf{D}}^{-1}_{f:1}\mathbf{D}_{b;1}[\delta^2\bar{\mathbf{D}}_{r;0}\mathbf{D}^{-1}_{f:0}\mathbf{D}_{r;1}-\bar{\mathbf{E}}_{f:-1}]^{-1}\bar{\mathbf{D}}_{r;0}\mathbf{D}^{-1}_{f:0}\mathbf{D}_{b;0}\hat{\mathbf{u}}^{i}_{-r(0);0}||\\
&\approx||-\delta\Delta^2\bar{\mathbf{D}}^{-1}_{f:1}\mathbf{D}_{b;1}\bar{\mathbf{D}}^{-1}_{f:-1}\bar{\mathbf{D}}_{r;0}\mathbf{D}^{-1}_{f:0}\mathbf{D}_{b;0}\hat{\mathbf{u}}^{i}_{-r(0);0}||\sim O(\delta\Delta^2),
\end{aligned}
$$

$$
||\hat{\mathbf{u}}^{i}_{-r(0);-1}||=||-\delta\Delta[\delta^2\bar{\mathbf{D}}_{r;0}\mathbf{D}^{-1}_{f;0}\mathbf{D}_{r;1}-\bar{\mathbf{E}}_{f;-1}]^{-1}\bar{\mathbf{D}}_{r;0}\mathbf{D}^{-1}_{r;0}\mathbf{D}_{b;0}\hat{\mathbf{u}}^{i}_{-r(0);0}||\approx||\delta\Delta\bar{\mathbf{D}}^{-1}_{f;-1}\bar{\mathbf{D}}_{r;0}\mathbf{D}^{-1}_{f;0}\mathbf{D}_{b;0}\hat{\mathbf{u}}^{i}_{-r(0);0}||\sim O(\delta\Delta),
$$

$$
||\mathbf{u}^{i}_{-r(0);0}||=||\Delta[\delta^2\mathbf{D}_{r;1}\bar{\mathbf{E}}^{-1}_{f;-1}\bar{\mathbf{D}}_{r;0}-\mathbf{D}_{f;0}]^{-1}\mathbf{D}_{b;0}\hat{\mathbf{u}}^{i}_{-r(0);0}||\approx||-\Delta\mathbf{D}^{-1}_{f;0}\mathbf{D}_{b;0}\hat{\mathbf{u}}^{i}_{-r(0);0}||\sim O(\Delta),
$$

$$
||\hat{\mathbf{u}}^{i}_{-r(0);1}||=||\delta\Delta\bar{\mathbf{D}}^{-1}_{f;1}\bar{\mathbf{D}}_{b;1}\mathbf{E}^{-1}_{f;-1}\mathbf{D}_{r;0}\hat{\mathbf{u}}^{i}_{-r(0);0}||\approx||\delta\Delta\bar{\mathbf{D}}v\bar{\mathbf{D}}_{b;1}\mathbf{D}^{-1}_{f;-1}\mathbf{D}_{r;0}\hat{\mathbf{u}}^{i}_{-r(0);0}||\sim O(\delta\Delta),
$$

$$
||\mathbf{u}^{i}_{-r(0);-1}||=||-\delta\mathbf{E}^{-1}_{f;-1}\mathbf{D}_{r;0}\hat{\mathbf{u}}^{i}_{-r(0);0}||\approx||-\delta\mathbf{D}^{-1}_{f;-1}\mathbf{D}_{r;0}\hat{\mathbf{u}}^{i}_{-r(0);0}||\sim O(\delta).
\tag{B.1}
$$

