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Nonlinear natural frequencies of an elastically restrained tapered beam

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Abstract

This paper presents the results of an analysis of an elastically restrained tapered cantilever beam using the harmonic balance and the time transformation methods. The results of the analysis show that the frequencies obtained from a two-term harmonic balance analysis are the most accurate and that the frequencies of the first and second modes of vibration change from a hardening mode (i.e. the frequency increases as the vibration amplitude increases) to a softening mode (i.e. the frequency decreases as the vibration amplitude increases) as the taper ratio of the beam is increased. The third mode is always softening regardless of the taper ratio.

1. Introduction

Many engineering structures, such as offshore structure piles, oil platform supports, oil-loading terminals, tower structures and moving arms, can be modelled as tapered beams. Since these structures are relatively flexible due to their high aspect ratio and as they are usually subjected to various excitation loads such as wind loads, wave loads and other excitations, the prediction of their nonlinear natural frequencies is required for design and analysis.

Most of the previous research in this direction has been oriented towards the calculation of linear natural frequencies and mode shapes [1–6], with different end conditions and with attached inertia elements at the free end of the beam. Nageswara Rao and Venkateswara Rao [7] presented a simple formulation for the large-amplitude free vibrations of tapered beams. The method is based on an iterative numerical scheme to obtain results for tapered beams with rectangular and circular cross sections.

The objective of the present work is to extend the analysis and the results obtained in Refs. [8–10] by studying the nonlinear, planar, large-amplitude free vibrations of an elastically restrained tapered beam for the cases of a double taper beam and a single taper "wedge-shaped beam". The mathematical model is derived using the

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Lagrange method and the resulting continuous equation is discretized using the assumed mode method. The inextensibility condition [11] is used to relate the axial shortening due to transverse deflection in the formulation of the kinetic energy of the beam and the nonlinear curvature is used in the potential energy expression.

2. Mathematical model

2.1. System description and assumptions

A schematic of the beam under study is shown in Fig. 1. The physical properties, modulus of elasticity, E, and density ρ, of the beam are constants. The beam thickness and width vary linearly along the beam axis. The restrained end of the beam is modelled by a torsional spring, K_r, in combination with a translational spring, K_t. The cross-sectional area and moment of inertia at the large end are A_1 and I_1, respectively.

The thickness of the beam is assumed to be small compared to the length of the beam, so that the effects of rotary inertia and shear deformation can be ignored. The beam transverse vibration can be considered to be purely planar and the amplitude of vibration may reach large values.

2.2. Derivation of the equation of motion

The potential energy of the system consists of the strain energy due to the bending deformation and the elastic energy stored in the torsional spring K_r and the translational spring K_t and can be written as

$$V = \frac{El}{2}\int_0^1 I(\zeta)R^2\,\mathrm{d}\zeta + \frac{1}{2}\{K_r v'^2 + K_t v^2\}\Big|_{\zeta=1}, \tag{1}$$

where $\zeta = s/l$ and R is the curvature of the beam neutral axis. R takes the form [8,10]

$$R = \lambda\phi', \tag{2}$$

where $\lambda = 1/l$, the prime is the derivative with respect to the dimensionless length, ζ, and ϕ is the change in slope along the beam (see Fig. 2). In order to express the exact curvature in terms of the transverse deflection, v, it is noted that $\cos\phi = \sqrt{1-\sin^2\phi}$. This implies that $\sin\phi = \mathrm{d}v/\mathrm{d}s = \lambda v'$ (see Fig. 2). Differentiating $\sin\phi = \lambda v'$ with respect to ζ, using the above trigonometric identities, expanding the resulted term in a power series and retaining the terms up to the fourth order, the nonlinear curvature R can be expressed as

$$R^2 = \lambda^4(v''^2 + \lambda^2 v''^2 v'^2). \tag{3}$$

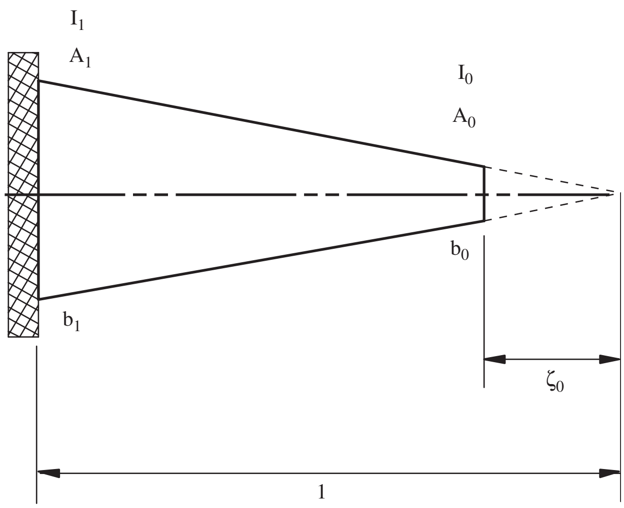

Fig. 1. A schematic for the tapered beam.

