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The dynamic stability of a simply supported beam with additional discrete elements

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Abstract

The dynamic stability of a simply supported beam with additional discrete elements was investigated in the paper. Those elements were an elastic spring, a concentrated mass and an undamped harmonic oscillator connected to the beam. All the discrete elements could be mounted at any chosen position along the beam length. The beam was axially loaded by a harmonic force. The problem of dynamic stability was solved by applying the mode summation method. The obtained Mathieu equation allowed the influence of additional elements on the position of solutions on a stability chart to be analysed. The analysis relied on testing the influence of individual discrete elements on the value of coefficient b in the Mathieu equation. The research carried out showed that both the concentrated mass and oscillator mass had a destabilising effect (maximum in the middle position) on the investigated system. The rigidity of the support and the oscillator had an influence on an increase in the stability of the investigated system. An increase in the loading force, independently of the relation between the mass and rigidity of discrete elements, had an influence on the increase in coefficient b in the Mathieu equation (the less stable system). The considered beam is treated as a Bernoulli–Euler beam in accordance with the small bending theory.

1. Introduction

There is a number of works dealing with the dynamic stability of beams and columns (compare Refs. [1–11]). These works deal both with the dynamic stability of beams or columns without additional discrete elements as well as with additional discrete elements applied at the end of the beam or column. Evensen and Evan-Iwanowski [1] carried out analytical and experimental research on the influence of a mass mounted at the end of a beam on the dynamic stability of this beam. Sato et al. [2] investigated the parametric vibrations of a horizontal beam loaded by a concentrated mass, which showed the influence of the beam weight and the inertia of a rotational mass on the beam vibrations. In Ref. [3] Ahmadi and Glockner determined criteria for the dynamic stability of a beam by assuming different types of changing load.

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Gürgöze [4] analysed the influence of a mass mounted at the end of an elastically supported beam along its axis. The dynamic stability of an elastic beam was analysed by Cederbaum and Mond [5]. Krawczuk and Ostachowicz [6] presented a mathematical model of parametrical vibrations of a beam with a closed gap. A few new types of parametric resonance have been found. Majorana and Pellegrino [7] analysed the dynamic stability of an elastically supported beam (rotation and translation springs at the ends). Beam vibrations were forced by the movements of the beam's second end. Sochacki and Tomski [8] solved the problem of parametric vibrations of a beam loaded by a follower force directed towards the positive pole. The same authors [9] considered the dynamic stability of divergence pseudo-flutter columns. Chen and Yen [10] analysed the instability of a column under oscillatory movement of a concentrated mass along the column axis. The same authors [11] considered analytically and experimentally the dynamic stability of an electromagnetically excited beam.

This paper takes into account a simply supported beam loaded by a longitudinal force in the form $P(t) = P_0 + S\cos\nu t$. Additionally, the beam is elastically supported and loaded by a concentrated mass in a chosen position along the beam length. An undamped harmonic oscillator was connected to the beam at a chosen position between the supports. The considered beam is treated as a Bernoulli–Euler beam and solved according to the small bending theory. The dynamic of the system was described with the use of the Mathieu equation. The problem of dynamic stability was solved using the mode summation method. The influence of additional mass and elasticity as well as an undamped harmonic oscillator on the position of solutions on the stability chart was investigated. The influence of additional elements mounted to the beam taking into account their values and positions on the value of coefficient b in the Mathieu equation was also investigated. In this way the possibility of a loss in dynamic stability by the investigated system was determined.

2. Mathematical model of beam vibrations

A scheme of the considered beam is presented in Fig. 1.