Likewise, we obtain, for $r(m) = r(-1)$ and $r > 0$, where $||\mathbf{D}_{f;-1}(\lambda^i_{r(-1)})\mathbf{u}^i_{r(-1);-1}|| \sim O(\delta, \Delta)$, by removing the second block matrix equation from Eq. (13a) and using $||\mathbf{u}^i_{r(-1);-1}|| \sim O(1)$,

$$||\mathbf{u}^i_{r(-1);0}|| = ||\delta\Delta\mathbf{E}^{-1}_{f;0}\mathbf{D}_{b;0}\bar{\mathbf{D}}^{-1}_{f;0}\bar{\mathbf{D}}_{r;1}\mathbf{u}^i_{r(-1);-1}|| \approx ||\delta\Delta\mathbf{D}^{-1}_{f;0}\mathbf{D}_{b;0}\bar{\mathbf{D}}^{-1}_{f;0}\bar{\mathbf{D}}_{r;1}\mathbf{u}^i_{r(-1);-1}|| \sim O(\delta\Delta),$$

$$||\hat{\mathbf{u}}^i_{r(-1);-1}|| = ||-\delta^2\Delta\bar{\mathbf{E}}^{-1}_{f;-1}\bar{\mathbf{D}}_{r;0}\mathbf{E}^{-1}_{f;0}\mathbf{D}_{b;0}\bar{\mathbf{D}}^{-1}_{f;0}\bar{\mathbf{D}}_{r;1}\mathbf{u}^i_{r(-1);-1}|| \approx ||-\delta^2\Delta\bar{\mathbf{D}}^{-1}_{f;-1}\bar{\mathbf{D}}_{r;0}\mathbf{D}^{-1}_{f;0}\mathbf{D}_{b;0}\bar{\mathbf{D}}^{-1}_{f;0}\bar{\mathbf{D}}_{r;1}\mathbf{u}^i_{r(-1);-1}|| \sim O(\delta^2\Delta),$$

$$||\hat{\mathbf{u}}^i_{r(-1);0}|| = ||-[\delta\Delta^2\bar{\mathbf{D}}^{-1}_{f;0}\bar{\mathbf{D}}_{b;0}\mathbf{E}^{-1}_{f;0}\mathbf{D}_{b;0}\bar{\mathbf{D}}^{-1}_{f;0}\bar{\mathbf{D}}_{r;1} + \delta\bar{\mathbf{D}}^{-1}_{f;0}\bar{\mathbf{D}}_{r;1}]\mathbf{u}^i_{r(-1);-1}|| \approx ||-\delta\bar{\mathbf{D}}^{-1}_{f;0}\bar{\mathbf{D}}_{r;1}\mathbf{u}^i_{r(-1);-1}|| \sim O(\delta),$$

$$\begin{aligned}||\mathbf{u}^i_{r(-1);1}|| &= ||\delta^2\Delta^2\mathbf{D}^{-1}_{f;1}\mathbf{D}_{b;1}\bar{\mathbf{E}}^{-1}_{f;-1}\bar{\mathbf{D}}_{r;0}\mathbf{E}^{-1}_{f;0}\mathbf{D}_{b;0}\bar{\mathbf{D}}^{-1}_{f;0}\bar{\mathbf{D}}_{r;1}\mathbf{u}^i_{r(-1);-1}||\\ &\approx ||\delta^2\Delta^2\mathbf{D}^{-1}_{f;1}\mathbf{D}_{b;1}\bar{\mathbf{D}}^{-1}_{f;-1}\bar{\mathbf{D}}_{r;0}\mathbf{D}^{-1}_{f;0}\mathbf{D}_{b;0}\bar{\mathbf{D}}^{-1}_{f;0}\bar{\mathbf{D}}_{r;1}\mathbf{u}^i_{r(-1);-1}|| \sim O(\delta^2\Delta^2),\end{aligned}$$

$$||\hat{\mathbf{u}}^i_{r(-1);1}|| = ||-\Delta\bar{\mathbf{D}}^{-1}_{f;1}\bar{\mathbf{D}}_{b;1}\mathbf{u}^i_{r(-1);-1}|| \sim O(\Delta) \tag{B.2}$$

with

$$\mathbf{E}_{f;0}(\lambda^i_{r(-1)}) = \mathbf{D}_{f;0}(\lambda^i_{r(-1)}) - \Delta^2\mathbf{D}_{b;0}(\lambda^i_{r(-1)})\bar{\mathbf{D}}^{-1}_{f;0}(\lambda^i_{r(-1)})\bar{\mathbf{D}}_{b;0} - \delta^2\mathbf{D}_{r;1}(\lambda^i_{r(-1)})\bar{\mathbf{E}}^{-1}_{f;-1}(\lambda^i_{r(-1)})\bar{\mathbf{D}}_{r;0}(\lambda^i_{r(-1)}) \approx \mathbf{D}_{f;0}(\lambda^i_{r(-1)})$$