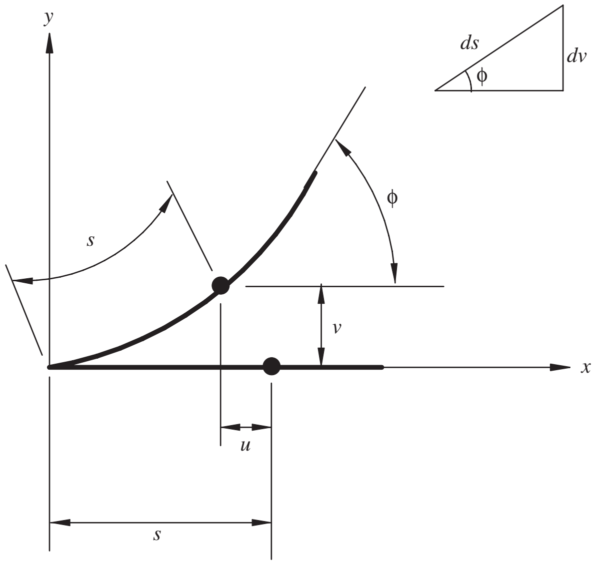

Fig. 2. The deformed inextensible beam.

The kinetic energy T of the beam can be written as

$$T = \frac{1}{2}\rho l \int_0^1 A(\zeta)[\dot{u}^2 + \dot{v}^2]\,\mathrm{d}\zeta, \tag{4}$$

where u is the axial shortening due to bending deformation as can be seen in Fig. 2. The inextensibility condition dictates that a total axial shortening u is given by [10]

$$\lambda u = \zeta - \int_0^\zeta \cos\phi\,\mathrm{d}\eta = \zeta - \int_0^\zeta \sqrt{1 - (\lambda v')^2}\,\mathrm{d}\eta. \tag{5}$$

Expanding the radical term in a power series, assuming that $(\lambda v')^2 \ll 1$, the axial shortening can be represented as

$$u = \frac{1}{2}\int_0^\zeta \left(\lambda v'^2 + \frac{\lambda^3}{4} v'^4\right)\mathrm{d}\zeta. \tag{6}$$

Differentiating Eq. (6) with respect to time yields

$$\dot{u} = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^\zeta (\lambda v'^2)\,\mathrm{d}\zeta. \tag{7}$$

The Lagrangian of the beam can expressed as

$$L = T - V. \tag{8}$$

The continuous system in Eq. (8), like most nonlinear systems, does not admit a closed-form solution. However, the interest here is in the case where the beam motion is dominated by single active mode. Therefore an assumed single mode approach is used to discretize the continuous Lagrangian. The assumption is made that

$$v(\zeta, t) = \phi_i(\zeta) q(t), \tag{9}$$

where $\phi(\zeta)$ is the normalized, self-similar (i.e. independent of the motion amplitude) assumed mode shape of the beam and $q(t)$ is an unknown time modulation of the assumed deflection mode $\phi_i(\zeta)$. In the present work $\phi_i(\zeta)$ for a double tapered beam is (see Ref. [6])

$$\phi_i(\zeta) = \zeta^{-1}[C_1 J_2(Z) + C_2 Y_2(Z) + C_3 I_2(Z) + C_4 K_2(Z)], \tag{10}$$

where $A(\zeta) = A_1(\zeta)^2$ and $I(\zeta) = I_1(\zeta)^4$, and for wedge-type beams (single taper)

$$\phi_i(\zeta) = \zeta^{-1/2}[C_1 J_1(Z) + C_2 Y_1(Z) + C_3 I_1(Z) + C_4 K_1(Z)], \tag{11}$$

where $A(\zeta) = A_1\zeta$ and $I(\zeta) = I_1(\zeta)^3$.

For both cases $Z = 2\beta\zeta^{1/2}$, $\beta^4 = \rho A_1 L_1^4 \omega_l^2 / EI_1$, ω_l is the linear frequency of vibration, J and Y are Bessel functions of the first and second kind, respectively, and I and K are modified Bessel functions of the first and second kind, respectively. C_1–C_4 are arbitrary constants to be determined by imposing the following boundary conditions to both ends of the beam:

$$EI(\zeta)\phi_i''(\zeta_0) = 0, \tag{12}$$

$$\frac{\mathrm{d}}{\mathrm{d}\zeta}(EI(\zeta)\phi_i''(\zeta_0)) = 0, \tag{13}$$

$$K_r \phi_i'(1) - EI(\zeta)\phi_i''(1) = 0, \tag{14}$$

$$\frac{\mathrm{d}}{\mathrm{d}\zeta}(EI(\zeta)\phi_i''(1)) - K_t \phi(1) = 0. \tag{15}$$

Using Eqs. (7), (9) and (10) or (11) the Lagrangian expression of the beam system can be written as

$$L = \rho l^3(\beta_1 \dot{q}^2 + \beta_2 q^2 \dot{q}^2 - \beta^2 \beta_3 q^2 - \beta^2 \beta_4 q^4), \tag{16}$$

where

$$\beta_1 = \int_0^1 A_1^* \phi^2 \, \mathrm{d}\zeta, \tag{17}$$

$$\beta_2 = \int_0^1 A_1^* \left\{ \int_0^\zeta \phi'^2 \, \mathrm{d}\chi \right\}^2 \mathrm{d}\zeta, \tag{18}$$

$$\beta_3 = \int_0^1 I_1^* \phi''^2 \, \mathrm{d}\zeta + \frac{K_t l^3}{EI_1}\phi(1)^2 + \frac{K_r l}{EI_1}\phi'(1)^2, \tag{19}$$

$$\beta_4 = \int_0^1 I_1^* \phi'^2 \phi''^2 \, \mathrm{d}\zeta. \tag{20}$$

For the double tapered beam $A_1^* = A_1\zeta^2$ and $I_1^* = I_1\zeta^4$, and for the single tapered wedge beam $A_1^* = A_1\zeta$ and $I_1^* = I_1\zeta^3$.

