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On the higher-order semi-discretizations for periodic delayed systems

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Abstract

Semi-discretization techniques of periodic delayed systems are presented using zeroth-, first- and higher-order approximations of the delayed term. It is shown that if the time-periodic coefficients in the equation are approximated by piecewise constant functions, then there is no need to use higher than first-order approximations of the delayed term. The results are demonstrated on construction of the stability chart of the delayed Mathieu equation.

1. Introduction

Systems governed by delay-differential equations (DDEs) often arise in different fields of science and engineering. Examples include control systems with feedback delay, laser dynamics, neuroscience, balancing with reflex delay, traffic modeling, or regenerative machine tool chatter. The qualitative investigation of these mechanical systems always includes stability analysis. In particular, stability charts are constructed displaying stable and unstable regions in the parameter space.

DDEs usually have an infinite dimensional state space representation [1,2], therefore, in general, no conditions involving system parameters are available to guarantee stability. One way to deal with stability-related issues for DDEs is to employ discretization techniques. For example, using piecewise constant arguments, DDEs can be approximated by discrete maps, for which stability conditions can be obtained in closed form.

An effective method, the so-called semi-discretization was introduced for delayed systems by Insperger and Stépán and its advantages compared to the full-discretization were discussed in Ref. [3]. In the process of semi-discretization of time-periodic DDEs, the delayed terms are discretized while the undelayed terms are unchanged and the time-periodic coefficients are approximated by piecewise constant functions. The convergence of the method was established in Ref. [4] for a large class of DDEs appearing in engineering applications. It was shown that semi-discretization preserves asymptotic stability of the original equation, therefore it can be used to construct approximate stability charts.

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The merit of the semi-discretization method is that it can effectively be used for determining stability of time-periodic DDEs arising in different engineering applications. Sheng et al. [5] used this technique for the stability analysis of periodic control systems with delayed feedback. Gradišek et al. [6] and Henninger and Eberhard [7] applied the method for constructing stability charts for milling operations. The method can also be used for systems with time-periodic delays, as it was shown in Ref. [8]. Recently, Long et al. [9] and Faassen et al. [10] applied the semi-discretization method for a detailed milling model including time-periodic regenerative delay.

The efficiency of the semi-discretization method can be increased by higher order approximations of the delayed term appearing in the model equations. Zeroth-, improved zeroth- and first-order versions of the semi-discretization technique were presented and analyzed by Elbeyly and Sun [11] for a second order periodic system with time delay. The improved zeroth-order method was also considered in Ref. [12] for general DDEs in state space form.

Following the work of Elbeyly and Sun [11], it is a straightforward step to introduce second- and higher-order approximations of the delayed term. In this paper, the convergence of different order semi-discretization schemes is investigated using rate of convergence estimates for the case when the time-periodic terms are approximated by piecewise constant functions. It is shown that in this case, second- and higher-order approximations of the delayed term does not provide better convergence than the first-order one. The results are demonstrated for the delayed Mathieu equation.

2. Zeroth-, first-, and higher-order semi-discretizations

Consider the linear delayed system

$$\dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t) + \mathbf{B}(t)\mathbf{u}(t-\tau), \tag{1}$$

$$\mathbf{u}(t) = \mathbf{D}\mathbf{x}(t), \tag{2}$$

where $\mathbf{x}(t) \in \mathbb{R}^n$, $\mathbf{u}(t) \in \mathbb{R}^m$ and the matrices $\mathbf{A}(t)$, $\mathbf{B}(t)$ are piecewise smooth, T-periodic functions, and $\mathbf{D}$ is a constant matrix. Such systems often arise in the field of delayed dynamic systems. When the linear stability properties of a nonlinear delayed system is investigated around a periodic orbit, then one usually ends up with equations quite similar to Eqs. (1) and (2).