for $r(m) = -r(1)$ and $r > 0$, where $||\bar{\mathbf{D}}_{f;-1}(\lambda^i_{-r(1)})\hat{\mathbf{u}}^i_{-r(1);-1}|| \sim O(\delta, \Delta)$, by removing the fifth block matrix equation from Eq. (13a) and using $||\hat{\mathbf{u}}^i_{-r(1);-1}|| \sim O(1)$,

$$||\hat{\mathbf{u}}^i_{-r(1);0}|| = ||\delta\Delta\bar{\mathbf{E}}^{-1}_{f;0}\bar{\mathbf{D}}_{b;0}\mathbf{D}^{-1}_{f;0}\mathbf{D}_{r;1}\hat{\mathbf{u}}^i_{-r(1);-1}|| \approx ||\delta\Delta\bar{\mathbf{D}}^{-1}_{f;0}\bar{\mathbf{D}}_{b;0}\mathbf{D}^{-1}_{f;0}\mathbf{D}_{r;1}\hat{\mathbf{u}}^i_{-r(1);-1}|| \sim O(\delta\Delta),$$

$$||\mathbf{u}^i_{-r(1);-1}|| = ||-\delta^2\Delta\mathbf{E}^{-1}_{f;-1}\mathbf{D}_{r;0}\bar{\mathbf{E}}^{-1}_{f;0}\bar{\mathbf{D}}_{b;0}\mathbf{D}^{-1}_{f;0}\mathbf{D}_{r;1}\hat{\mathbf{u}}^i_{-r(1);-1}|| \approx ||-\delta^2\Delta\mathbf{D}^{-1}_{f;-1}\mathbf{D}_{r;0}\bar{\mathbf{D}}^{-1}_{f;0}\bar{\mathbf{D}}_{b;0}\mathbf{D}^{-1}_{f;0}\mathbf{D}_{r;1}\hat{\mathbf{u}}^i_{-r(1);-1}|| \sim O(\delta^2\Delta),$$

$$||\mathbf{u}^i_{-r(1);0}|| = ||-[\delta\Delta^2\mathbf{D}^{-1}_{f;0}\mathbf{D}_{b;0}\bar{\mathbf{E}}^{-1}_{f;0}\bar{\mathbf{D}}_{b;0}\mathbf{D}^{-1}_{f;0}\mathbf{D}_{r;1} + \delta\mathbf{D}^{-1}_{f;0}\mathbf{D}_{r;1}]\hat{\mathbf{u}}^i_{-r(1);-1}|| \approx ||-\delta\mathbf{D}^{-1}_{f;0}\mathbf{D}_{r;1}\hat{\mathbf{u}}^i_{-r(1);-1}|| \sim O(\delta),$$

$$\begin{aligned}||\hat{\mathbf{u}}^i_{-r(1);1}|| &= ||\delta^2\Delta^2\bar{\mathbf{D}}^{-1}_{f;1}\bar{\mathbf{D}}_{b;1}\mathbf{E}^{-1}_{f;-1}\mathbf{D}_{r;0}\bar{\mathbf{E}}^{-1}_{f;0}\bar{\mathbf{D}}_{b;0}\mathbf{D}^{-1}_{f;0}\mathbf{D}_{r;1}\hat{\mathbf{u}}^i_{-r(1);-1}||\\ &\approx ||\delta^2\Delta^2\bar{\mathbf{D}}^{-1}_{f;1}\bar{\mathbf{D}}_{b;1}\mathbf{D}^{-1}_{f;-1}\mathbf{D}_{r;0}\bar{\mathbf{D}}^{-1}_{f;0}\bar{\mathbf{D}}_{b;0}\mathbf{D}^{-1}_{f;0}\mathbf{D}_{r;1}\hat{\mathbf{u}}^i_{-r(1);-1}|| \sim O(\delta^2\Delta^2),\end{aligned}$$

$$||\mathbf{u}^i_{-r(1);1}|| = ||-\Delta\mathbf{D}^{-1}_{f;1}\mathbf{D}_{b;1}\hat{\mathbf{u}}^i_{-r(1);-1}|| \sim O(\Delta) \tag{B.3}$$