Applying the Euler–Lagrangian relation to the system Lagrangian,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = 0, \tag{21}$$

the following nonlinear non-dimensional uni-modal equation of motion is obtained:

$$\beta_1 \ddot{q} + \beta_2(q^2\ddot{q} + q\dot{q}^2) + \beta^2(\beta_3 q + 2\beta_4 q^3) = 0. \tag{22}$$

It is to be noted that some of the coefficients β_i in Eq. (22), defined by Eqs. (17)–(20), in general may have large values. Therefore, for convenience, Eq. (22) is scaled and converted to the dimensionless form

$$\ddot{q} + q + \varepsilon_1(q^2\ddot{q} + q\dot{q}^2) + \varepsilon_2 q^3 = 0. \tag{23}$$

A dot is used to denote a derivative with respect to the non-dimensional time. $t^* = (\beta^2\beta_3/\beta_1)^{1/2}t$, $\varepsilon_1 = \beta_2/\beta_1$ and $\varepsilon_2 = 2\beta_4/\beta_3$ are dimensionless coefficients.

Eq. (23) describes the nonlinear, planar, flexural free vibration of the inextensible tapered beam. In this equation, the terms $\varepsilon_1\ddot{q}q^2$ and $\varepsilon_1 q\dot{q}^2$ are inertial nonlinearities due to the kinetic energy of axial motion which arise as a result of using the inextensibility condition and they are of softening type (i.e., they lead to a decrease in the natural frequency when the vibration amplitude increases). The nonlinear term $\varepsilon_2 q^3$ is due to the potential energy stored in bending and arises as a result of using nonlinear curvature and it is of hardening static type (i.e., it leads to an increase in the natural frequency when the vibration amplitude increases). The nonlinear natural frequencies of the beam are dominated by the two competing nonlinearities mentioned above, and the behaviour of the elastically restrained tapered beam considered in this paper is either hardening or softening depending on the ratio $\varepsilon_1/\varepsilon_2$ [9].

3. Method of solution

The calculations of the coefficients β_i in Eqs. (17)–(20), ε_1 and ε_2, indicate that the nonlinear oscillator described in Eq. (23) is strongly nonlinear, and the nonlinear natural frequencies are calculated using two methods: the harmonic balance method (HB) and the time transformation method (TT). The initial conditions are taken to be $q(0) = A$ and $\dot{q}(0) = 0$, where A is the amplitude of the motion.

3.1. The harmonic balance method

According to the HB method, an approximate single-term solution (SHB) takes the form [4]

$$q(t^*) = A\cos(\omega t^*), \tag{24}$$

where ω is the nonlinear natural frequency. Substituting Eq. (24) and its derivatives into Eq. (23) and equating coefficients, one obtains

$$\omega^2 = \frac{1 + (3/4)\varepsilon_2 A^2}{1 + (\varepsilon_1 A^2/2)}. \tag{25}$$

To improve the accuracy of the assumed solution, more terms can be added and a two-term solution is sought (2THB), such that

$$q(t^*) = A_1\cos(\omega t^*) + A_3\cos(3\omega t^*). \tag{26}$$

Using the above-mentioned initial conditions yields

$$A = A_1 + A_3. \tag{27}$$

Substituting Eq. (26) and their derivatives into Eq. (23) and equating the coefficient of each of the assumed harmonics, one obtains

$$A_3 = \frac{(\varepsilon_2/4)(A_1^3 + 3A_3^3) - (\varepsilon_1\omega^2/2)(A_1^3 + 9A_3^3)}{(1 + (3\varepsilon_2 A_1^2/2) - \omega^2(9 + 5\varepsilon_1 A_1^2)}, \tag{28}$$

$$\omega^2 = \frac{1 + (3\varepsilon_2/4)(A_1^2 + A_1A_3 + 2A_3^2)}{1 + (\varepsilon_1/2)(A_1^2 + 3A_1A_3 + 10A_3^2)}. \tag{29}$$

Eqs. (28) and (29) are solved numerically for a given amplitude A, using an iterative technique with an accuracy of 10^{-6}. To speed up convergence of the two-term (2THB) method the initial iteration uses the results found from the single-term harmonic balance (SHB) method.

3.2. The time transformation method

The time transformation method, described in detail in Ref. [8], is used to obtain an approximation to the frequency–amplitude relation of the nonlinear oscillator given in Eq. (23). Accordingly, a single-valued transformation $T(t^*)$ is sought between the time t^* and a new time domain T and the solution of Eq. (23) is simple harmonic with a period equal to 2π, i.e. one assumes $q(T) = A\cos T$, where $T(0) = 0$. Transforming Eq. (23) to the new time domain, T, defining $F = \mathrm{d}T/\mathrm{d}t^*$ and substituting for $q(T) = A\cos T$ and its derivatives into Eq. (23), one obtains

$$\left\{1 - F^2\left(1 + \frac{\varepsilon_1}{2}A^2\right) + \frac{3}{4}\varepsilon_2 A^2\right\}\cos T + \frac{A^2}{4}(\varepsilon_2 - 2\varepsilon_1 F^2)\cos 3T - FF'\left\{\left(1 + \frac{\varepsilon_1}{4}A^2\right)\sin T + \frac{\varepsilon_1}{4}A^2 \sin 3T\right\} = 0, \tag{30}$$

where a prime denotes a derivative with respect to time T. Eq. (30) can be solved for F^2 by noting that, since the nonlinear equation (23) does not involve even nonlinearities, a series solution of period 2π may be assumed in the form, using two terms only:

$$F^2 = \sum_{n=0,2}^{\infty} G_n \cos nT = G_0 + G_2 \cos 2T + G_4 \cos 4T. \tag{31}$$