The approximating semi-discrete system is formulated as

$$\dot{\mathbf{y}}(t) = \tilde{\mathbf{A}}_i\mathbf{y}(t) + \mathbf{B}(t)\boldsymbol{\sigma}_i^{(p)}(t-\tau), \quad t \in [ih,(i+1)h), \tag{3}$$

$$\boldsymbol{\sigma}_i^{(p)}(t-\tau) = \sum_{j=0}^{p}\left(\prod_{l=0,l\neq j}^{p}\frac{t-\tau-(i+l-r)h}{(j-l)h}\right)\mathbf{w}_{i+j-r}, \tag{4}$$

$$\mathbf{w}_i = \mathbf{w}(ih) = \mathbf{D}\mathbf{y}(ih) = \mathbf{D}\mathbf{y}_i, \tag{5}$$

where $h = T/k$ is the discretization step with $k \in \mathbb{Z}^+$, and

$$\tilde{\mathbf{A}}_i = \frac{1}{h}\int_{ih}^{(i+1)h}\mathbf{A}(t)\,\mathrm{d}t, \quad i \in \mathbb{Z} \tag{6}$$

is the piecewise constant approximation of $\mathbf{A}(t)$ in Eq. (1). The delayed term $\boldsymbol{\sigma}_i^{(p)}(t-\tau)$ is a pth-order polynomial interpolation of $\mathbf{D}\mathbf{y}(t-\tau)$ using the discrete values of $\mathbf{w}_i = \mathbf{D}\mathbf{y}(ih)$ and integer r is defined by

$$r = \mathrm{int}(\tau/h + p/2), \tag{7}$$

where the function $\mathrm{int}(a)$ denotes the integer part of a. Thus, we have two approximation parameters: the number k of discrete steps over the period T and the order p of the approximation of the delayed term. Note that integer r is defined such that $(r+p/2)h \approx \tau$. The concept of the approximation scheme is illustrated in Fig. 1, where the dashed curve denotes the approximating polynomial $\boldsymbol{\sigma}_i^{(p)}(t-\tau)$.

One reason for using the semi-discretization method is that the approximate system (3)–(5) can be solved analytically over the discretization interval $t \in [ih,(i+1)h)$ for given initial values $\mathbf{y}(ih)$ and

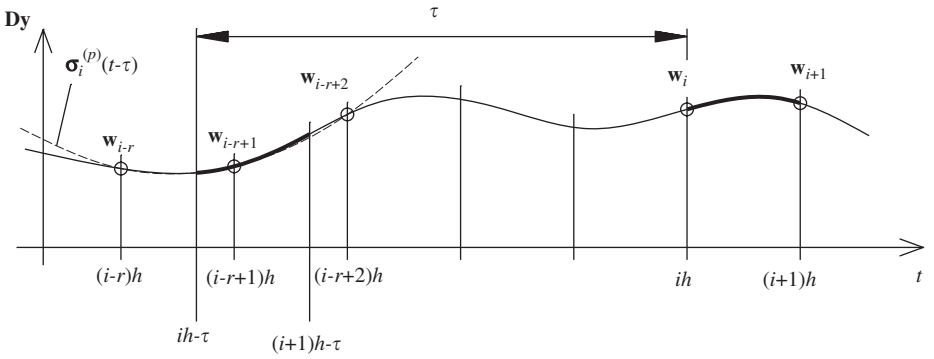

Fig. 1. The semi-discretization scheme.

$\mathbf{w}_{i+j-r}$, $j = 0, 1, \ldots, p$:

$$\mathbf{y}((i+1)h) = \mathbf{P}_i \mathbf{y}(ih) + \sum_{j=0}^{p} \mathbf{R}_{i,j} \mathbf{w}_{i+j-r}, \tag{8}$$

where

$$\mathbf{P}_i = \mathrm{e}^{\tilde{\mathbf{A}}_i h} \tag{9}$$

and

$$\mathbf{R}_{i,j} = \int_0^h \mathrm{e}^{\tilde{\mathbf{A}}_i (h-s)} \mathbf{B}(s) \left(\prod_{l=0, l\neq j}^{p} \frac{s - \tau + (r-l)h}{(j-l)h} \right) \mathrm{d}s. \tag{10}$$