for $r(m) = r(1)$ and $r > 0$, where $||\mathbf{D}_{f;1}(\lambda^i_{r(1)})\mathbf{u}^i_{r(1);1}|| \sim O(\delta, \Delta)$, by removing the sixth block matrix equation from Eq. (13a) and using $||\mathbf{u}^i_{r(1);1}|| \sim O(1)$,

$$||\mathbf{u}^i_{r(1);0}|| = ||\delta\Delta\mathbf{F}^{-1}_{f;0}\mathbf{D}_{r;1}\bar{\mathbf{D}}_{f;-1}\bar{\mathbf{D}}_{b;-1}\mathbf{u}^i_{r(1);1}|| \approx ||\delta\Delta\mathbf{D}^{-1}_{f;0}\mathbf{D}_{r;1}\bar{\mathbf{D}}_{f;-1}\bar{\mathbf{D}}_{b;-1}\mathbf{u}^i_{r(1);1}|| \sim O(\delta\Delta),$$

$$\begin{aligned}||\mathbf{u}^i_{r(1);-1}|| &= ||\delta^2\Delta^2\mathbf{F}^{-1}_{f;-1}\mathbf{D}_{r;0}\bar{\mathbf{D}}^{-1}_{f;0}\bar{\mathbf{D}}_{b;0}\mathbf{F}^{-1}_{f;0}\mathbf{D}_{r;1}\bar{\mathbf{D}}_{f;-1}\bar{\mathbf{D}}_{b;-1}\mathbf{u}^i_{r(1);1}||\\ &\approx ||\delta^2\Delta^2\mathbf{D}^{-1}_{f;-1}\mathbf{D}_{r;0}\bar{\mathbf{D}}^{-1}_{f;0}\bar{\mathbf{D}}_{b;0}\mathbf{D}^{-1}_{f;0}\mathbf{D}_{r;1}\bar{\mathbf{D}}_{f;-1}\bar{\mathbf{D}}_{b;-1}\mathbf{u}^i_{r(1);1}|| \sim O(\delta^2\Delta^2),\end{aligned}$$

$$\begin{aligned}||\hat{\mathbf{u}}^i_{r(1);1}|| &= ||-\delta^2\Delta^3\bar{\mathbf{D}}_{f;1}\bar{\mathbf{D}}_{b;1}\mathbf{F}^{-1}_{f;-1}\mathbf{D}_{r;0}\bar{\mathbf{D}}^{-1}_{f;0}\bar{\mathbf{D}}_{b;0}\mathbf{F}^{-1}_{f;0}\mathbf{D}_{r;1}\bar{\mathbf{D}}_{f;-1}\bar{\mathbf{D}}_{b;-1}\mathbf{u}^i_{r(1);1}||\\ &\approx ||-\delta^2\Delta^3\bar{\mathbf{D}}_{f;1}\bar{\mathbf{D}}_{b;1}\mathbf{D}^{-1}_{f;-1}\mathbf{D}_{r;0}\bar{\mathbf{D}}^{-1}_{f;0}\bar{\mathbf{D}}_{b;0}\mathbf{D}^{-1}_{f;0}\mathbf{D}_{r;1}\bar{\mathbf{D}}_{f;-1}\bar{\mathbf{D}}_{b;-1}\mathbf{u}^i_{r(1);1}|| \sim O(\delta^2\Delta^3),\end{aligned}$$

$$||\hat{\mathbf{u}}^i_{r(1);-1}|| = ||-\bar{\mathbf{D}}^{-1}_{f;-1}[\delta^2\Delta\bar{\mathbf{D}}_{r;0}\mathbf{F}^{-1}_{f;0}\mathbf{D}_{r;1}\bar{\mathbf{D}}_{f;-1}\bar{\mathbf{D}}_{b;-1} + \Delta\bar{\mathbf{D}}_{b;-1}]\mathbf{u}^i_{r(1);1}|| \approx ||-\Delta\bar{\mathbf{D}}^{-1}_{f;-1}\bar{\mathbf{D}}_{b;-1}\mathbf{u}^i_{r(1);1}|| \sim O(\Delta)\chi,$$