Substituting Eq. (31) into Eq. (30), equating to zero the coefficient of each harmonic in the resulting equation, and solving for G_0, G_2 and G_4, one obtains the following expressions for the coefficients G_0, G_2 and G_4:

$$G_0 = \frac{1}{4\Delta}\{(96 + 4\varepsilon_1 A^2(20 + 3\varepsilon_1 A^2)) + \varepsilon_2(3A^2 - A^4(3\varepsilon_1 + \varepsilon_1^2 A^2))\}, \tag{32}$$

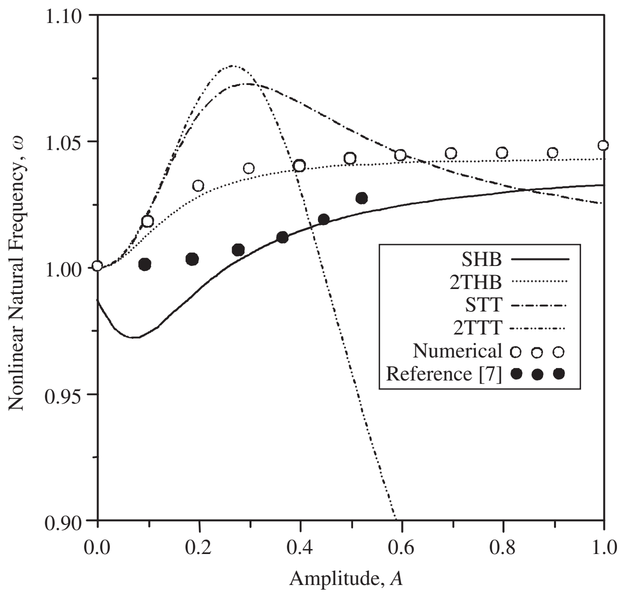

Fig. 3. Nonlinear natural frequency versus amplitude of the first mode for a single tapered beam.

Table 1
Comparison of nonlinear natural frequencies obtained from different models with the numerical solution

Amplitude	0.0	0.095	0.188	0.279	0.366	0.448	0.523
Exact (numerical)	4.410	4.485	4.543	4.575	4.585	4.592	4.600
Rao and Rao [7]	4.292 (2.67%)	4.299 (4.14%)	4.317 (4.97%)	4.348 (4.96%)	4.392 (4.20%)	4.452 (3.05%)	4.528 (1.56%)
SHB	4.355 (1.36%)	4.293 (4.14%)	4.360 (4.02%)	4.423 (3.32%)	4.462 (2.68%)	4.488 (2.26%)	4.504 (2.09%)
2THB	4.410 (0.00%)	4.465 (0.45%)	4.530 (0.29%)	4.561 (0.31%)	4.577 (0.17%)	4.585 (0.15%)	4.589 (0.24%)
STT	4.410 (0.00%)	4.494 (0.20%)	4.665 (2.69%)	4.729 (3.37%)	4.716 (2.86%)	4.673 (1.76%)	4.637 (0.80%)
2TTT	4.410 (0.00%)	4.497 (0.27%)	4.686 (3.15%)	4.759 (4.02%)	4.635 (1.09%)	4.391 (4.38%)	4.161 (9.54%)

Results in parenthesis represent the error. Amplitude values are the same as those in Ref. [7].

$$G_2 = \frac{1}{16\Delta}\{\varepsilon_1 A^2(3-\varepsilon_1 A^2)(4-3A^2\varepsilon_2) - (6+\varepsilon_1 A^2(5+\varepsilon_1 A^2))(4A^2\varepsilon_2)\}, \tag{33}$$

$$G_4 = \frac{3}{64\Delta}\{\varepsilon_1^2 A^4(4+3A^2\varepsilon_2) - \varepsilon_1\varepsilon_2 A^4(2+\varepsilon_2 A^2)\}, \tag{34}$$

$$\Delta = \frac{1}{32}(48+\varepsilon_1 A^2(16+20\varepsilon_1 A^2+\varepsilon_1^2 A^4)). \tag{35}$$