The vibration equation for two parts of a beam loaded by a force is known and has the following form:

$$E_iJ_i\frac{\partial w_i^4(x_i,t)}{\partial x_i^4} + P(t)\frac{\partial w_i^2(x_i,t)}{\partial x_i^2} + \rho_iA_i\frac{\partial^2 w_i(x_i,t)}{\partial t^2} = 0, \quad \text{(1a,b)}$$

where $P(t) = P_0 + S\cos\nu t$, ν is the forcing frequency, E_iJ_i the flexural rigidity of beam, ρ_i the density, A_i the cross-section area and $i = 1,2$ ith part of the beam

Eq. (1) is accompanied by the following boundary and matching conditions:

$$w_1(0,t) = 0, \quad w_2(l_2,t) = 0, \quad \text{(2a,b)}$$

$$w_1^{\mathrm{II}}(0,t) = 0, \quad w_2^{\mathrm{II}}(l_2,t) = 0, \quad \text{(2c,d)}$$

$$E_1J_1w_1^{\mathrm{III}}(l_1,t) + P(t)w_1^{\mathrm{I}}(l_1,t) - k_1w_1(l_1,t) - m_1\ddot{w}_1(l_1,t) - m_2\ddot{z} - E_2J_2w_2^{\mathrm{III}}(0,t) - P(t)w_2^{\mathrm{I}}(0,t) = 0, \quad \text{(2e)}$$

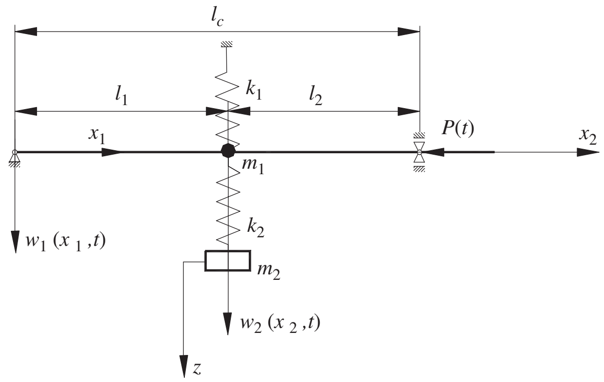

Fig. 1. Model of the beam with additional discrete elements (k_1, m_1, oscillator k_2, m_2) mounted in selected positions along the beam length.

$$w_1(l_1,t) = w_2(0,t), \tag{2f}$$

$$w_1^{\mathrm{I}}(l_1,t) = w_2^{\mathrm{I}}(0,t), \tag{2g}$$

$$E_1J_1w_1^{\mathrm{II}}(l_1,t) = E_2J_2w_2^{\mathrm{II}}(0,t), \tag{2h}$$

$$m_2\ddot{z} + k_2(z - w_1(l_1,t)) = 0, \tag{2i}$$

in which the Roman numerals denote differentiation with respect to x_i, and dots denote differentiation with respect to time t.

During the vibrations the displacement of the beam and oscillator mass take the form:

$$w_i(x_i,t) = W_i(x_i)\cos(\omega t) \quad (i = 1,2) \tag{3}$$

and

$$z = Z\cos(\omega t), \tag{4}$$

where $W_i(x_i)$ and Z are displacement amplitudes w_i and z, while ω is the natural frequency of the beam with discrete elements.

Substituting Eqs. (3) and (4) into Eq. (1a,b) and into conditions (2a–i) one can obtain (for $S = 0$):

$$E_iJ_iW_i^{\mathrm{IV}}(x_i) + P_0W_i^{\mathrm{II}}(x_i) - \rho_iA_i\omega^2W_i(x_i) = 0 \quad (i = 1,2) \tag{5a,b}$$

and

$$W_1(0) = 0, \quad W_2(l_2) = 0, \tag{6a,b}$$

$$W_1^{\mathrm{II}}(0) = 0, \quad W_2^{\mathrm{II}}(l_2) = 0, \tag{6c,d}$$

$$E_1J_1W_1^{\mathrm{III}}(l_1) - k_1W_1(l_1) + m_1\omega^2W_1(l_1) + m_2\omega^2Z - E_2J_2W_2^{\mathrm{III}}(0) = 0, \tag{6e}$$

$$W_1(l_1) = W_2(0), \tag{6f}$$

$$W_1^{\mathrm{I}}(l_1) = W_2^{\mathrm{I}}(0), \tag{6g}$$

$$E_1J_1W_1^{\mathrm{II}}(l_1) = E_2J_2W_2^{\mathrm{II}}(0), \tag{6h}$$

$$k_2(Z - W_1(l_1)) - m_2\omega^2Z = 0. \tag{6i}$$

The general solution to Eqs. (5a,b) takes the form:

