Rapid Communication

Isolated sub-harmonic resonance branch in the frequency response of an oscillator with slight asymmetry in the clearance

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Abstract

A family of isolated sub-harmonic branches in nonlinear frequency response of piecewise linear system is examined in this paper. Given their peculiarity, they could be easily overlooked in frequency response studies in numerical integration or laboratory experiments when the quasi-static frequency sweeping technique is employed. An increase in the viscous damping ratio could shrink the isolated branches and ultimately the sub-harmonic resonance may vanish. Asymmetry does not appear to be the key factor that generates the isolated branches. Rather, the relationship between the linear mean operating and transition point seems to dictate their occurrence.

1. Introduction

Piecewise linear systems, with applications to geared systems and the like, have been widely studied over the past decade. In particular, nonlinear frequency response characteristics have been investigated using analytical, numerical and experimental approaches; Refs. [1–5] are typical citations. In addition to the primary resonance, super- and sub-harmonic resonant peaks are also found. Recently, we [5] found a family of isolated sub-harmonic branches for a mechanical oscillator with near pre-load nonlinearity (the stiffness of the first stage is extremely high). Such solutions appear to posses interesting dynamic characteristics though sub-harmonics could be harmful since most real-life devices are designed to operate away from the key resonance(s). Tomlinson and Lam [6] and Ing et al. [7] have examined asymmetric clearance nonlinearity problems but not the isolated sub-harmonic branches. Doole and Hogan [8] found isola in a piecewise linear (bilinear) suspension bridge model. Recently, Takacs et al. [9] found that bifurcation branches of large amplitude periodic motions could be isolated following an isola birth in a shimming wheel system (for certain parameter regions). However, the occurrence of such isolated branches is yet to be understood. Accordingly, this communication will conceptually examine the evolution of isolated sub-harmonic branches by using a mechanical oscillator with an asymmetric piecewise nonlinearity as shown in Fig. 1a. The piecewise restoring force in Fig. 1b is written as follows where α is the ratio between the first and second stage stiffness, b is the

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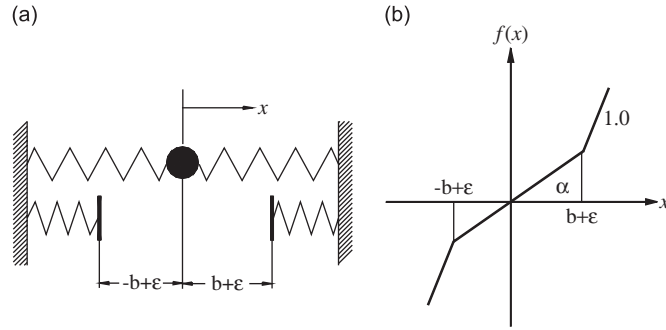*E-mail address:* singh.3@osu.edu (R. Singh).

Fig. 1. A single-degree-of-freedom oscillator with slight asymmetry in the clearance: (a) schematic of the system and (b) force–displacement relationship.

transition point (or clearance), ε is the asymmetry in the first stage and geometrically, $\varepsilon < b$:

$$f(x) = \begin{cases} x - (1-\alpha)b + (\alpha-1)\varepsilon, & x \geqslant (b+\varepsilon), \\ \alpha x, & (-b+\varepsilon) \leqslant x < (b+\varepsilon), \\ x + (1-\alpha)b + (\alpha-1)\varepsilon, & x \leqslant (-b+\varepsilon). \end{cases} \quad (1)$$

We may write Eq. (1) in a simplified form using the signum (sgn) function as

$$f(x) = x + \frac{1-\alpha}{2}[(x-b)\mathrm{sgn}(x-(b+\varepsilon)) - (x+b)\mathrm{sgn}(x-(-b+\varepsilon))]$$
$$+ \frac{(\alpha-1)\varepsilon}{2}[\mathrm{sgn}(x-(b+\varepsilon)) + \mathrm{sgn}(x-(-b+\varepsilon))]\mathrm{sgn}(x). \quad (2)$$

Using Eq. (1) or (2), we write the governing equation of an oscillator under harmonic excitation as follows in the dimensionless form:

$$\ddot{x} + 2\zeta\dot{x} + f(x) = F_m + F_p \sin(\omega t). \quad (3)$$

Here, ζ is the damping ratio, F_m is the mean load (excitation) and F_p is the alternating force amplitude. Further, $b = 1/\alpha$ is presumed by normalizing F_m and F_p, thereby making the stiffness transition point coincide with $F_m = 1$.