Eqs. (8) and (5) lead to the $(n + rm)$-dimensional discrete map

$$\mathbf{z}_{i+1} = \mathbf{C}_i \mathbf{z}_i, \tag{11}$$

where

$$\mathbf{z}_i = \begin{pmatrix} \mathbf{y}_i \\ \mathbf{w}_{i-1} \\ \mathbf{w}_{i-2} \\ \vdots \\ \mathbf{w}_{i-r} \end{pmatrix}, \quad \mathbf{C}_i = \begin{pmatrix} \mathbf{P}_i & \mathbf{0} & \ldots & \mathbf{0} & \mathbf{R}_{i,p} & \ldots & \mathbf{R}_{i,1} & \mathbf{R}_{i,0} \\ \mathbf{D} & \mathbf{0} & \ldots & \mathbf{0} & \mathbf{0} & \ldots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} & \ldots & \mathbf{0} & \mathbf{0} & \ldots & \mathbf{0} & \mathbf{0} \\ \vdots & & & & & & & \vdots \\ \mathbf{0} & \mathbf{0} & \ldots & \mathbf{0} & \mathbf{0} & \ldots & \mathbf{I} & \mathbf{0} \end{pmatrix}. \tag{12}$$

Since $T = kh$, k-multiple recursive applications of Eq. (11) with initial state $\mathbf{z}_0$ result in

$$\mathbf{z}_n = \mathbf{\Phi} \mathbf{z}_0, \tag{13}$$

where

$$\mathbf{\Phi} = \mathbf{C}_{n-1} \mathbf{C}_{n-2} \ldots \mathbf{C}_0 \tag{14}$$

is the monodromy matrix (or Floquet transition matrix) of system (3)–(5). $\mathbf{\Phi}$ is a finite dimensional approximation of the infinite dimensional monodromy operator of the original systems (1) and (2).

The stability of the approximate system (3)–(5) can be assessed by the eigenvalue analysis of matrix $\mathbf{\Phi}$. If all the eigenvalues are inside the unit circle of the complex plane then system (3)–(5) are asymptotically stable. Since semi-discretization preserves asymptotic stability of the original system (1)–(2) as it was shown in Ref. [4], the method can be used to construct approximate stability charts.

The above formulae give the steps of the semi-discretization method for arbitrary approximation order p. For the sake of completeness, in the next subsections, special cases of these formulae are presented for the zeroth- and the first-order approximations.

2.1. Zeroth-order semi-discretization ($p = 0$)

For the zeroth-order case, Eqs. (4) and (7) give $\boldsymbol{\sigma}_i^{(0)}(t-\tau) = \mathbf{w}_{i-r}$ and $r = \mathrm{int}(\tau/h)$, thus, the approximate system reads

$$\dot{\mathbf{y}}(t) = \tilde{\mathbf{A}}_i \mathbf{y}(t) + \mathbf{B}(t)\mathbf{w}_{i-r}, \quad t \in [ih, (i+1)h), \tag{15}$$

$$\mathbf{w}_i = \mathbf{w}(ih) = \mathbf{D}\mathbf{y}(ih). \tag{16}$$

The solution over one discrete step is given as

$$\mathbf{y}((i+1)h) = \mathbf{P}_i \mathbf{y}(ih) + \mathbf{R}_{i,0}\mathbf{w}_{i-r}, \tag{17}$$

where $\mathbf{P}_i$ is given by Eq. (9) and

$$\mathbf{R}_{i,0} = \int_0^h \mathrm{e}^{\tilde{\mathbf{A}}_i(h-s)}\mathbf{B}(s)\,\mathrm{d}s. \tag{18}$$

If $\tilde{\mathbf{A}}_i^{-1}$ exists and $\mathbf{B}(s) \equiv \mathbf{B}$ is constant, then integration gives

$$\mathbf{R}_{i,0} = (\mathrm{e}^{\tilde{\mathbf{A}}_i h} - \mathbf{I})\tilde{\mathbf{A}}_i^{-1}\mathbf{B}. \tag{19}$$

Thus, the coefficient matrix in Eq. (11) reads

$$\mathbf{C}_i = \begin{pmatrix} \mathbf{P}_i & \mathbf{0} & \dots & \mathbf{0} & \mathbf{R}_{i,0} \\ \mathbf{D} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} & \dots & \mathbf{0} & \mathbf{0} \\ \vdots & & \ddots & & \vdots \\ \mathbf{0} & \mathbf{0} & \dots & \mathbf{I} & \mathbf{0} \end{pmatrix}. \tag{20}$$

Now, the approximate monodromy matrix is given by Eq. (14).