$$W_i(x_i) = C_{i1}\sinh(\alpha_ix_i) + C_{i2}\cosh(\alpha_ix_i) + C_{i3}\sin(\beta_ix_i) + C_{i4}\cos(\beta_ix_i), \tag{7a,b}$$

where C_{ik} are integration constants (k = 1, 2, 3, 4) and:

$$\alpha_i^2 = -\frac{\lambda_i}{2} + \sqrt{\frac{\lambda_i^2}{4} + \Omega_i}, \quad \beta_i^2 = \frac{\lambda_i}{2} + \sqrt{\frac{\lambda_i^2}{4} + \Omega_i}, \tag{8a,b}$$

where

$$\Omega_i^2 = \omega^2\frac{\rho_iA_i}{E_iJ_i}, \quad \lambda_i = \frac{P_0}{E_iJ_i}.$$

The equations of vibrations (5a,b) together with the boundary and matching conditions (6a–i) are used in the formulation of the boundary value problem of the investigated beam. The natural frequency ω, amplitude Z and eigenfunctions of the beam $W_i(x_i)$ are determined by solving the boundary value problem (see Appendix A).

3. The solution to the problem of the dynamic stability of the beam

Using the method of assumed modes [12], the transverse deflection of the beam (in Eq. (1a,b)) can be expressed as

$$w_i(x_i,t)=\sum_{n=1}^{\infty} W_{in}(x_i)T_n(t) \quad (i=1,2), \tag{9a,b}$$

where $T_n(t)$ is an unknown time function and $W_{in}(x_i)$ is nth form of free vibrations of ith part of the beam which satisfies

$$\sum_{i=1}^{2}\rho_i A_i \int_0^{l_i} W_{im}(x_i)W_{in}(x_i)\,\mathrm{d}x_i + W_{1m}(l_1)W_{1n}(l_1)(m_1+m_2k_2^*) = \begin{cases} 0 & \text{for} \quad m\neq n, \\ \gamma_n^2 & \text{for} \quad m=n, \end{cases} \tag{10a}$$

where

$$k_2^*=\frac{k_2^2}{(k_2-m_2\omega_m^2)(k_2-m_2\omega_n^2)}, \quad \gamma_n^2=\sum_{i=1}^{2}\rho_i A_i\int_0^{l_i} W_{in}^2(x_i)\,\mathrm{d}x_i + W_{1n}^2(l_1)(m_1+m_2k_2^{**}), \tag{10b,c}$$

$$k_2^{**}=\frac{k_2^2}{(k_2-m_2\omega_n^2)^2}. \tag{10d}$$

The derivation of an orthogonality condition (10) is shown in Appendix B.

Substituting Eqs. (9a,b) into Eq. (1a,b) leads to

$$\sum_{n=1}^{\infty}[E_iJ_iW_{in}^{\mathrm{IV}}(x_i)T_n(t)+(P_0+S\cos vt)W_{in}^{\mathrm{II}}(x_i)T_n(t)+\rho_iA_iW_{in}(x_i)\ddot{T}_n(t)]=0 \quad (i=1,2). \tag{11}$$

Multiplying Eq. (11) by mth eigenfunction one can obtain:

$$\sum_{n=1}^{\infty}[E_iJ_iW_{in}^{\mathrm{IV}}(x_i)W_{im}(x_i)T_n(t)+P_0W_{in}^{\mathrm{II}}(x_i)W_{im}(x_i)T_n(t)$$
$$+S\cos vt\,W_{in}^{\mathrm{II}}(x_i)W_{im}(x_i)T_n(t)+\rho_iA_iW_{in}(x_i)W_{im}(x_i)\ddot{T}_n(t)]=0. \tag{12}$$