Substituting the result into Eq. (31), using the relation $F = \mathrm{d}T/\mathrm{d}t^*$, integrating the resulting equation from 0 to 2π for T and noting that the period in the time T domain is 2π lead to the nonlinear frequency–amplitude relation in the dimensionless t^* domain:

$$\omega = \sqrt{G_0}\left\{1+\frac{3}{16}(H_2^2+H_4^2)+\frac{105}{1024}(H_2^4+H_4^4)+\frac{1155}{116,384}(H_2^6+H_4^6)+\cdots\right\}, \tag{36}$$

where

$$H_2 = \frac{G_2}{G_0}, \tag{37}$$

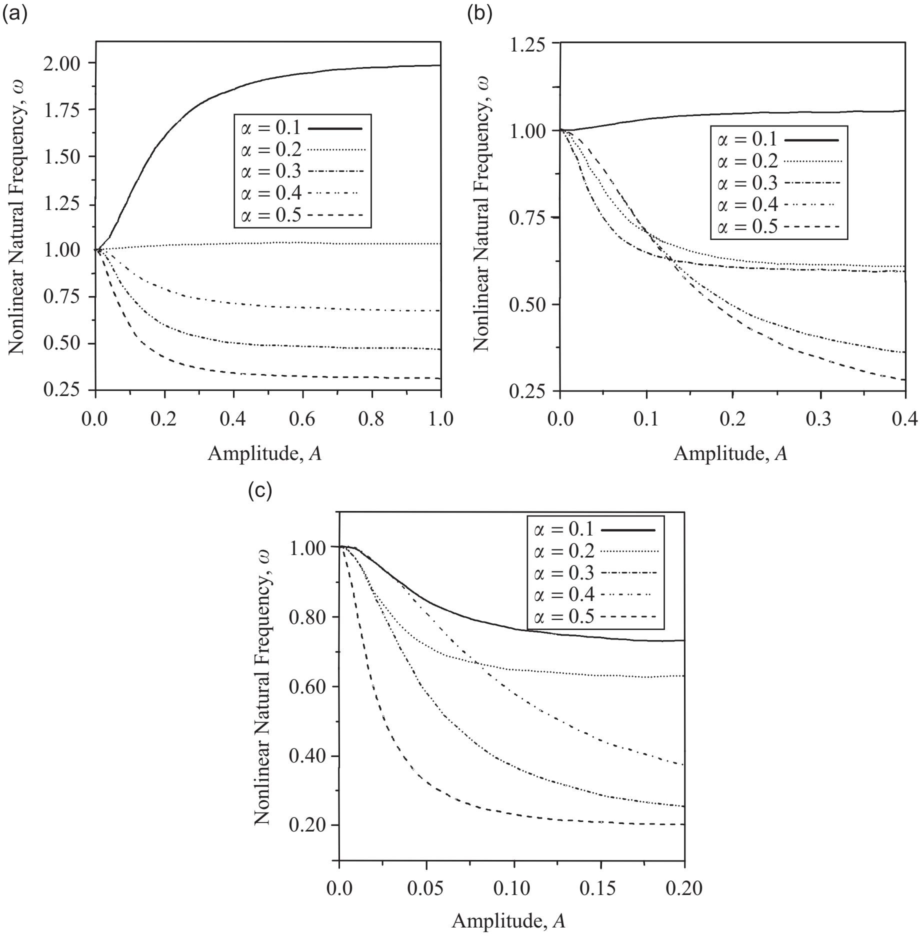

Fig. 4. Nonlinear natural frequency versus amplitude for the double tapered beam, $C_r = C_t = 0$: (a) refers to the first mode; (b) refers to the second mode; and (c) refers to the third mode.

$$H_4 = \frac{G_4}{G_0}. \quad (38)$$

The dimensionless nonlinear natural frequency ω calculated from Eqs. (25), (29) and (36) using the HB and TT methods is the ratio of the nonlinear natural frequency in time t to the natural frequency of the linear beam.

4. Results and discussion

The theory derived above is valid for vibrations with small rotations but with large amplitude as $(\lambda v')^2 \ll 1$ in the derivation of the equation of motion, and this is consistent with the theory presented by Wagner [12]. In addition, harmonics above the third are ignored, because they have a negligible effect either on qualitative or on quantitative dynamical behaviour as shown and presented in earlier publications [9–11].

The coefficients of the terms β_i given in Eq. (23) are calculated by integrating numerically the coefficients given in Eqs. (17)–(20). Also, it is worth mentioning that the range of motion amplitudes to be considered in the present work (i.e., the values of vibration amplitude A) is assumed up to 1.0 for the first mode, 0.4 for the second mode and 0.2 for the third mode, to be consistent with the assumption of large-amplitude vibration. A vibration amplitude of 1 corresponds to a ratio of tip displacement/length of the beam equal to 1.

The accuracy of the calculated nonlinear natural frequencies was first examined by comparing the results obtained using: the harmonic balance method using single (SHB) and two terms (2THB) given in Eqs. (25) and