An improved or updated zeroth-order semi-discretization method was suggested in Refs. [11,12], where integer r is defined as $r = \mathrm{int}(\tau/h + 1/2)$ instead of $r = \mathrm{int}(\tau/h)$. It was shown that for some cases, e.g., when $k\tau/T$ is an integer, this choice of r improves convergence. However, this does not hold for general τ and T values. For instance, if $k\tau/T + 1/2$ is an integer, then $r = \mathrm{int}(\tau/h)$ gives better convergence than $r = \mathrm{int}(\tau/h + 1/2)$. Therefore, in this study, we assume that r is defined as in Eq. (7).

2.2. First-order semi-discretization ($p = 1$)

For the first-order case, Eq. (7) gives $r = \mathrm{int}(\tau/h + 1/2)$ and Eq. (4) gives

$$\boldsymbol{\sigma}_i^{(1)}(t-\tau) = \frac{t-\tau-(i-r)h}{h}\mathbf{w}_{i+1-r} - \frac{t-\tau-(i+1-r)h}{h}\mathbf{w}_{i-r}. \tag{21}$$

In this case, the approximate system is

$$\dot{\mathbf{y}}(t) = \tilde{\mathbf{A}}_i \mathbf{y}(t) + \mathbf{B}(t)\boldsymbol{\sigma}_i^{(1)}(t-\tau), \quad t \in [ih, (i+1)h), \tag{22}$$

$$\mathbf{w}_i = \mathbf{w}(ih) = \mathbf{D}\mathbf{y}(ih). \tag{23}$$

The solution over one discrete step reads

$$\mathbf{y}((i+1)h) = \mathbf{P}_i \mathbf{y}(ih) + \mathbf{R}_{i,0}\mathbf{w}_{i-r} + \mathbf{R}_{i,1}\mathbf{w}_{i+1-r}, \tag{24}$$

where $\mathbf{P}_i$ is given by Eq. (9) and

$$\mathbf{R}_{i,0} = -\int_0^h \frac{s-\tau+(r-1)h}{h} \mathrm{e}^{\tilde{\mathbf{A}}_i(h-s)}\mathbf{B}(s)\,\mathrm{d}s, \tag{25}$$

$$\mathbf{R}_{i,1} = \int_0^h \frac{s-\tau+rh}{h} \mathrm{e}^{\tilde{\mathbf{A}}_i(h-s)}\mathbf{B}(s)\,\mathrm{d}s. \tag{26}$$

If $\tilde{\mathbf{A}}_i^{-1}$ exists and $\mathbf{B}(s) \equiv \mathbf{B}$ is constant, then integration gives

$$\mathbf{R}_{i,0} = \left(\tilde{\mathbf{A}}_i^{-1} + \frac{1}{h}(\tilde{\mathbf{A}}_i^{-2} - (\tau-(r-1)h)\tilde{\mathbf{A}}_i^{-1})(\mathbf{I}-\mathrm{e}^{\tilde{\mathbf{A}}_i h})\right)\mathbf{B}, \tag{27}$$

$$\mathbf{R}_{i,1} = \left(-\tilde{\mathbf{A}}_i^{-1} + \frac{1}{h}(-\tilde{\mathbf{A}}_i^{-2} + (\tau-rh)\tilde{\mathbf{A}}_i^{-1})(\mathbf{I}-\mathrm{e}^{\tilde{\mathbf{A}}_i h})\right)\mathbf{B}. \tag{28}$$

In this case, the coefficient matrix in Eq. (11) reads

$$\mathbf{C}_i = \begin{pmatrix} \mathbf{P}_i & \mathbf{0} & \dots & \mathbf{0} & \mathbf{R}_{i,1} & \mathbf{R}_{i,0} \\ \mathbf{D} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} & \dots & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \vdots & & & & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{I} & \mathbf{0} \end{pmatrix} \tag{29}$$

and the approximate monodromy matrix is given by Eq. (14).