From Eqs. (5a,b) for the nth eigenfunction $W_{in}(x_i)$, after multiplying by $W_{im}(x_i)$, one can receive:

$$E_iJ_iW_{in}^{\mathrm{IV}}(x_i)W_{im}(x_i)+P_0W_{in}^{\mathrm{II}}(x_i)W_{im}(x)=\rho_iA_i\omega_n^2W_{in}(x_i)W_{im}(x_i) \quad (i=1,2). \tag{13}$$

Then Eq. (12) becomes

$$\sum_{n=1}^{\infty}[\rho_iA_i\omega_n^2W_{in}(x)W_{im}(x)T_n(t)+S\cos vt\,W_{in}^{\mathrm{II}}(x_i)W_{im}(x_i)T_n(t)+\rho_iA_iW_{in}(x_i)W_{im}(x_i)\ddot{T}_n(t)]=0. \tag{14}$$

Research by Evensen and Evan-Iwanowski [1] shows that only the first term of sum in Eqs. (9a,b) is of significance, so integrating to Eq. (14) gives the following form (for the first term):

$$T_1(t)\left(\omega_1^2\rho_iA_i\int_0^{l_i}W_{i1}^2(x_i)\,\mathrm{d}x_i+S\cos vt\int_0^{l_i}W_{i1}^{\mathrm{II}}(x_i)W_{i1}(x_i)\,\mathrm{d}x_i\right)+\ddot{T}_1(t)\rho_iA_i\int_0^{l_i}W_{i1}^2(x_i)\,\mathrm{d}x_i=0 \quad (i=1,2). \tag{15}$$

Appropriate transformations of Eq. (15) lead to the form of Mathieu equations

$$\ddot{T}_1(t)+(a_1+b_1S\cos vt)T_1(t)=0, \tag{16a}$$

where

$$a_1=\omega_1^2, \quad b_1=\frac{\sum_{i=1}^{2}\int_0^{l_i}W_{i1}^{\mathrm{II}}(x_i)W_{i1}(x_i)\,\mathrm{d}x_i}{\sum_{i=1}^{2}\rho_iA_i\int_0^{l_i}W_{i1}^2(x_i)\,\mathrm{d}x_i}. \tag{16b,c}$$

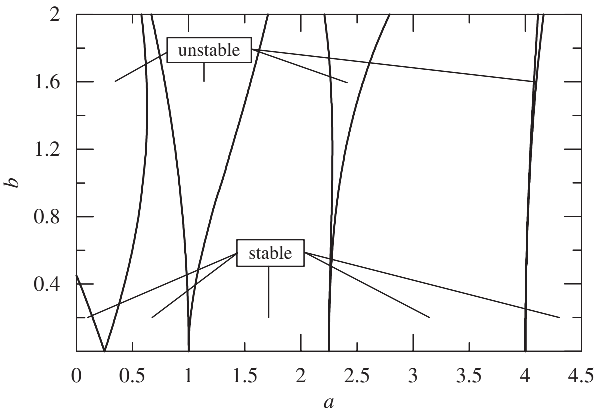

Fig. 2. Stable and unstable regions of solutions for the Mathieu equation (Timoshenko and Gere [14]).

Hence, the substitution of t in Eq. (16) by a new variable τ according to the relation $\tau = \nu t$ leads to the following form of the equation for the whole beam system (the subscript 1 is omitted):

$$\ddot{T}(\tau) + (a + b\cos\tau)T(\tau) = 0, \tag{17a}$$

where

$$a = \frac{\omega_1^2}{\nu^2}, \quad b = b_1\frac{S}{\nu^2}, \tag{17b,c}$$

dots denote differentiation with respect to dimensionless time τ.

The periodical solutions of the Mathieu equation (17) are known (compare Refs. [13–15]). These solutions allow us to determine the stable and unstable regions of solutions as in Fig. 2.

As shown in Fig. 2, the numerical values of a and b each time decide the position of solution in the stable or unstable region. It can be seen that the highest probability of obtaining a stable solution occurs for the smaller value of coefficient b at determined value a. However, it must be remembered that in the case of the relation of the forcing frequency towards the natural frequency (expressed by coefficient a) equal to $a = 0.25$, or $a = 1$, the solutions of the equation will be placed decidedly more often in an unstable region. The influence of additional discrete elements of the system on the value of coefficient b at the determined values of coefficient a was determined in this paper.