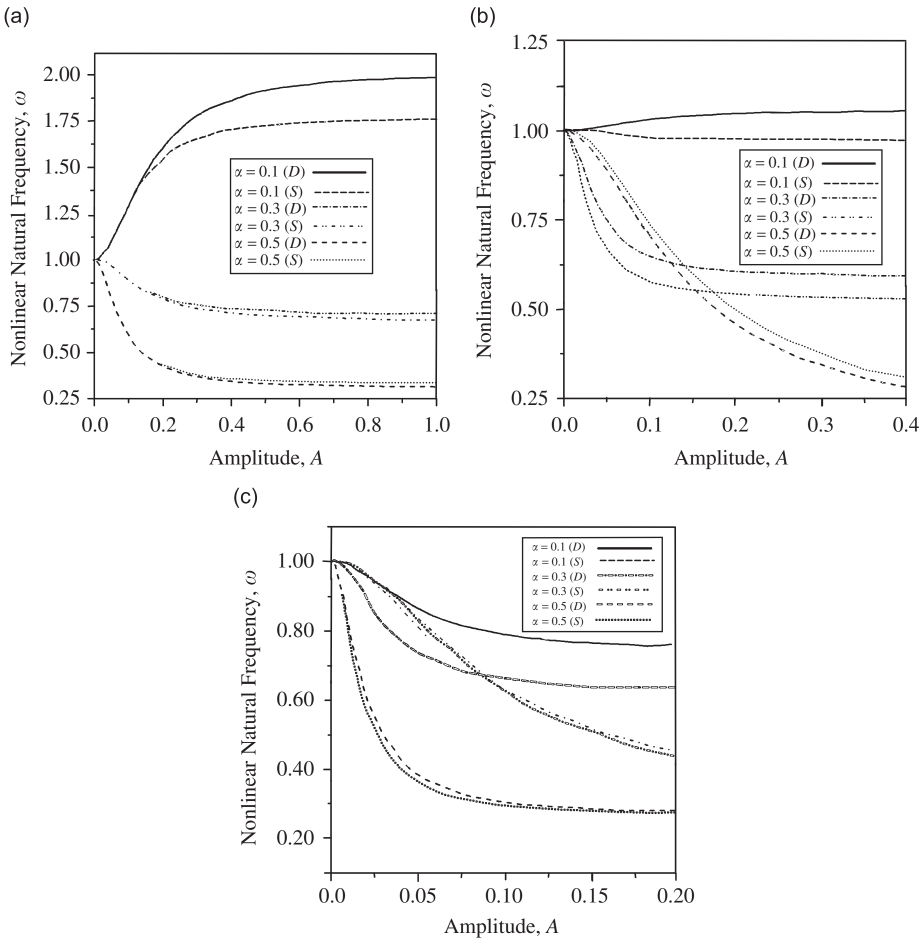

Fig. 5. Comparison of nonlinear natural frequency between single (S) and double (D) tapered beams, $C_r = C_t = 0$: (a) refers to the first mode; (b) refers to the second mode; and (c) refers to the third mode.

(29); the TT using single (STT) and two terms (2TTT) given in Eqs. (30) and (36); and numerically integrating Eq. (23).

The results obtained for the first mode of a cantilever single tapered beam, i.e. $C_r = C_t = 0$, and taper ratio $\alpha = 0.2$ are presented in Fig. 3 and shown in Table 1. The errors quoted are the differences from the approximate solutions and the numerical solution which is assumed to be exact. They are also compared with those obtained by Nageswara Rao and Venkateswara Rao [7]. As can be seen, the most accurate approximate results when compared to a numerical solution are obtained using the 2THB method. These results almost equal those of the 'exact' "numerical solution". The difference in the linear frequencies for the SHB from the numerical solution for low amplitudes is due to the fact that in nonlinear systems with strong nonlinearities, such as the one under consideration, "Eq. (23)", the solution should contain higher harmonics in the assumed solution. The results obtained from the TT failed to give the expected accuracy and the method gave incorrect results for amplitude values more than 0.1. It is therefore recommended that this method be not used for amplitudes greater than 1. The differences between the linear frequencies obtained by Nageswara Rao and Venkateswara Rao [7] and those of the authors are due to the method of solution. Ref. [7] used expressions with a single harmonic time degree to obtain equation of motion and also converted the boundary value problem into an initial value problem before solving the differential equation by the Runge–Kutta method. The authors have used analytical methods to obtain their approximate solutions which are close to the numerical solutions.