$$\begin{aligned}||\hat{\mathbf{u}}^i_{r(1);0}|| &= ||-\bar{\mathbf{D}}^{-1}_{f;0}[\delta\Delta^2 + \delta^3\Delta^2\bar{\mathbf{D}}_{r;1}\mathbf{F}^{-1}_{f;-1}\mathbf{D}_{r;0}\bar{\mathbf{D}}^{-1}_{f;0}]\bar{\mathbf{D}}_{b;0}\mathbf{F}^{-1}_{f;0}\mathbf{D}_{r;1}\bar{\mathbf{D}}_{f;-1}\bar{\mathbf{D}}_{b;-1}\mathbf{u}^i_{r(1);1}||\\ &\approx ||-\delta\Delta^2\bar{\mathbf{D}}^{-1}_{f;0}\bar{\mathbf{D}}_{b;0}\mathbf{D}^{-1}_{f;0}\mathbf{D}_{r;1}\bar{\mathbf{D}}_{f;-1}\bar{\mathbf{D}}_{b;-1}\mathbf{u}^i_{r(1);1}|| \sim O(\delta\Delta^2)\end{aligned} \tag{B.4}$$

with

$$\mathbf{F}_{f;-1}(\lambda^i_{r(1)}) = \mathbf{D}_{f;-1}(\lambda^i_{r(1)}) - \Delta^2\mathbf{D}_{b;-1}(\lambda^i_{r(1)})\bar{\mathbf{D}}_{f;1}(\lambda^i_{r(1)})\bar{\mathbf{D}}_{b;1}(\lambda^i_{r(1)}) - \delta^2\mathbf{D}_{r;0}(\lambda^i_{r(1)})\bar{\mathbf{D}}^{-1}_{f;0}(\lambda^i_{r(1)})\bar{\mathbf{D}}_{r;1}(\lambda^i_{r(1)}) \approx \mathbf{D}_{f;-1}(\lambda^i_{r(1)}),$$

$$\begin{aligned}\mathbf{F}_{f;0}(\lambda^i_{r(1)}) = {} & \mathbf{D}_{f;0}(\lambda^i_{r(1)}) - \Delta^2\mathbf{D}_{b;0}(\lambda^i_{r(1)})\bar{\mathbf{D}}^{-1}_{f;0}(\lambda^i_{r(1)})\bar{\mathbf{D}}_{b;0}(\lambda^i_{r(1)}) - \delta^2\mathbf{D}_{r;1}(\lambda^i_{r(1)})\bar{\mathbf{D}}^{-1}_{f;-1}(\lambda^i_{r(1)})\bar{\mathbf{D}}_{r;0}(\lambda^i_{r(1)}) \\ & - \delta^2\Delta^2\mathbf{D}_{b;0}(\lambda^i_{r(1)})\bar{\mathbf{D}}^{-1}_{f;0}(\lambda^i_{r(1)})\bar{\mathbf{D}}_{r;1}(\lambda^i_{r(1)})\mathbf{h}_3^{-1}(\lambda^i_{r(1)})\mathbf{D}_{r;0}(\lambda^i_{r(1)})\bar{\mathbf{D}}^{-1}_{f;0}(\lambda^i_{r(1)})\bar{\mathbf{D}}_{b;0}(\lambda^i_{r(1)}) \approx \mathbf{D}_{f;0}(\lambda^i_{r(1)})\end{aligned}$$

and for $r(m) = -r(-1)$ and $r>0$, where $||\bar{\mathbf{D}}_{f;1}(\lambda^i_{-r(-1)})\hat{\mathbf{u}}^i_{-r(-1);1}||\sim O(\delta,\Delta)$, by removing the first block matrix equation from Eq. (13a) and using $||\hat{\mathbf{u}}^i_{-r(-1);1}||\sim O(1)$,