3. Rate of convergence estimates

Rigorous proof of the convergence was established for the zeroth-order case in Ref. [4], and the convergence of the higher-order techniques can be proved similarly. In this section, the rate of convergence will be investigated for the zeroth-, first- and higher-order approximations.

First, the discretization error over a single interval $[0,h]$ is analyzed. The difference between the discretized and the exact solution is

$$\begin{aligned} \mathbf{x}(h)-\mathbf{y}(h) &= \int_0^h \mathbf{A}(s)\mathbf{x}(s) - \tilde{\mathbf{A}}_0\mathbf{y}(s)\,\mathrm{d}s + \int_0^h \mathbf{B}(s)\mathbf{u}(s-\tau) - \mathbf{B}(s)\boldsymbol{\sigma}_0^{(p)}(s-\tau)\,\mathrm{d}s \\ &= \underbrace{\int_0^h \mathbf{A}(s)\mathbf{x}(s) - \tilde{\mathbf{A}}_0\mathbf{x}(s)\,\mathrm{d}s}_{E_1} + \int_0^h \tilde{\mathbf{A}}_0\mathbf{x}(s) - \tilde{\mathbf{A}}_0\mathbf{y}(s)\,\mathrm{d}s \\ &\quad + \underbrace{\int_0^h \mathbf{B}(s)\mathbf{u}(s-\tau) - \mathbf{B}(s)\boldsymbol{\sigma}_0^{(p)}(s-\tau)\,\mathrm{d}s}_{E_2}. \end{aligned} \tag{30}$$

Terms E_1 and E_2 can be analyzed using Taylor expansions over $[0,h]$ of $\mathbf{u}$, $\boldsymbol{\sigma}_0^{(p)}$, $\mathbf{A}$ and $\mathbf{B}$ with $\mathbf{x}(t) = \mathbf{y}(t)$ in $t \in [-\tau, 0]$. Long, but straightforward computations show that $E_1 = \mathcal{O}(h^3)$ independently on the order of the approximation, while $E_2 = \mathcal{O}(h^2)$ if $p = 0$ and $E_2 = \mathcal{O}(h^3)$ if $p = 1$.

Taking the norm of both sides of Eq. (30), we obtain the local discretization error

$$E_{\mathrm{local}} = \|\mathbf{x}(h)-\mathbf{y}(h)\| \leqslant \|E_1+E_2\| + \int_0^h K\|\mathbf{x}(s)-\mathbf{y}(s)\|\,\mathrm{d}s, \tag{31}$$

with $K = \|\tilde{\mathbf{A}}_0\|$. Due to the Gronwall inequality, Eq. (31) yields

$$\|\mathbf{x}(s) - \mathbf{y}(s)\| \leqslant \|E_1 + E_2\| \exp(Ks) \tag{32}$$

for all $s \in [0, h]$. This implies that $E_{\text{local}} = \mathcal{O}(h^2)$ if $p = 0$ and $E_{\text{local}} = \mathcal{O}(h^3)$ if $p = 1$.

If $p > 1$, then the order of the term E_2 increases, but the term E_1 remains third order, consequently, $E_{\text{local}} = \mathcal{O}(h^3)$ for all $p > 1$. In order to achieve higher order convergence for $p > 1$, the approximation of matrix $\mathbf{A}(t)$ should be improved, e.g., higher-order Magnus series [13] should be used for the approximation of $\mathbf{A}(t)$ instead of Eq. (6). (Note that Eq. (6) corresponds to the first-order Magnus series.) Here, however, we only investigate the cases $p = 0$ and 1 with Eq. (6). So, we can conclude that if piecewise constant approximation of the periodic coefficients is used, then the first-order approximation of the delayed term is the optimal choice.

