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Erosion of safe basins in a nonlinear oscillator under bounded noise excitation

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Abstract

The erosion of the safe basins of a quadratic nonlinearity oscillator under harmonic or bounded random noise excitations is studied in detail by the Monte-Carlo method. It is found that a small random disturbance may destroy the integrity of the safe basins, thus making the system less safe. However, numerical results show that increasing the system's damping and decreasing the system's nonlinearity may enlarge the original integral safe basins. As an alternative definition, stochastic bifurcation may be defined as a sudden change in the character of stochastic safe basins when the bifurcation parameter of the system passes through a critical value, which is different from the previous ones by the authors, where stochastic bifurcation may be defined as a sudden change in the character of a stochastic attractor when the bifurcation parameter of the system passes through a critical value.

1. Introduction

Engineering structures are often subjected to time-dependent loadings of both deterministic and stochastic nature, such as the natural phenomena due to wind gusts, earthquakes, ocean waves, and random disturbance or noise, which always exists in a physical system. The influence of random disturbance on the dynamical behavior, especially bifurcation phenomena, of a nonlinear dynamical system has caught the attention of many researchers. At present, there are mainly two kinds of definitions for stochastic bifurcation available. One is based on the sudden change of the stationary probability density function—the so-called *P*-bifurcation [1]—and the other one is based on the sudden change of sign of the largest Lyapunov exponent—the so-called *D*-bifurcation [1]. The lack of a certain relationship between the shape variation of the stationary probability density function of the random response and the quantitative variation of the random excitation is the difficulty encountered by *P*-bifurcation. On the other hand, the lack of an efficient and accurate algorithm for calculating the Lyapunov exponent is the difficulty encountered by *D*-bifurcation. Besides, several studies show that these two kinds of definitions may lead to different results [2,3]. For instance, Baxendale [2] provides an example in which the shape of the stationary probability density does not depend on the bifurcation

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parameter, while the largest Lyapunov exponent changes its sign. On the contrary, Crauel and Flandoli [3] presented an example in which the stationary probability density function does change its shape from a mono peak into a double peak at a critical parameter value, while the largest Lyapunov exponent does not change its sign. Thus, one cannot help thinking about what really occurs in stochastic bifurcation, what is the topological property of a stochastic system, what kind of invariance is suitable for predicting stochastic bifurcation, and so on. In the recent researches [4–7], the authors of this study provide an alternative topological definition, in which stochastic bifurcation may be defined as a sudden change in the character of a stochastic attractor when the bifurcation parameter of the system passes through a critical value.

Alternative to the stationary probability density function, the largest Lyapunov exponent, and the stochastic attractor, there is another quantitative method to identify the increase of bifurcation and chaos. We know that one of the goals of studying dynamical systems is to determine their global structures, and one of these global structures is the boundary of a basin. For some time, the limitation of the vibration amplitude may be more important, since the structure of the system will be destroyed when the amplitude of the vibration passes through a critical value. Studies on safe basins in deterministic oscillators can be found in Refs. [8–12]. It is well known that the coexistence of periodic and chaotic attractors often leads to fractal basin boundaries. On the contrary, a smooth basin boundary being eroded into a fractal one may imply the generation of chaos in the system. Up to now, few studies [13] have focused on the influences of random disturbance on the safe basins. In this paper, the erosion of the safe basins of a quadratic nonlinearity oscillator under harmonic or bounded random noise is studied in detail by the Monte-Carlo method. It is found that a small bounded random noise may destroy the integrity of the safe basins and make the system less safe.

2. Erosion of safe basins in a deterministic system

Consider a quadratic nonlinearity oscillator under deterministic harmonic excitations governed by the following equation:

$$\ddot{x} + \mu\dot{x} + x - \alpha x^2 = f \cos \Omega t, \tag{1}$$

where overhead dots indicate differentiation with respect to time t, μ is the damping ratio of the oscillator, α is a positive constant which denotes the density of the nonlinearity of the oscillator, and f and $\Omega > 0$ are the amplitude and frequency of the deterministic excitation, respectively.

System (1) is a typical nonlinear dynamical system. While dealing with vibration problems of shipping and circumgyrating axial, the coupling of the lengthwise and transverse vibrations in a pillar may be modeled as a free vibration problem with quadratic nonlinearity as shown in system (1). According to Refs. [8,9], the safe basins of the system may be defined using a bounded area D in the space of phase trajectories. The trajectory start from the safe basins will remain in the area D when the time t tends to infinity. Otherwise, the trajectory start beyond the safe basins will escape the area D; such a trajectory is unstable and may destroy or collapse the system. The structure of the safe basins is similar to some attractor basins [10]. The acreage and shape of the safe basins will change when the parameter of the system changes.