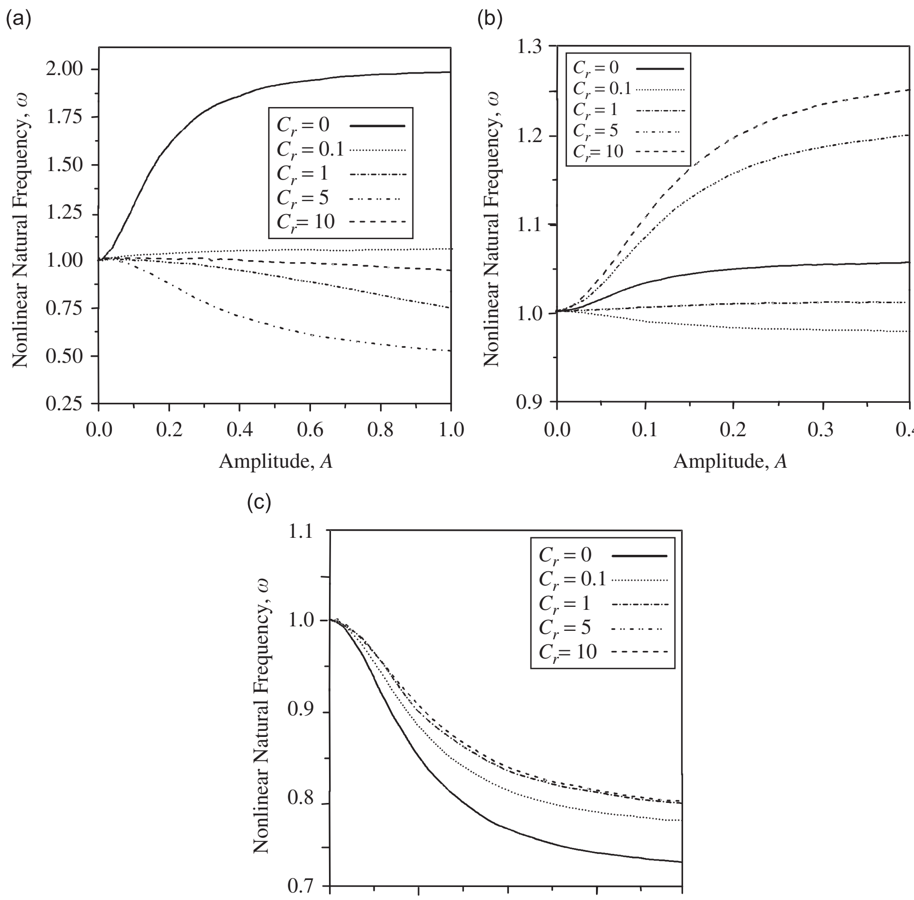

Fig. 6. Nonlinear natural frequency for a restrained double tapered beam, $\alpha = 0.1$, $C_t = 0$: (a) refers to the first mode; (b) refers to the second mode; and (c) refers to the third mode.

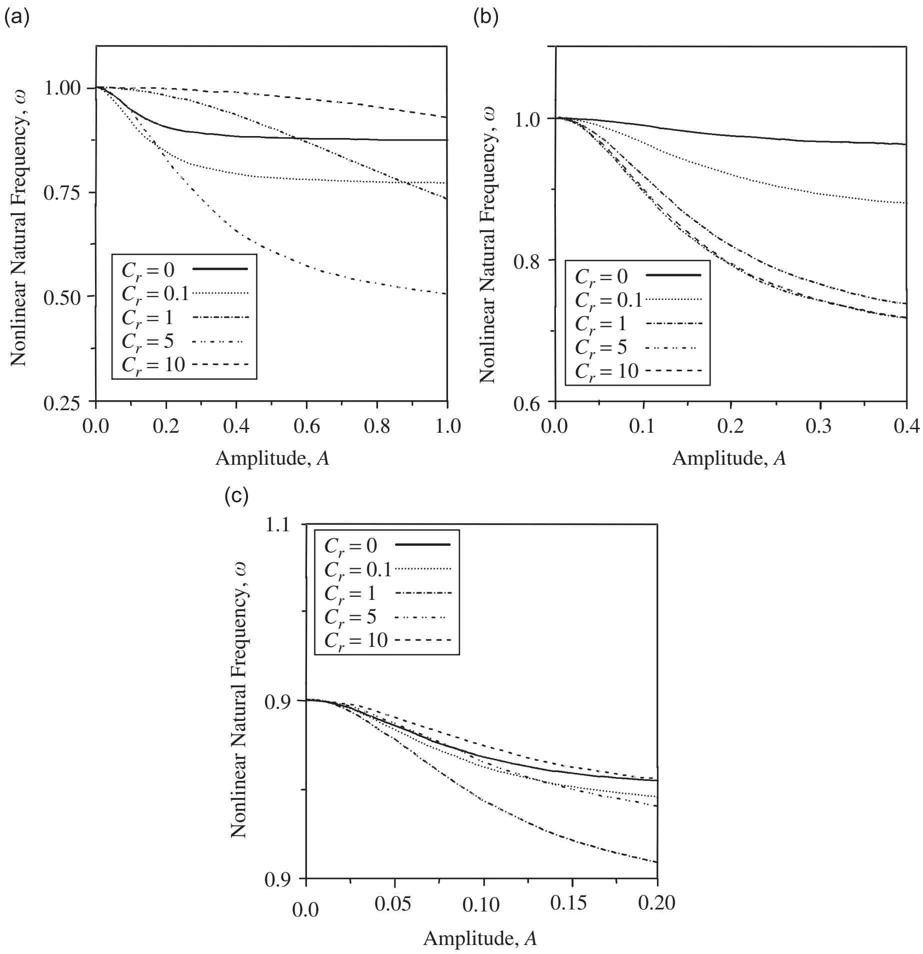

Fig. 7. Nonlinear natural frequency for a restrained double tapered beam, $\alpha = 0.1$, $C_t = 0.1$: (a) refers to the first mode; (b) refers to the second mode; and (c) refers to the third mode.

From the results shown in Fig. 3, the 2THB method is the most accurate approximate method. Consequently, all the remaining results were obtained using the method of harmonic balance method with two terms (2THB).

In Fig. 4, results were obtained for the double-tapered cantilevered beam, i.e. $C_r = C_t = 0$, where $C_r = EI_1/(K_r l)$ and $C_t = EI_1/(K_r l^3)$, and for different values of the taper ratio $\alpha = b_0/b_1$. Results have shown that the behaviour of the first and second modes is changed from hardening to softening when the taper ratio is increased, while the third mode is of a softening type regardless of the value of the taper ratio, α. This is due to the fact that when the taper ratio α increases the mode shape is modified accordingly, which in turn affects the values of the calculated coefficients β_1 given in Eq. (23) and the values of ε_1 and ε_2.

This type of nonlinear oscillator given in Eq. (23) is dominated by two competing nonlinearities ($\varepsilon_1 \ddot{q} q^2$ and $\varepsilon_1 q \dot{q}^2$) and $\varepsilon_2 q^3$, and the behaviour is hardening when the ratio $(\varepsilon_1/\varepsilon_2) \leqslant 1.6$ and softening when $(\varepsilon_1/\varepsilon_2) > 1.6$ [9].