$$||\hat{\mathbf{u}}^i_{-r(-1);0}|| = ||\delta\Delta\bar{\mathbf{F}}^{-1}_{f;0}\bar{\mathbf{D}}_{r;1}\mathbf{D}_{f;-1}\mathbf{D}_{b;-1}\hat{\mathbf{u}}^i_{-r(-1);1}|| \approx ||\delta\Delta\bar{\mathbf{D}}^{-1}_{f;0}\bar{\mathbf{D}}_{r;1}\mathbf{D}_{f;-1}\mathbf{D}_{b;-1}\hat{\mathbf{u}}^i_{-r(-1);1}||\sim O(\delta\Delta),$$

$$\begin{aligned}||\hat{\mathbf{u}}^i_{-r(-1);-1}|| &= ||\delta^2\Delta^2\bar{\mathbf{F}}^{-1}_{f;-1}\bar{\mathbf{D}}_{r;0}\mathbf{D}^{-1}_{f;0}\mathbf{D}_{b;0}\bar{\mathbf{F}}^{-1}_{f;0}\bar{\mathbf{D}}_{r;1}\mathbf{D}_{f;-1}\mathbf{D}_{b;-1}\hat{\mathbf{u}}^i_{-r(-1);1}|| \\ &\approx ||\delta^2\Delta^2\bar{\mathbf{D}}^{-1}_{f;-1}\bar{\mathbf{D}}_{r;0}\mathbf{D}^{-1}_{f;0}\mathbf{D}_{b;0}\bar{\mathbf{D}}^{-1}_{\mathbf{f};0}\bar{\mathbf{D}}_{r;1}\mathbf{D}_{f;-1}\mathbf{D}_{b;-1}\hat{\mathbf{u}}^i_{-r(-1);1}||\sim O(\delta^2\Delta^2),\end{aligned}$$

$$\begin{aligned}||\mathbf{u}^i_{-r(-1);1}|| &= ||-\delta^2\Delta^3\mathbf{D}_{f;1}\mathbf{D}_{b;1}\bar{\mathbf{F}}^{-1}_{\mathbf{f};-1}\bar{\mathbf{D}}_{r;0}\mathbf{D}^{-1}_{f;0}\mathbf{D}_{b;0}\bar{\mathbf{F}}^{-1}_{f;0}\bar{\mathbf{D}}_{r;1}\mathbf{D}_{f;-1}\mathbf{D}_{b;-1}\hat{\mathbf{u}}^i_{-r(-1);1}|| \\ &\approx ||-\delta^2\Delta^3\mathbf{D}_{f;1}\mathbf{D}_{b;1}\bar{\mathbf{D}}^{-1}_{f;-1}\bar{\mathbf{D}}_{r;0}\mathbf{D}^{-1}_{f;0}\mathbf{D}_{b;0}\bar{\mathbf{D}}^{-1}_{f;0}\bar{\mathbf{D}}_{r;1}\mathbf{D}_{f;-1}\mathbf{D}_{b;-1}\hat{\mathbf{u}}^i_{-r(-1);1}||\sim O(\delta^2\Delta^3),\end{aligned}$$

$$\begin{aligned}||\mathbf{u}^i_{-r(-1);0}|| &= ||-\mathbf{D}^{-1}_{f;0}\left[\delta\Delta^2 + \delta^3\Delta^2\mathbf{D}_{r;1}\bar{\mathbf{F}}^{-1}_{f;-1}\bar{\mathbf{D}}_{r;0}\mathbf{D}^{-1}_{f;0}\right]\mathbf{D}_{b;0}\bar{\mathbf{F}}^{-1}_{f;0}\bar{\mathbf{D}}_{r;1}\mathbf{D}_{f;-1}\mathbf{D}_{b;-1}\hat{\mathbf{u}}^i_{-r(-1);1}|| \\ &\approx ||-\delta\Delta^2\mathbf{D}^{-1}_{f;0}\mathbf{D}_{b;0}\bar{\mathbf{D}}^{-1}_{f;0}\bar{\mathbf{D}}_{r;1}\mathbf{D}_{f;-1}\mathbf{D}_{b;-1}\hat{\mathbf{u}}^i_{-r(-1);1}||\sim O(\delta\Delta^2),\end{aligned}$$

$$||\mathbf{u}^i_{-r(-1);-1}|| = ||-\mathbf{D}^{-1}_{f;-1}[\delta^2\Delta\mathbf{D}_{r;0}\bar{\mathbf{F}}^{-1}_{f;0}\bar{\mathbf{D}}_{r;1}\mathbf{D}_{f;-1}\mathbf{D}_{b;-1} + \Delta\mathbf{D}_{b;-1}]\hat{\mathbf{u}}^i_{-r(-1);1}|| \approx ||-\Delta\mathbf{D}^{-1}_{f;-1}\mathbf{D}_{b;-1}\hat{\mathbf{u}}^i_{-r(-1);1}||\sim O(\Delta). \quad \text{(B.5)}$$