2. Calculation of isolated sub-harmonic branches

The piecewise linear system of Eq. (3) can be solved in several ways. For instance, Pavlovskaia and Wiercigroch [10] developed a periodic solution finder for an impact oscillator with a drift. The well-developed numerical solver AUTO97 with path following function [11] could also be utilized. Nevertheless, we intend to construct the frequency responses by using the semi-analytical multi-term harmonic balance method. This method is a very efficient in generating results directly in the frequency domain while yielding some insight [5]. The multi-term harmonic balance method essentially seeks periodic responses that are represented by a truncated Fourier series as shown below where υ is the sub-harmonic index:

$$x(t) = a_0 + \sum_{n=1}^{N_h\upsilon} a_{2n-1}\sin\left(\frac{n}{\upsilon}\omega t\right) + a_{2n}\cos\left(\frac{n}{\upsilon}\omega t\right) \xrightarrow{\text{discretize}} \underline{x} = \underline{\underline{\Delta}}\,\underline{a}, \quad (4a)$$

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Here, $\underline{\underline{\Delta}}$ is the discrete inverse Fourier transform matrix:

$$\underline{\underline{\Delta}} = \begin{bmatrix} 1 & \sin\left(\dfrac{n}{\upsilon}\omega t_0\right) & \cos\left(\dfrac{n}{\upsilon}\omega t_0\right) & \cdots & \cos\left(\dfrac{N_h\upsilon}{\upsilon}\omega t_0\right) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & \sin\left(\dfrac{n}{\upsilon}\omega t_{K-1}\right) & \cos\left(\dfrac{n}{\upsilon}\omega t_{K-1}\right) & \cdots & \cos\left(\dfrac{N_h\upsilon}{\upsilon}\omega t_{K-1}\right) \end{bmatrix}. \tag{5}$$

By matching the harmonic coefficients between the excitation (with coefficient F_m and F_p and denoted as $\underline{p}$ in vector format) and responses, the numerical integration of Eq. (3) is transformed into an algebraic problem. Then numerical continuation or path following technique is then employed to minimize the residue $\underline{R}$ where $\underline{\underline{D}}$ is a differentiator:

$$\underline{R} = \omega^2 \underline{\underline{D}}^2 \underline{a} + 2\zeta\omega \underline{\underline{D}}\,\underline{a} + \underline{b} - \underline{p}. \tag{6}$$

To ensure the minimization calculation following the deepest descent, Jacobian ($\underline{\underline{J}}$) matrix has to be defined (readers should refer to Ref. [4] for a more detailed discussion of the multi-term harmonic balance method and
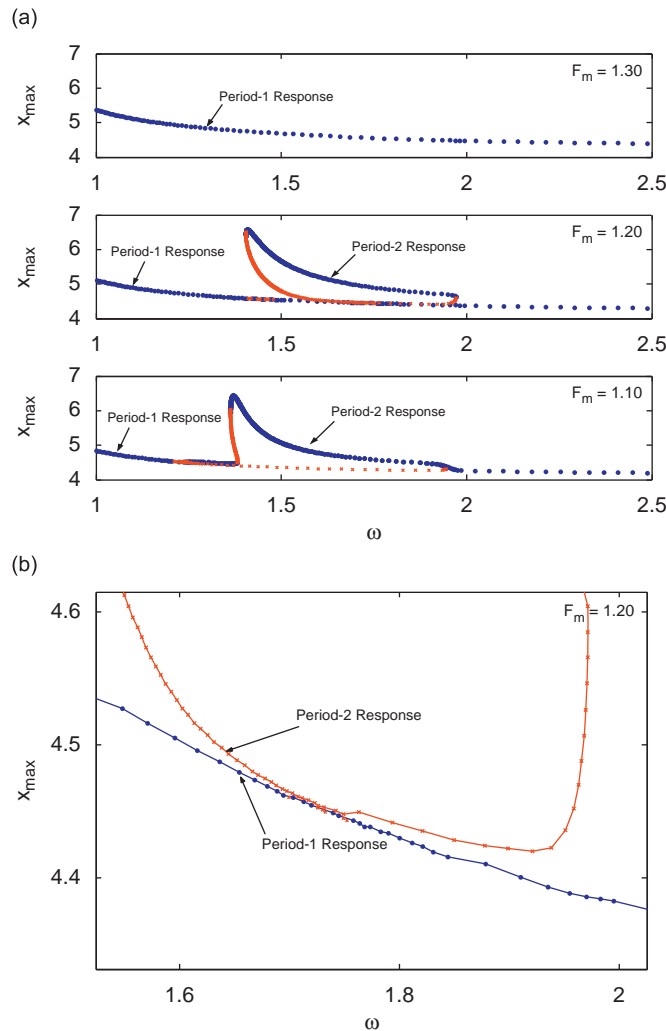