In Fig. 5 a comparison between the double tapered and wedge-type beams for different values of taper ratio $\alpha = b_0/b_1$ is presented. Results have shown that, for a given value of taper ratio α, the natural frequency of a double tapered beam is higher than that of a wedge-type beam.

The effect of the beam's root flexibility on the nonlinear natural frequency is studied and presented in Figs. 6–8. Results were obtained for a given value of taper ratio $\alpha = 0.1$ and for the first three modes.

As shown in Figs. 6–8, the qualitative and quantitative behaviour of the nonlinear natural frequency changes from hardening to a softening depending on the value of the beam's root flexibility, i.e. the values of

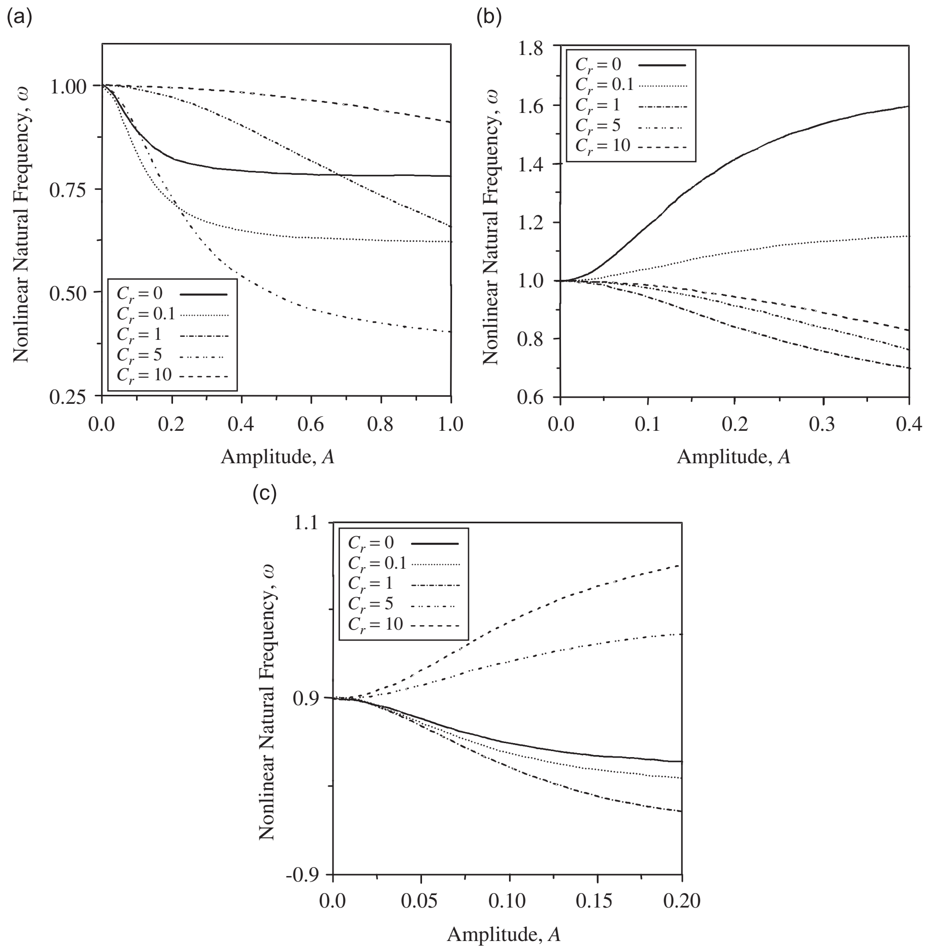

Fig. 8. Nonlinear natural frequency for a restrained double tapered beam, $\alpha = 0.1$, $C_t = 1$: (a) refers to the first mode; (b) refers to the second mode; and (c) refers to the third mode.

$C_r = EI_1/(K_r l)$, $C_t = EI_1/(K_r l^3)$ and the vibration amplitude A. This is due to the nonlinear interaction between these values and their effect on the modes' shape and on the values of β_i, $i = 1$–4, ε_1 and ε_2.

5. Conclusions

A mathematical model for calculating the nonlinear natural frequencies of a tapered beam elastically restrained is derived. The axial shortening due to transverse deflection and the nonlinear curvature are used in the formulation of the kinetic and potential energy, respectively. The assumed mode method is used to discretize the continuous Lagrangian of the system and the resultant uni-modal nonlinear differential equation of motion is solved using the harmonic balance method (HB) and the time transformation method (TT) to calculate the nonlinear natural frequencies for the first three modes of vibrations. The comparisons made with a numerical solution of the differential equation show that harmonic balance method with two terms yields accurate solutions for all amplitudes up to $A = 1$, the TT fails to yield accurate solutions for all amplitudes.

Results have shown that for the first and second modes the behaviour is changed from hardening to softening type when the taper ratio α is increased, while the third mode is of a softening type regardless of the value of the taper ratio α. Also, for a given value of a taper ratio, the nonlinear natural frequency of a double tapered beam is higher than that of a single tapered beam.

From the results presented for the effect of the beam's root flexibility, it was shown that the nonlinear natural frequency changes from softening to hardening behaviour depending on combinations of the physical parameters of the beam C_r, C_t and the vibration amplitude A. This would require a more detailed analysis to study the forced vibration of the beam, which is currently under consideration.

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