Appendix C. Matrix norm [8]

Let us consider a matrix relation given by

$$\mathbf{H} = \mathbf{u}\mathbf{v}^{\mathrm{T}}, \quad \text{(C.1)}$$

where $\mathbf{H}$ is an $m \times n$ matrix, and $\mathbf{u}$ and $\mathbf{v}$ are $m \times 1$ and $n \times 1$ vectors, respectively. Post-multiplying an $n \times 1$ arbitrary vector $\mathbf{f}$ with $||\mathbf{f}|| = 1$ in Eq. (C.1), we obtain

$$\mathbf{H}\mathbf{f} = \mathbf{u}\mathbf{v}^{\mathrm{T}}\mathbf{f}. \quad \text{(C.2)}$$

Taking the vector norm of Eq. (C.2), we have

$$||\mathbf{H}\mathbf{f}|| = ||\mathbf{u}\mathbf{v}^{\mathrm{T}}\mathbf{f}|| = ||\mathbf{u}||||\mathbf{v}^{\mathrm{T}}\mathbf{f}|| \leqslant ||\mathbf{H}||||\mathbf{f}|| = ||\mathbf{H}|| \quad \text{(C.3)}$$

and

$$||\mathbf{v}^{\mathrm{T}}\mathbf{f}|| = |\mathbf{v}^{\mathrm{T}}\mathbf{f}| \leqslant ||\mathbf{v}^{\mathrm{T}}||||\mathbf{f}|| = ||\mathbf{v}^{\mathrm{T}}||. \quad \text{(C.4)}$$

It also holds, by definition of norm

$$||\mathbf{H}|| = ||\mathbf{H}_1 + \mathbf{H}_2|| \leqslant ||\mathbf{H}_1|| + ||\mathbf{H}_2||. \quad \text{(C.5)}$$

References

[1] C.W. Lee, *Vibration Analysis of Rotors*, Kluwer Academic Publishers, Dordrecht, 1993.

[2] C.W. Lee, Stability and strength of modes in rotating machinery, Keynote Paper, in: *The Third International Symposium on Stability Control of Rotating Machinery*, Cleveland, 2005.

[3] C.W. Lee, J.H. Suh, S.W. Hong, Modal analysis of rotor system using modulated coordinates for asymmetric rotor system with anisotropic stator, in: *The Eighth International Conference on Vibrations in Rotating Machinery*, Swansea, September 2004.
[4] C.W. Lee, D.J. Han, J.H. Suh, S.W. Hong, Modal analysis of periodically time-varying linear rotor systems, *Journal of Sound and Vibration* 303 (2007) 553–574.
[5] J.S. Rao, *Turbomachine Blade Vibration*, Wiley Eastern Ltd., New Delhi, 1991.
[6] M. Lalanne, G. Ferraris, *Rotordynamics Prediction in Engineering*, Wiley, New York, 1990.
[7] J.H. Suh, S.W. Hong, C.W. Lee, Modal analysis of asymmetric rotor system with isotropic stator using modulated coordinates, *Journal of Sound and Vibration* 284 (2005) 651–671.
[8] I. Stakgold, *Boundary Value Problems of Mathematical Physics*, Vol. I, The MacMillan Co., London, 1967 (Chapter 2).
[9] Y.G. Jei, C.W. Lee, Modal analysis of continuous asymmetrical rotor-bearing systems, *Journal of Sound and Vibration* 152 (2) (1992) 245–262.
[10] C.A. Papadopoulos, A.D. Dimarogonas, Coupled longitudinal and bending vibrations of a rotating shaft with an open crack, *Journal of Sound and Vibration* 117 (1) (1987) 81–93.