In this paper, the evolution of the safe basins of system (1) is studied numerically when the parameter f changes its value, firstly. In the numerical simulation, the parameters in system (1) are chosen as $\mu = 0.5$, $\alpha = 1.0$, and $\Omega = 1.0$. The bounded area D is defined as follows:

$$D = \{(x, y) : -4 \leqslant x \leqslant 4, \ \ -4 \leqslant y \leqslant 4, \ \ y = \dot{x}\},$$

then D is divided into 200×200 lattices, and the lattice points are taken as the initial values for the solutions of system (1). If the solution of system (1) remains in the area D for a sufficiently long time up to $t = 2000$, such a solution can be approximately taken as a safe solution, and the corresponding lattice may be taken as part of the safe basins; if the solution of system (1) escapes the area D, such a solution is taken to be an unsafe solution, and the corresponding lattice is beyond the safe basins. The governing equation (1) is numerically integrated by the fourth-order Runge–Kutta algorithm, and the numerical results are shown in Fig. 1(a)–(j). The black region denotes the safe basins while the blank region represents the unsafe area in Figs. 1, 3, 7, and 8.

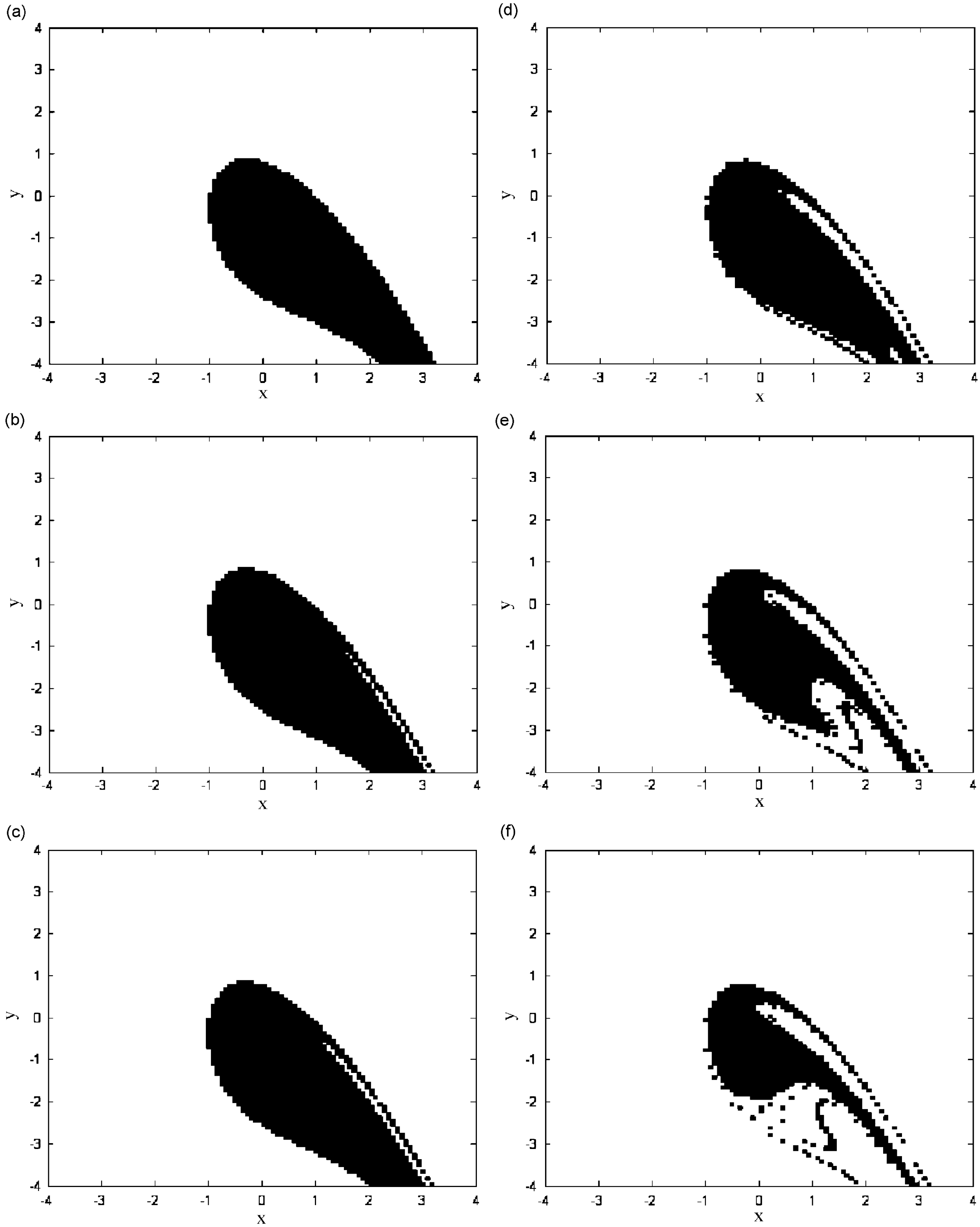

Fig. 1. Erosion of safe basins in system (1): $\mu = 0.5$, $\alpha = 1.0$, $\Omega = 1.0$. (a) $f = 0.300$, (b) $f = 0.350$, (c) $f = 0.360$, (d) $f = 0.380$, (e) $f = 0.390$, (f) $f = 0.395$, (g) $f = 0.397$, (h) $f = 0.400$, (i) $f = 0.410$ and (j) $f = 0.420$.