Repeated applications of Eqs. (30)–(32) over the intervals $[h, 2h], [2h, 3h], \ldots, [(k-1)h, kh]$, where $k = T/h$, give that the maximum error over the period T is $\mathcal{O}(h)$ if $p = 0$, and $\mathcal{O}(h^2)$ if $p = 1$.

4. Semi-discretization of the delayed Mathieu equation

The demonstration of the rate of convergence of the semi-discretization method is problematic, since the exact stability properties of the underlying systems are usually unknown. To the best knowledge of the authors, the only time-delayed time-periodic system with analytically available stability chart is the delayed Mathieu equation [14]. Here, this equation will be analyzed as a case study, so the approximate stability charts can be compared to the exact one.

We consider the delayed Mathieu equation in the form

$$\ddot{x}(t) + \kappa\dot{x}(t) + (\delta + \varepsilon\cos\omega t)x(t) = bx(t-\tau). \tag{33}$$

If $\kappa = 0$ and $\tau = T = 2\pi/\omega$ (i.e., the period T is equal to the time delay τ), then the exact stability boundaries can be given analytically, as it was shown in Ref. [14]. The corresponding nonlinear perturbation of the delayed Mathieu equation was considered in Ref. [15].

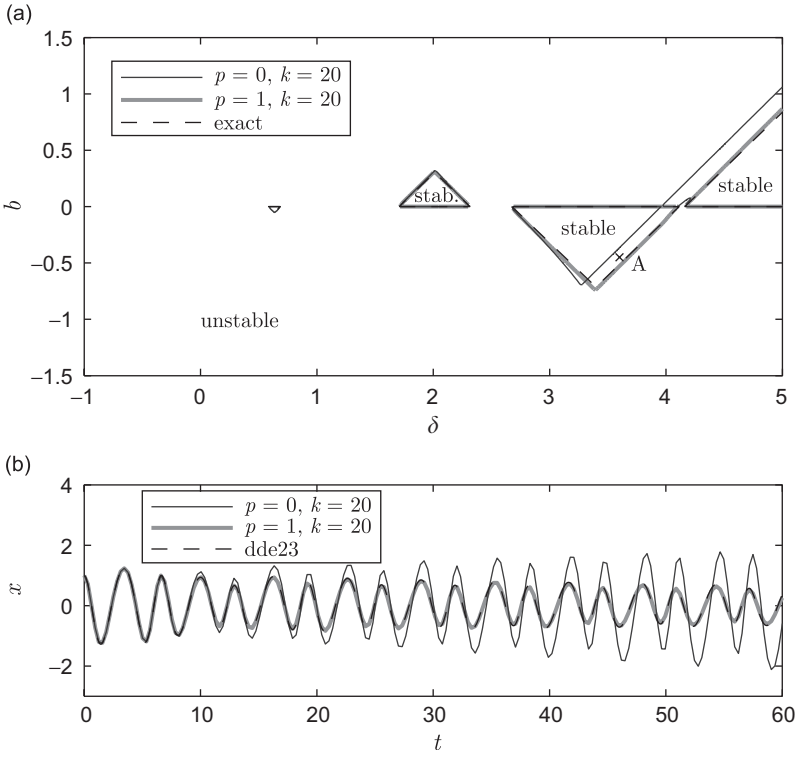

Fig. 2. (a) Stability chart of Eq. (33) with $\varepsilon = 2$, $\kappa = 0$, $T = \tau = 2\pi$; (b) simulations for parameter point A.