Fig. 2. Evolution of the sub-harmonic resonance for a piecewise linear system given $\alpha = 0.25$, $F_p = 0.5$, $\zeta = 0.02$, $\varepsilon = 0$. Key: $\bullet\bullet\bullet$, stable multi-term harmonic balance method (MHBM) solution; $\times\times\times$, unstable multi-term harmonic balance method solution.

determination of stability):

$$\underline{\underline{J}} = \frac{\partial \underline{R}}{\partial \underline{a}} = \omega^2 \underline{\underline{D}}^2 + 2\zeta\omega\,\underline{\underline{D}} + \frac{\partial \underline{b}}{\partial \underline{a}}, \tag{7a}$$

$$\nabla \underline{a} = \underline{a}^k - \underline{a}^{k+1} = \underline{\underline{J}}^{-1}\,\underline{R}. \tag{7b}$$

Fig. 2a shows a typical sub-harmonic resonance evolution. It is clearly observed that as F_m moves closer to the stiffness transition point, sub-harmonic resonance is excited. Duan et al. [5] have stated that as a special case when $F_m = 1$, even minimal effort could excite the sub-harmonic resonance. A closer look in Fig. 2b reveals that the sub-harmonic resonance is indeed a closed branch isolated from the period-1 solution. Seydel [12] has discussed a similar loci evolution for a fictitious model. He has explained that the transition from a fully isolated to a continuous branch is attributed to the transcritical bifurcation. That is, there exists a range $1.1 < F_m^B < 1.2$ from which the isolated branch touches the period-1 locus. Nevertheless, the transcritical bifurcation has never been regarded as a common phenomenon since it is rather hard to find it by numerical simulation.
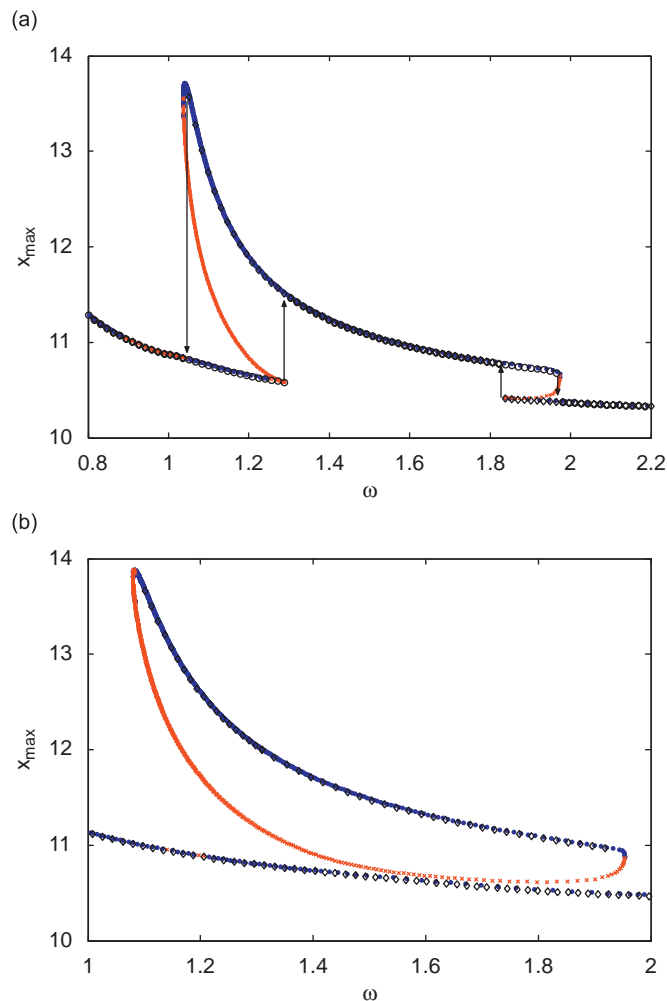