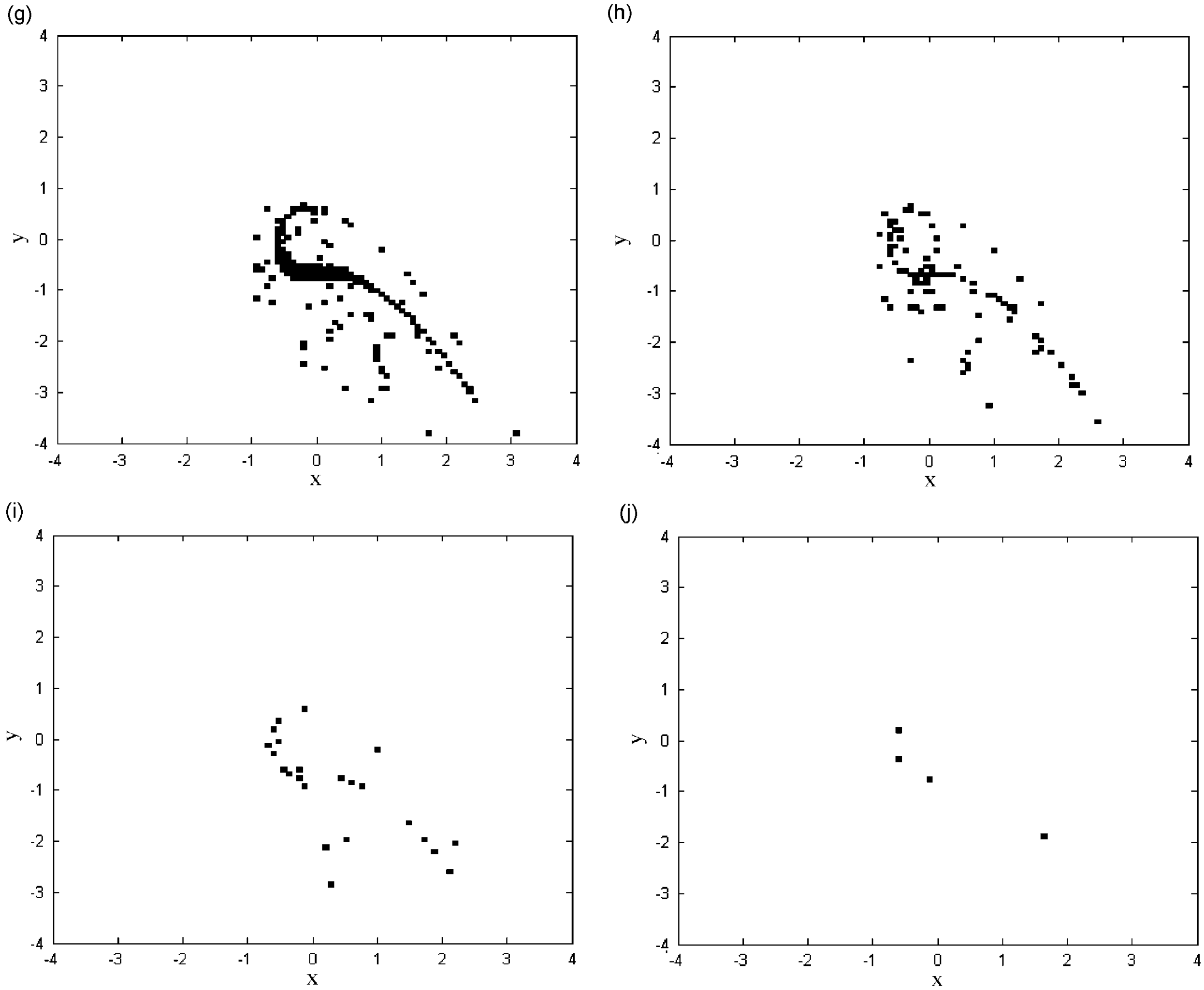

Fig. 1. (*Continued*)

The safe basins shown in Fig. 1(a) are a densely packed, integral ones, while the safe basins shown in Fig. 1(b)–(j) are eroded ones. Calculation results show that in the case when $f \leqslant f_1 = 0.321$, the boundaries of the safe basins of system (1) are smooth without any erosion as shown in Fig. 1(a); in the case when $f > f_1$, the boundaries of the safe basins are eroded more and more with an increase of f as shown in Fig. 1(b)–(j); and in the case when $f > f_1 = 0.443$, the safe basins disappear completely. Hence, f_1 and f_2 are two significant critical points for the evolution of erosion.

To quantify the erosion process more clearly, we introduce the measure G_m to assess the engineering integrity of the safe basins following the methodology established by Soliman and Thompson [9]. Using a grid of N starts, we write the proportions that fall within D in m forcing cycles number as G_m. The numerical results are shown in Fig. 2.