4. The results of numerical computations and discussion

The results of the solution to the dynamic stability problem allows us to determine the values of coefficient b in the Mathieu equation for changeable values of mass m_1 mounted at a randomly selected position on the beam and the changeable coefficient of support elasticity k_1. The values of coefficient b for different values of mass m_2 and elasticity coefficient k_2 for different positions of the oscillator on the beam were similarly determined.

Calculations were carried out assuming the following dimensionless quantities:

$$K_1 = \frac{k_1 l_c^3}{E_1 J_1}, \quad K_2 = \frac{k_2 l_c^3}{E_1 J_1}, \quad M_1 = \frac{m_1}{\sum_{i=1}^{2}\rho_i A_i l_i},$$
$$M_2 = \frac{m_2}{\sum_{i=1}^{2}\rho_i A_i l_i}, \quad l = \frac{l_1}{l_c}, \quad p = \frac{P_0}{P_c}, \quad s = \frac{S}{P_c}, \tag{18}$$

where P_c is the critical load of the beam without additional discrete elements

To depict the influence of individual elements added to the beam on its dynamic stability, research was conducted assuming: Case 1; $K_2 = 0$ and $M_2 = 0$ (Figs. 3–10) and Case 2; $K_1 = 0$ and $M_1 = 0$ (Figs. 11–17). In Figs. 3–9 and 12–16 the results were obtained for $p = 0.05$ and $s = 0.05$.

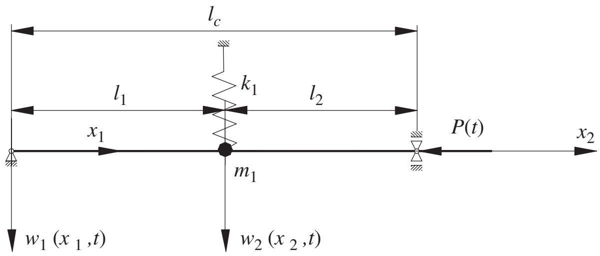

Fig. 3. Beam model with additional discrete elements (k_1, m_1) mounted at a chosen position on beam length.

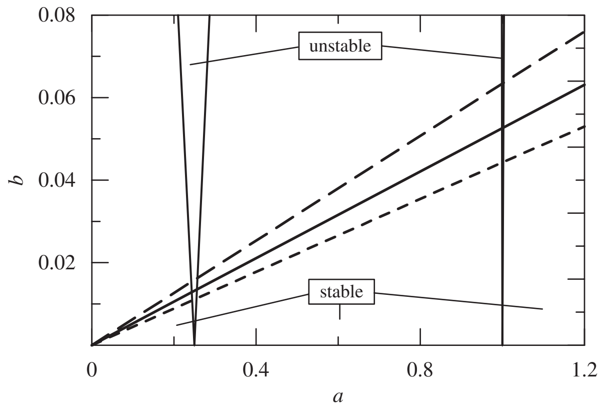

Fig. 4. Exemplary positions of solutions to the Mathieu equation (17) for chosen values K_1 and M_1, $l = 0.1$: $K_1 = 0$, $M_1 = 0$ ——, $K_1 = 0$, $M_1 = 1$– – – –, $K_1 = 100$, $M_1 = 0$- - - - - -.

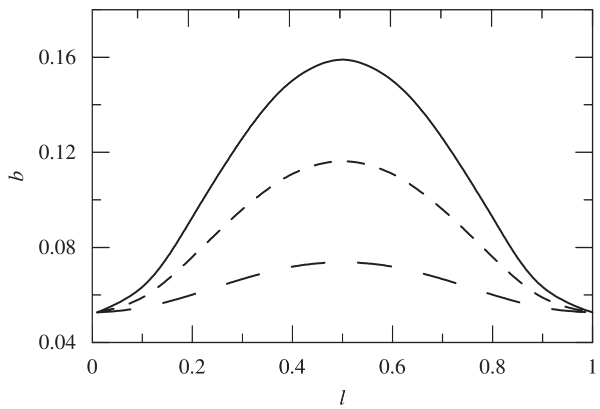

Fig. 5. The influence of the mounted position of mass M_1 on the beam and its values on the value of coefficient b for $a = 1$, $K_1 = 0$: $M_1 = 0.2$– – – –, $M_1 = 0.6$- - - - - -, $M_1 = 1$ ——.