Fig. 3. Multi-term harmonic balance method and numerical integration results given $\alpha = 0.1$, $F_p = 0.5$, $\zeta = 0.02$ and $\varepsilon = 0$: (a) $F_m = 1.20$ and (b) $F_m = 1.3$. Key: •••, stable multi-term harmonic balance method solution; × × ×, unstable multi-term harmonic balance method solution; ◇◇◇, numerical integration with downward sweep; ○○○, numerical integration with upward sweep.

To validate the multi-term harmonic balance method results and to show that the isolated branch is not just a numerical artifact, Fig. 3 presents the numerical integration results (based on the Runge–Kutta scheme). First, both cases show a very good agreement between the numerical integration and multi-term harmonic balance method solutions. Fig. 3a shows a typical jump phenomenon and the sub-harmonic resonance can be easily solved with either multi-term harmonic balance method (employing the excitation perturbation approach as proposed in an earlier paper [5]), or numerical integration. When an isolated branch occurs as in Fig. 3b, system dynamics is obviously more complicated. Traditional numerical integration tends to bypass the isolated resonant peak and thus the period-1 locus cannot jump to the period-2 branch. Instead, one must exercise a massive search using many initial conditions to successfully latch on the isolated branch. Since laboratory experiments tend to adopt frequency sweep technique (like the numerical integration in a quasi-static manner), isolated peak could be easily overlooked unless its occurrence is somehow known *a priori*. Algebraically, the isolated branch exists in a sub-space that cannot be spanned by the period-1 Jacobian in the multi-term harmonic balance method or the predictor–corrector technique in numerical integration.

3. Plausible cause and conclusion

First, we examine the effect of viscous damping ratio ζ on the isolated branches in Fig. 4. As expected, an increase in ζ decreases the peak amplitude at the sub-harmonic resonance. The period-1 solutions under various ζ values have minimal difference because the frequency regime is away from the primary resonance. Further, we also observe that the isolated branch shrinks as ζ increases and it migrates away from the period-1 solution. The isolated branch disappears when $\zeta = 8\%$ and then the sub-harmonic resonance disappears from the nonlinear frequency response plots.

By varying the value of ε, while keeping the other system parameters and excitation unchanged, the system switches from symmetric to asymmetric response. Fig. 5a shows sub-harmonic response for $\varepsilon = 0.05b$. Similar to the symmetric case in Fig. 3b, isolated branches occur again. As ε is further increased to $0.10b$, isolated branches degenerates to a continuous curve in Fig. 5b. Nevertheless, an examination of maximum and minimum responses reveals that the system still experiences the single-side impact condition in both cases. This implies that the response only crosses the positive transition ($x_{c+} = b + \varepsilon$) and no crossing of the negative transition ($x_{c-} = -b + \varepsilon$) occurs. Therefore, the system asymmetry (as shown in Fig. 1a) alone does not generate any isolated branches. Revisiting the results in Fig. 3, we would rather postulate that it is the relative position between the external excitation and the transition that could be the root cause of an isolated branch. To verify this, we decrease F_m to 1.2. This makes the distance between the linear mean operating ($x_{m0} = F_m - (b + \varepsilon)\alpha$) and positive transition equal to the asymmetric case. Now we find in Fig. 3a that the
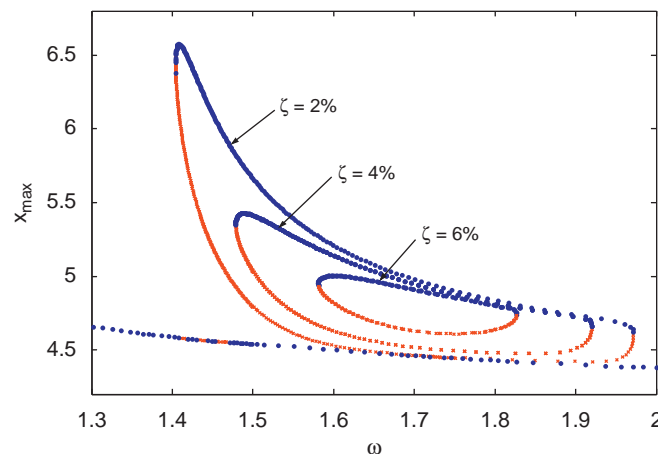