3. Erosion of safe basins in a stochastic system

Next, we consider the effect of the random noise on the safe basins. System (1) is rewritten as follows:

$$\ddot{x} + \mu\dot{x} + x - \alpha x^2 = \xi(t), \tag{2}$$

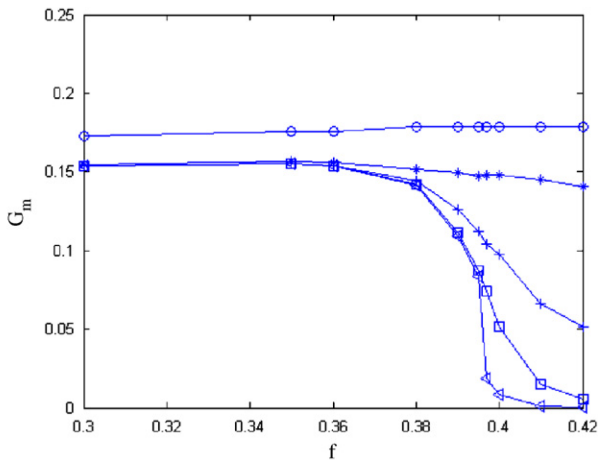

Fig. 2. Integrity measure curves of system (1): ○: G_1, *: G_2, ×: G_3, □: G_4, △: G_5.

where $\xi(t)$ is a bounded random process which is governed by the following equations:

$$\xi(t) = f\cos\varphi(t), \quad \dot{\varphi} = \Omega + \gamma\dot{W}(t), \tag{3}$$

where $f>0$ is the amplitude of the random excitation, Ω the center frequency, $W(t)$ the standard Wiener process, and $\gamma \geqslant 0$ the noise intensity. Herein, the external dynamic force $\xi(t)$ is modeled by a cosine function with deterministic amplitude f, and an angle $\phi(t) = \Omega t + \gamma W(t)$ whose rotating speed is a constant Ω which is superimposed by white noise $\dot{W}(t)$ of intensity γ. According to Wedig [14], the power spectrum $S_\xi(\omega)$ of $\xi(t)$ is

$$S_\xi(\omega) = \frac{1}{2}\frac{f^2\gamma^2(\Omega^2+\omega^2+(\gamma^4/4))}{(\Omega^2-\omega^2+(\gamma^4/4))^2+\omega^2\gamma^4}. \tag{4}$$

The generalized fluctuation model (3) covers both opposite limit cases from Eq. (4). Obviously, the limit procedure $f = (\gamma/\sqrt{2}) \to \infty$ leads to the uniformly distributed power spectrum of white noise. However, if $\gamma \to 0$, the fluctuation spectrum $S_\xi(\omega)$ is disappearing over the entire frequency range except at two discrete frequencies $\omega = \pm\Omega$ where $S_\xi(\pm\Omega)$ goes to infinity, implying a harmonic excitation. Obviously $|\xi(t)| \leqslant f$, and hence $\xi(t)$ is a bounded random process. When $\gamma = 0$, we have $\xi(t) = f\cos\Omega t$, which is exactly the harmonic excitation we considered in Eq. (1). In this paper, only the case when γ is small, say $\gamma < 0.1$, is discussed. In order to compare the results with those of the deterministic system (1), the parameters in system (2) are chosen as follows:

$$\mu = 0.5,\ \alpha = 1.0,\ \Omega = 1.0,\ \gamma = 0.03.$$

For the method of numerical simulation, the reader is referred to Shinozuka [15], and the Monte-Carlo method is used to generate random samples. Eq. (3) can be written as follows:

$$\begin{cases} \xi(t) = f\cos(\phi(t)), \\ \dot{\phi}(t) = \Omega + \gamma\zeta(t), \quad \zeta(t) = \dot{W}(t). \end{cases} \tag{5}$$

The formal derivative $\zeta(t)$ of the unit Wiener process is Gaussian white noise, which has a uniform power spectrum and is physically unrealized. However, for the numerical simulation in this paper, the power spectrum of $\zeta(t)$ is taken as

$$S_\zeta(\omega) = \begin{cases} 1, & 0 < \omega \leqslant 2\Omega, \\ 0, & \omega > 2\Omega. \end{cases} \tag{6}$$

For numerical simulation, it is more convenient to use the pseudorandom signal given by [15]

$$\zeta(t) = \sqrt{\frac{4\Omega}{N}} \sum_{k=1}^{N} \cos\left[\frac{\Omega}{N}(2k-1)t + \varphi_k\right], \tag{7}$$

where φ_k's are mutually independent and uniformly distributed in (0, 2π], and N is a large integer number. By the center limit theorem, it can be proved [15] that when $N \to \infty$, the random process $\xi(t)$ given by Eq. (7) will converge to an ergodic Gaussian stationary process with the same correlation function and spectrum density given by Eq. (6) as that of the expected process.

Here, only 500 random samples are used in this paper due to the limitation of calculation capacity. If the solution of system (2) remains in the area D for a sufficiently long time t not less than 2000 in all the 500 random samples, which can be approximately taken as a safe solution, then the corresponding lattice may be considered to be a part of the safe basins, which is defined in a similar way as for the deterministic one; if the solution of system (1) escapes from the area D, such a solution is considered to be an unsafe solution, and the corresponding lattice is not belonging to the safe basins. One may call such safe basins as stochastic safe basins. The governing equation (2) is numerically integrated by the fourth-order Runge–Kutta algorithm, and the numerical results are shown in Fig. 3(a)–(h).