Case 1. ($K_2 = 0$ and $M_2 = 0$) Fig. 3.

The values of the elasticity coefficient of the spring K_1 were each time assumed to be below the values determined by the curve of change in shape of the first form of system vibrations (“boundary” values of K_1) for the chosen mounting position of the spring to the beam (Fig. 8). The curve of “boundary” values of K_1 near mounting places ($l = 0$ and 1) approach infinity.

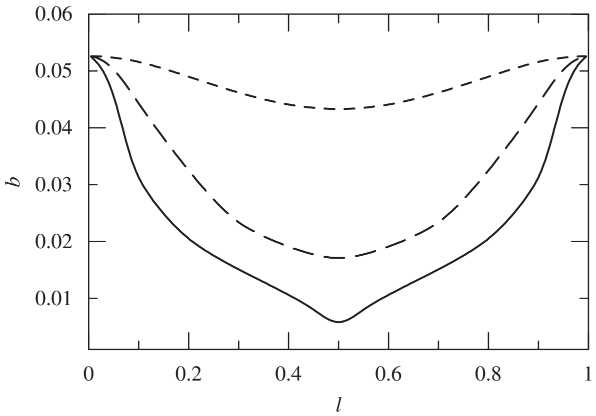

Fig. 6. The influence of the position of a spring with elasticity coefficient K_1 mounted on the beam on the value of coefficient b for $a = 1$, $M_1 = 0$: $K_1 = 10$------, $K_1 = 100$–––, $K_1 = 500$ ——.

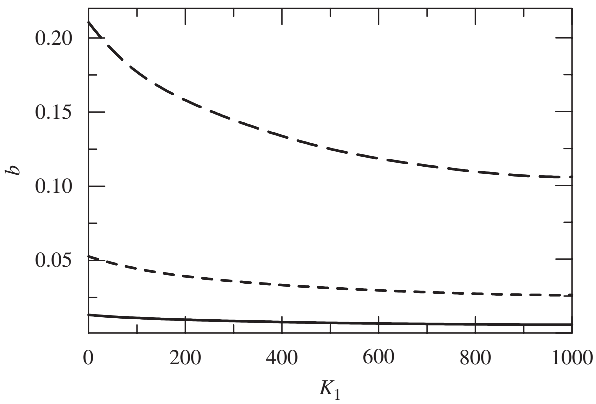

Fig. 7. The influence of the value of the elasticity coefficient of the spring K_1 on the value of coefficient b for chosen values of coefficient a and $M_1 = 0$, $l = 0.1$: $a = 0.25$ ——, $a = 1$------, $a = 4$–––.

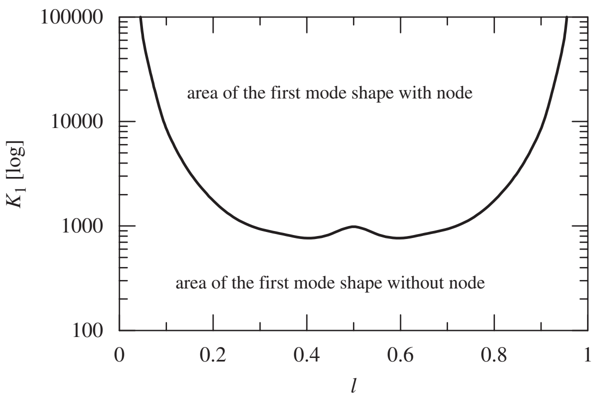

Fig. 8. The boundary values of the elasticity coefficient of the spring K_1 depending on the mounting position of the spring to the beam, at which the change in shape of the first form of system vibrations of beam occurs: $p = 0.05$, $M_1 = 0$.