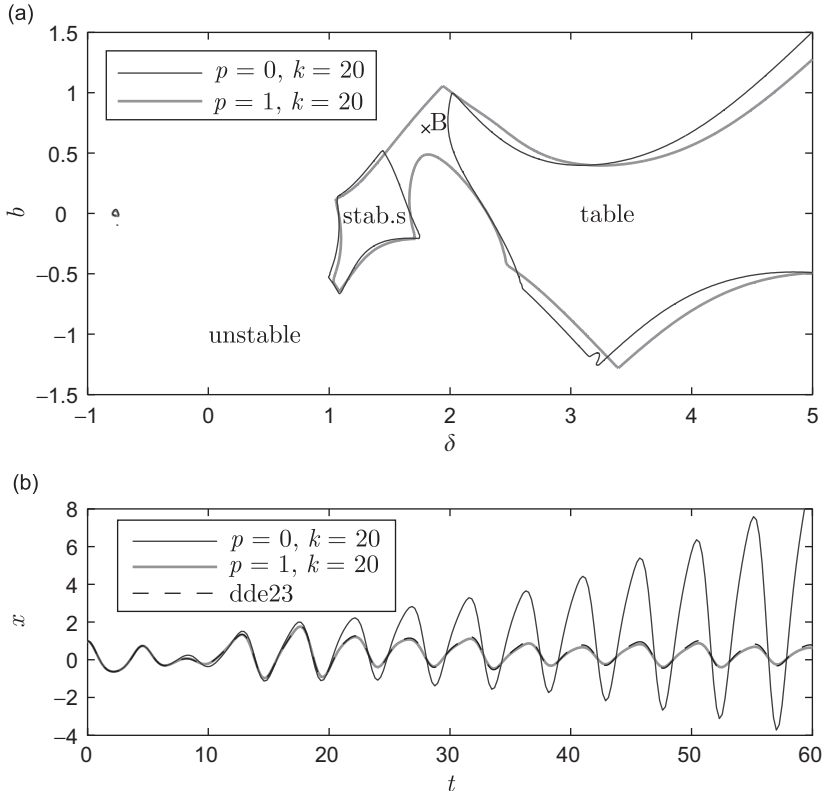

Fig. 3. (a) Stability chart of Eq. (33) with $\varepsilon = 2$, $\kappa = 0.2$, $T = 3\pi/2$, $\tau = 2\pi$; (b) simulations for parameter point B.

Eq. (33) can be written in the state space forms Eqs. (1) and (2) with

$$\mathbf{x}(t) = \begin{pmatrix} x(t) \\ \dot{x}(t) \end{pmatrix}, \quad \mathbf{A}(t) = \begin{pmatrix} 0 & 1 \\ -(\delta + \varepsilon \cos \omega t) & -\kappa \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} 0 \\ b \end{pmatrix}, \quad \mathbf{D} = (1\ 0).$$

Zeroth- and first-order semi-discretizations can be obtained as described in Section 2. We do not investigate higher-order cases, since they do not give any improvements in the convergence, as it was shown in Section 3.

The top panel of Fig. 2 shows the stability chart of the delayed Mathieu equation with $\varepsilon = 2$, $\kappa = 0$, $T = \tau = 2\pi$ obtained by zeroth- and first-order semi-discretizations using $k = 20$ discrete steps over the period T. The charts were determined via point-by-point evaluation of the Floquet transition matrices and the associated critical eigenvalues over a 400×200-sized grid of parameters δ and b. The exact stability boundaries are also presented for reference. It can be seen, that the first-order method results in a smaller error than the zeroth-order one. The bottom panel shows three different simulations obtained by the zeroth- and the first-order semi-discretization, and the dde23 module of Matlab. The associated parameters are $\delta = 3.6$ and $b = -0.45$, this is denoted by point A in the stability chart. It can be seen that the zeroth-order method predicts an unstable process, while the first-order one shows that the system is stable.

The top panel of Fig. 3 shows the stability chart for the parameters $\varepsilon = 2$, $\kappa = 0.2$, $\tau = 2\pi$, $T = 3\pi/2$. In this case, no closed form stability boundaries are available. The bottom panel shows the simulations for the parameters $\delta = 1.8$, $b = 0.7$ that correspond to point B in the stability chart.

5. Conclusion

Semi-discretization of periodic delayed systems was considered using zeroth-, first-, and higher-order approximations of the delayed term. It was shown that if the time-periodic coefficients in the equation are

approximated by piecewise constant functions, then there is no need to use higher than first-order approximations of the delayed term. In order to achieve higher order convergence, the time-periodic coefficients should also be approximated using higher-order techniques, like the Magnus series method [13]. The results were demonstrated by constructing approximate stability charts of the delayed Mathieu equation.

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