Fig. 3(a)–(h) shows that the stochastic safe basins are eroded more and more with an increase of f, which is similar to the deterministic case as shown in Fig. 1(a)–(j), and yet with significant differences in f_1 and f_2. Calculation results show that in the case when $f \leqslant f_1 = 0.313$, the boundaries of the stochastic safe basins of system (2) are smooth without any erosion as shown in Fig. 1(a); in the case when $f > f_1$, the boundary of the stochastic safe basins begins to be eroded more and more with an increase of f as shown in Fig. 3(b)–(h); and in the case when $f > f_2 = 0.391$, the stochastic safe basins disappear completely. The random disturbance $\gamma W(t)$ causes f_1 and f_2 to decrease from $f_1 = 0.321$, $f_2 = 0.443$ to $f_1 = 0.313$, $f_2 = 0.391$ and makes the system more unsafe in comparison with the deterministic case.

The integrity measure curves of system (2) are shown in Fig. 4. For different γ, the pictures of f_1 and f_2 as functions of γ are drawn in Figs. 5 and 6. Figs. 5 and 6 show that both f_1 and f_2 are decreasing functions of γ, which means random disturbance may destroy the integrity of the safe basins, thus making the system less safe. Here, the ratio of the safe basins acreage and the acreage of area D is used to calculate the two critical points f_1 and f_2. The ratio will be a constant, hence $f < f_1$ when f is small; the ratio will become small when the boundary of the stochastic safe basins begins to be eroded hence $f = f_1$; however the ratio will be zero when $f = f_2$.

4. Effect of other parameters on safe basins

Herein, the effect of the other parameters such as μ and α on the safe basins is discussed. The effect of the parameter μ on the deterministic safe basins is discussed firstly. The safe basins of system (1) are shown in Fig. 7, when the parameters of system (1) are chosen as follows:

$$f = 0.395, \ \alpha = 1.0, \ \mu = 1.0, \ \gamma = 0.0,$$

which are the same as in Fig. 1(f), except μ increases from 0.5 to 1.0 for comparison. Clearly, the safe basins shown in Fig. 7 are larger than that in Fig. 1. The two critical values of the deterministic safe basins of system (1) are $f_1 = 0.747$ and $f_2 = 1.099$ in the case when $\alpha = 1.0$, $\mu = 1.0$, $\gamma = 0.0$, while the two critical values in system (2) are $f_1 = 0.525$, $f_2 = 0.685$ in the case when $\alpha = 1.0$, $\mu = 1.0$, $\gamma = 0.03$. The area of the safe basins in Fig. 7 is larger than the area of the safe basins in Fig. 3. Obviously, the random noise $\gamma W(t)$ makes the system less safe, but this can be improved by increasing the system's damping.

Then the effect of α on the deterministic safe basins is discussed secondly. The safe basins of system (1) are shown in Fig. 8, when the parameters of system (1) are chosen as follows:

$$f = 0.395, \ \alpha = 0.5, \ \mu = 0.5, \ \gamma = 0.0,$$

which are the same as in Fig. 1(f), except α decreases from 1.0 to 0.5 for comparison. Clearly, the safe basins shown in Fig. 8 are larger than that in Fig. 1. The two critical values of the deterministic safe basins of system (1) are $f_1 = 0.681$ and $f_2 = 0.881$ in the case when $\alpha = 0.5$, $\mu = 0.5$, $\gamma = 0.0$, while the two critical values in