A violation of ''boundary'' values K_1 results in a change in the system vibration from a non-nodal form into a one-nodal form of vibration. The phenomenon of a change in the vibration form with an increase in the elasticity coefficient of the beam support was studied in detail by Albarracín et al. [16]. The description

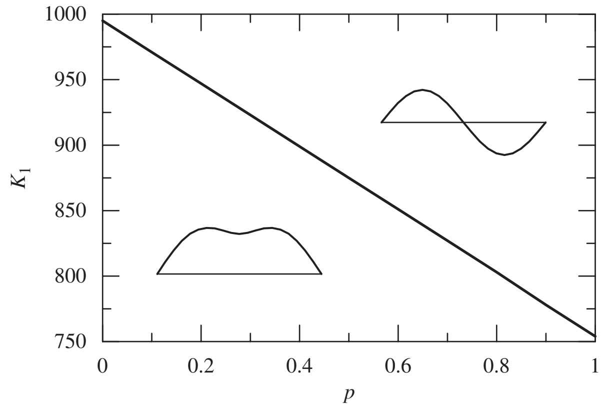

Fig. 9. The dependence of the boundary values of the elasticity coefficient of spring K_1 on the value of the static force loading the beam p, at which point the change of the first form of beam vibrations takes position, $M_1 = 0$, $l = 0.5$.

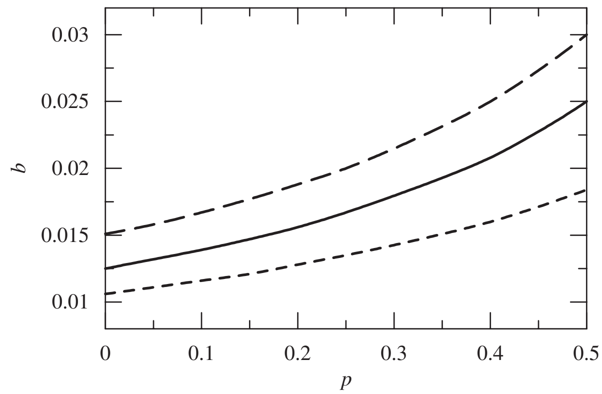

Fig. 10. The influence of the value of the static force loading the beam p on the value of coefficient b at determined relations K_1 and M1: $K_1 = 0$ and $M_1 = 0$ ———, $K_1 = 0$ and $M_1 = 1$– – – –, $K_1 = 100$ and $M_1 = 0$- - - - - -.

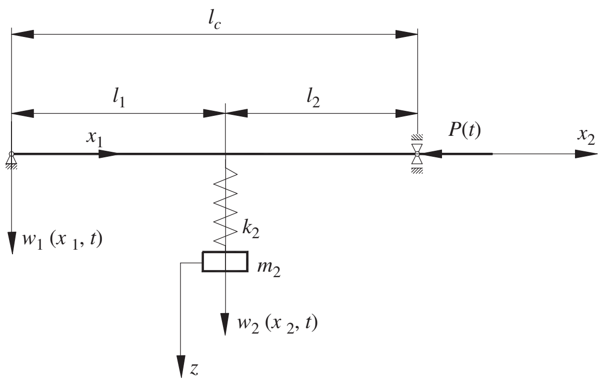

Fig. 11. Model of the beam with oscillator (k_2, m_2) mounted at selected location on the beam length.

concerned a beam elastically supported in the middle of a beam without a load. If the beam is statically loaded by a force p, a change in the form of vibrations takes place at lower values of the elasticity coefficient of the spring K_1. The boundary values K_1 for increasing values of the loaded force are shown in Fig. 9. The first form