Fig. 4. Effect of viscous damping ratio ζ on the isolated sub-harmonic branch given $\alpha = 0.25$, $F_m = 1.20$, $F_p = 0.5$ and $\varepsilon = 0$. Key: $\bullet\bullet\bullet$, stable multi-term harmonic balance method solution; $\times\times\times$, unstable multi-term harmonic balance method solution.
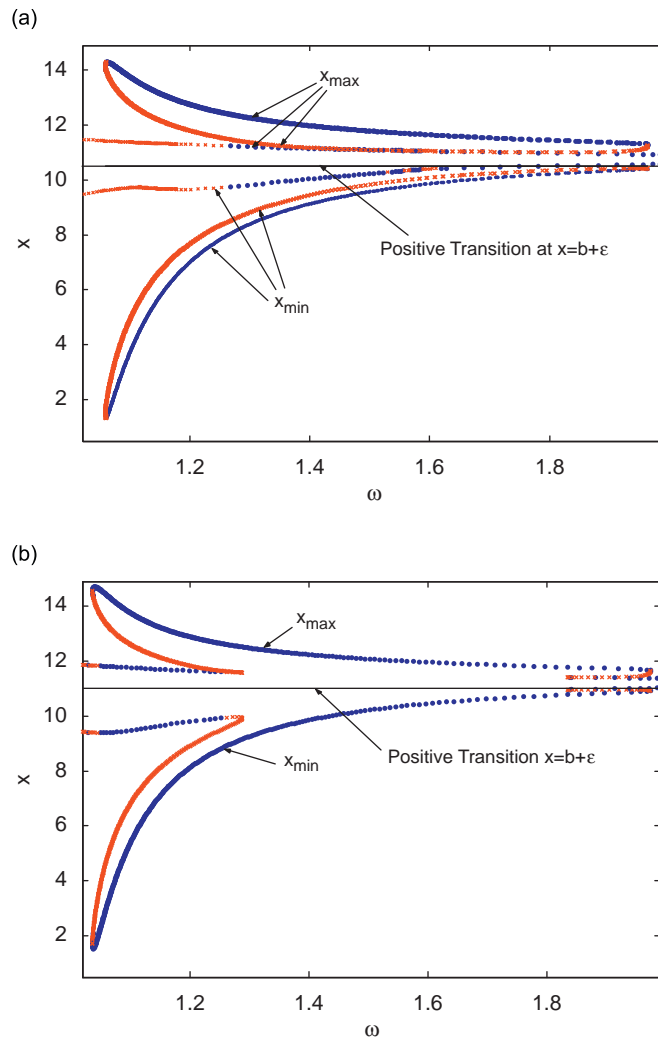
(a)

(b)

Fig. 5. Effect of asymmetry on the isolated sub-harmonic branch given $\alpha = 0.1$, $F_m = 1.3$, $F_p = 0.5$ and $\zeta = 0.02$: (a) $\varepsilon = 0.05b$ and (b) $\varepsilon = 0.10b$. Key: •••, stable multi-term harmonic balance method solution; $\times \times \times$, unstable multi-term harmonic balance method solution.

isolated branch fades back to a smooth curve connecting with the period-1 solution. Further work in this area is suggested especially when two or more clearances are present in a system.

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