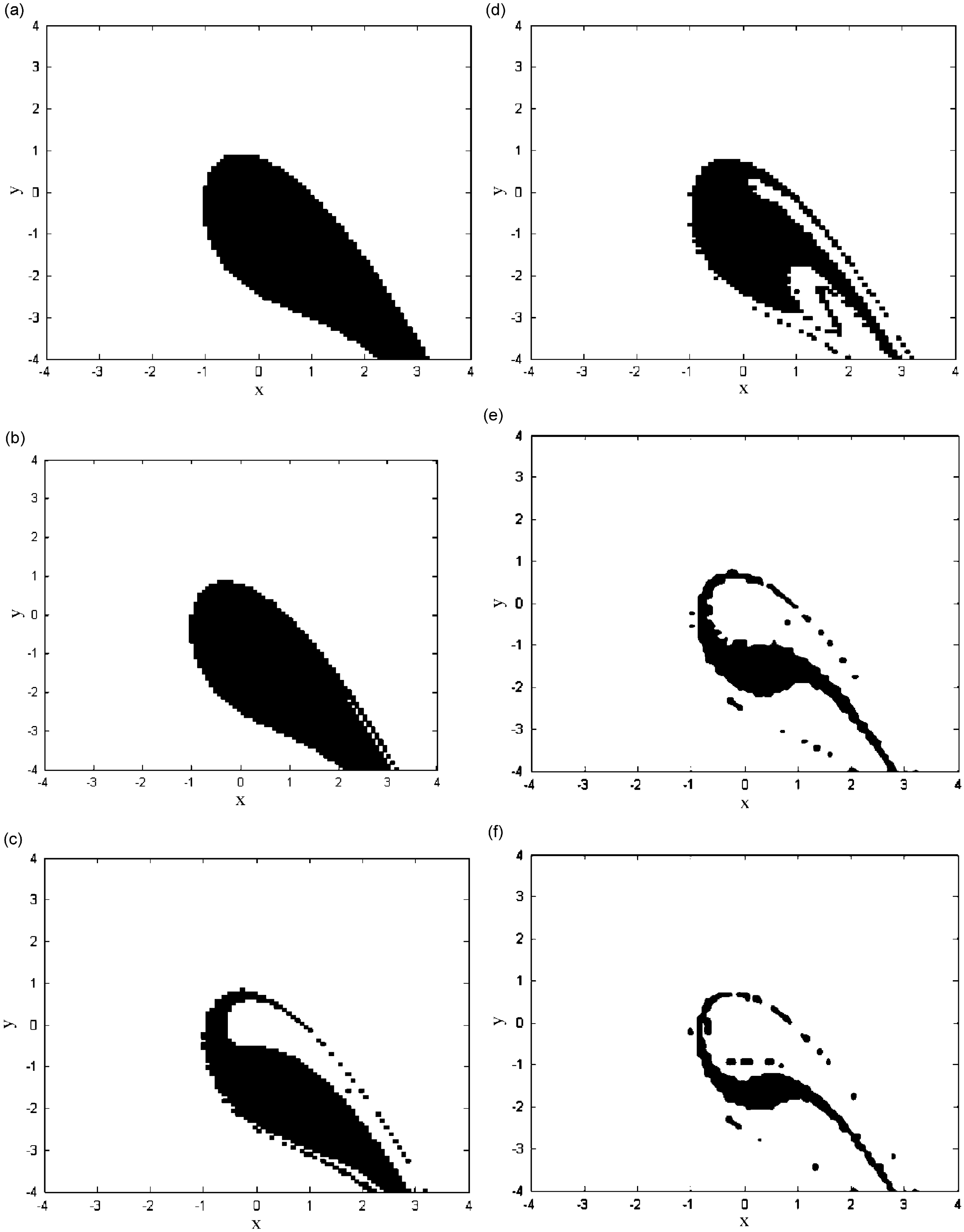

Fig. 3. Erosion of safe basins in system (2): $\mu = 0.5$, $\alpha = 1.0$, $\Omega = 1.0$, $\gamma = 0.03$. (a) $f = 0.300$, (c) $f = 0.370$, (d) $f = 0.372$, (e) $f = 0.377$, (f) $f = 0.380$, (g) $f = 0.385$ and (h) $f = 0.389$.

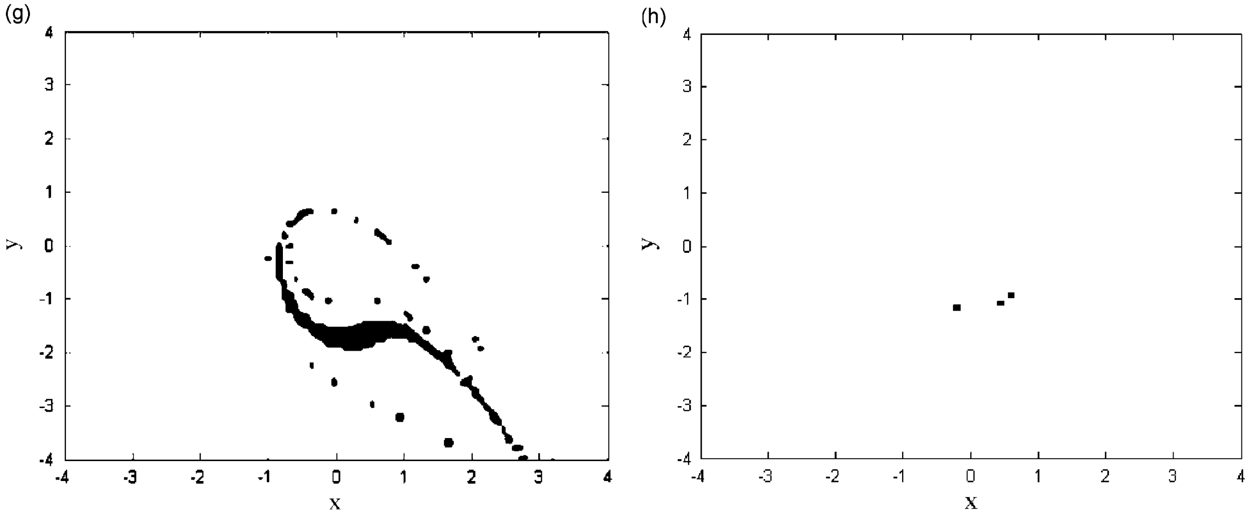

Fig. 3. (*Continued*)

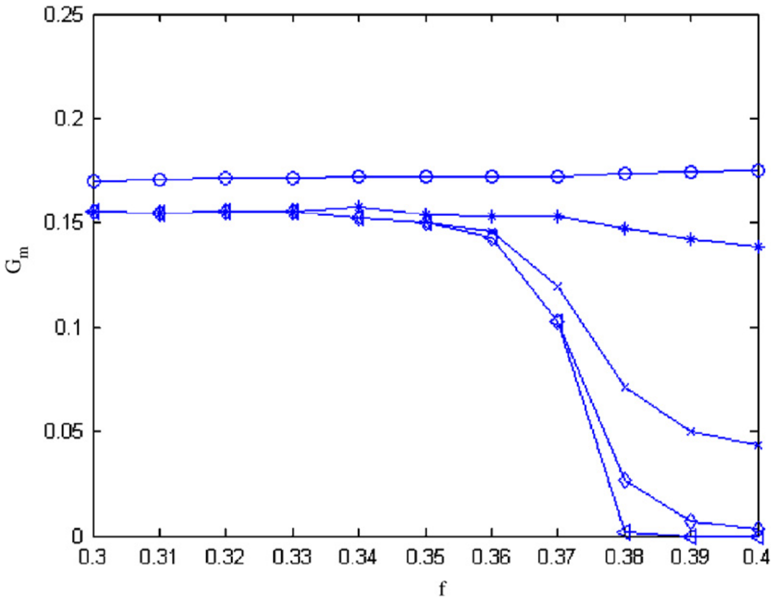

Fig. 4. Integrity measure curves of system (2): ○: G_1, *: G_2, ×: G_3, □: G_4, △: G_5.

system (2) are $f_1 = 0.597, f_2 = 0.692$ in the case when $\alpha = 0.5$, $\mu = 0.5$, $\gamma = 0.03$. Obviously, the random noise $\gamma W(t)$ makes the system less safe, but this can be improved by decreasing the system's nonlinearity.

5. Conclusion and discussion

The analysis shows that random disturbances lead to a decrease of two critical values f_1 and f_2, and make the system less safe; however, this can be improved by increasing the system's damping and decreasing the system's nonlinearity. The stochastic safe basins discussed here should be attached to some probability measure, since the random samples 500 are limited due to the limitation of calculation capacity. More random samples should be taken if one wants to have a more reliable probabilistic analysis.

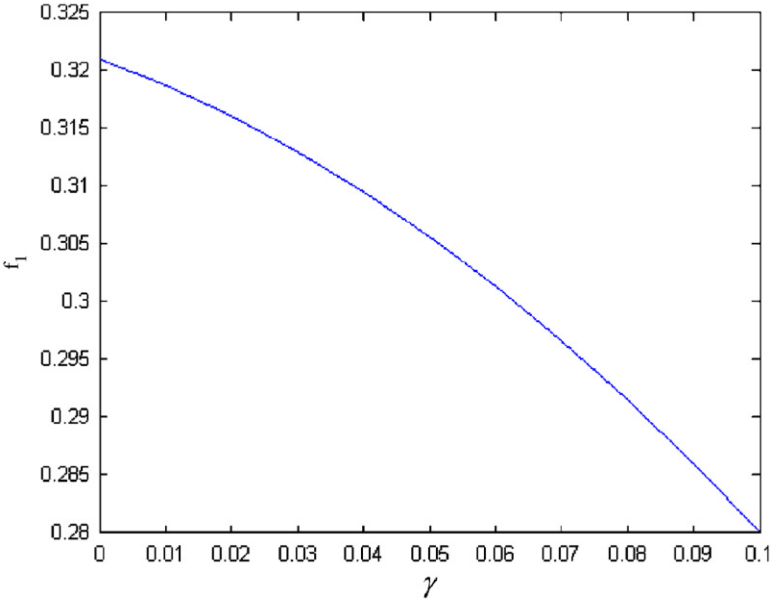

Fig. 5. f_1 as a function of γ in system (2): $\mu = 0.5$, $\alpha = 1.0$, $\Omega = 1.0$.

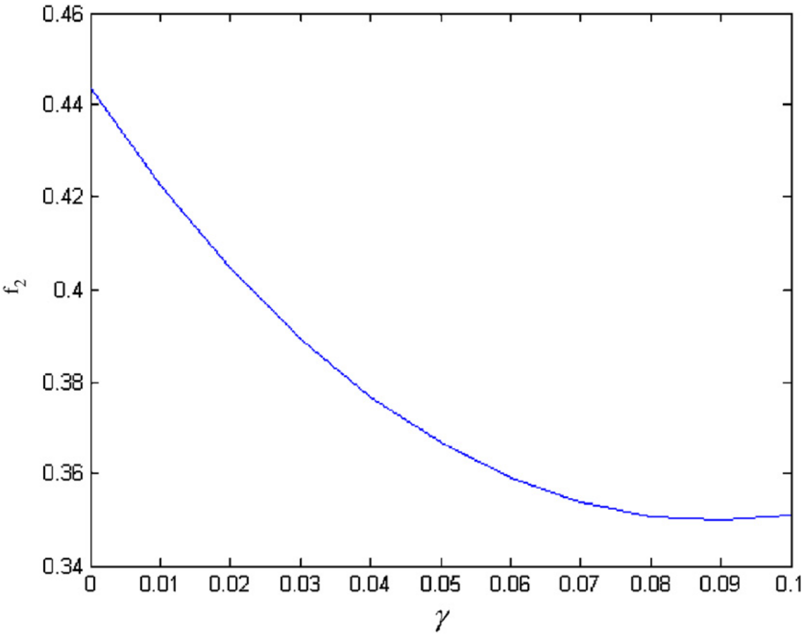

Fig. 6. f_2 as a function of γ in system (2): $\mu = 0.5$, $\alpha = 1.0$, $\Omega = 1.0$.

It is likely that a small value of damping μ can be more realistic and will provide more compact safe basins, but the large values of the damping $\mu = 0.5$, 1.0 taken here may reduce the CPU time since the solution needs a shorter time to escape from the potential well. It is known that the solution needs a short time to escape from the potential well, when it does not correspond to a safe initial condition; hence, $T = 2000$ is a sufficiently long time for the transient time.

One may take the parameters μ, α, f, and γ as adjustable parameters to study the evolution of erosion of the safe basins in the further research.

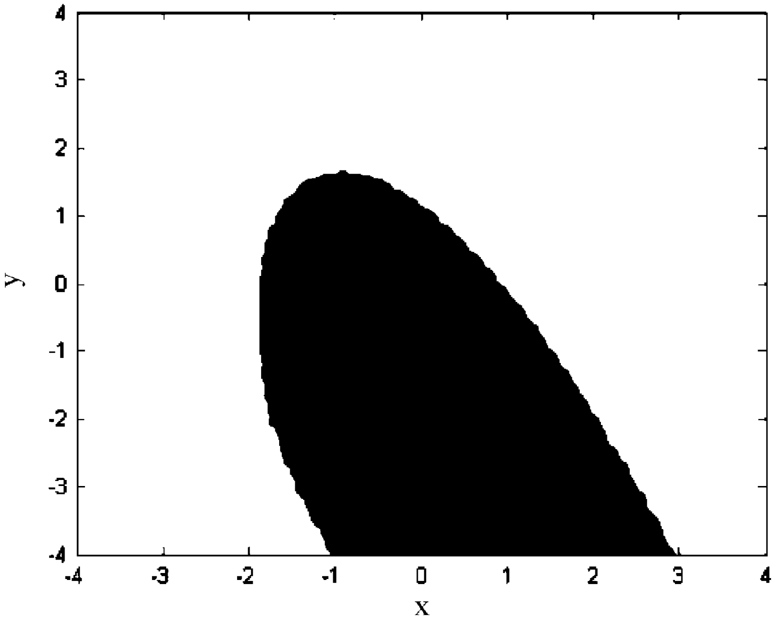

Fig. 7. Safe basins in system (1) $f = 0.395$, $\alpha = 1.0$, $\mu = 1.0$, $\gamma = 0.0$.

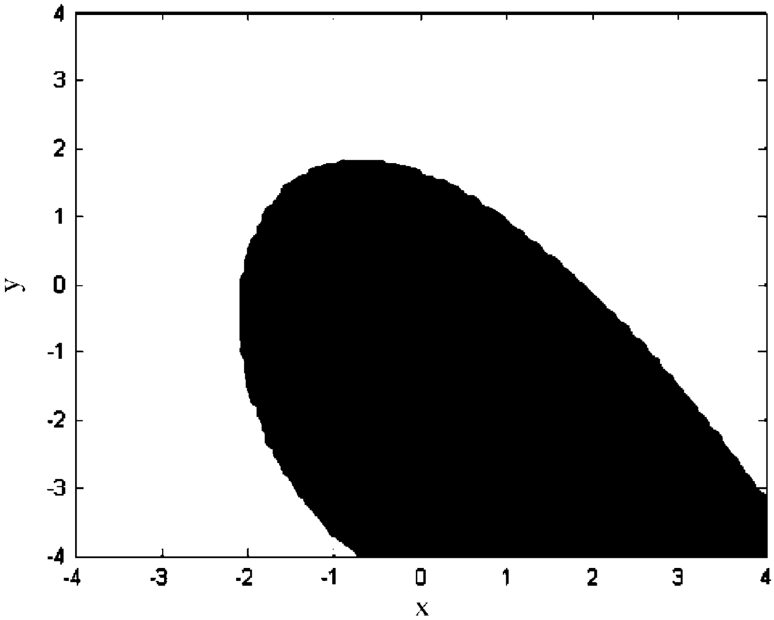

Fig. 8. Safe basins in system (1) $f = 0.395$, $\alpha = 0.5$, $\mu = 0.5$, $\gamma = 0.0$.

As an alternative definition, stochastic bifurcation may be defined as a sudden change in the character of stochastic safe basins when the bifurcation parameter of the system passes through a critical value, which is different from the previous ones by the authors, where stochastic bifurcation may be defined as a sudden change in the character of a stochastic attractor when the bifurcation parameter of the system passes through a critical value. One may call such phenomena that occur in the sudden change of the safe basins, which transform from an integrated one to an eroded one or from an eroded one to nothingness when f passes through the critical values f_1 and f_2, as safe basin bifurcation; then the two critical values f_1 and f_2 may be take as stochastic bifurcation points. This definition applies equally well either to randomly perturbed motions or to purely deterministic motions.

Acknowledgments

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