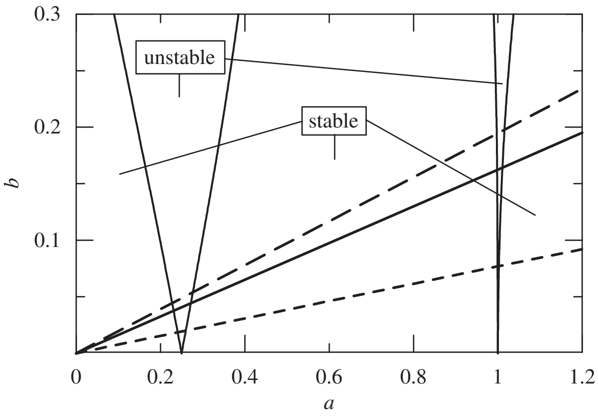

Fig. 12. Exemplary positions of solutions to the Mathieu equation (17) for chosen values K_2 and M_2, $l = 0.5$: $K_2 = 100$ and $M_2 = 0.2$------, $K_2 = 100$ and $M_2 = 1$ ———, $K_2 = 1000$ and $M_2 = 1$– – – –.

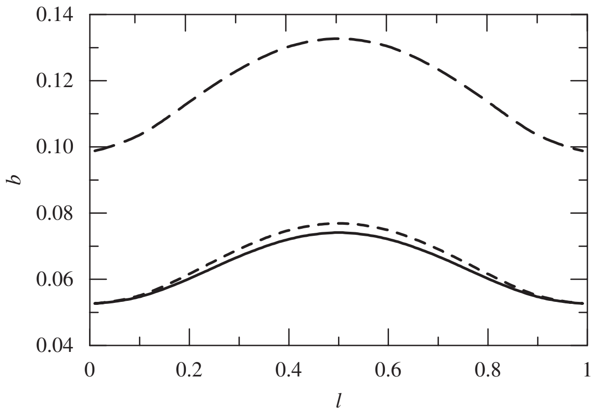

Fig. 13. The influence of oscillator mounting location on the beam and the value of the elasticity coefficient of oscillator K_2 on the value of coefficient b for $a = 1$ and $M_2 = 0.2$: $K_2 = 10$– – – –, $K_2 = 100$------, $K_2 = 1000$ ———.

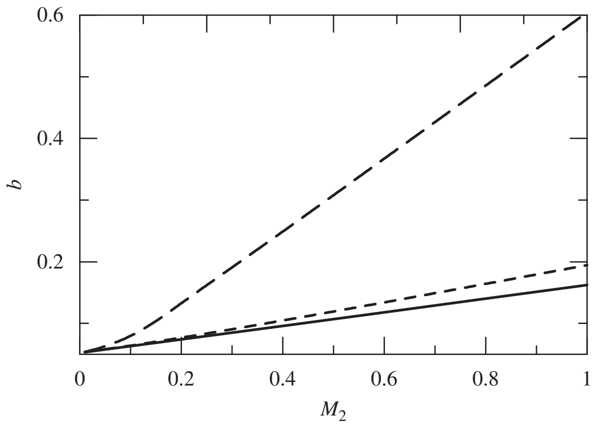

Fig. 14. The influence of the value of the oscillator mass M_2 and its elasticity coefficient K_2 on the value of coefficient b for $a = 1$ and $l = 0.5$: $K_2 = 10$– – – –, $K_2 = 100$------, $K_2 = 100$ ———.

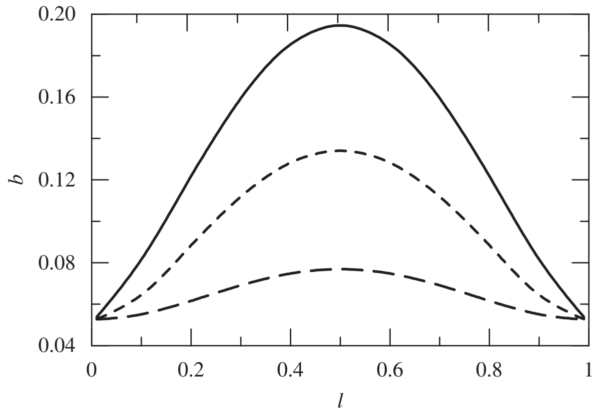

Fig. 15. The influence of the oscillator mounting location on the beam and its mass M_2 on the value of coefficient b for $a = 1$ and $K_2 = 100$: $M_2 = 1$ ———, $M_2 = 0.6$------, $M_2 = 0.2$– – – –.

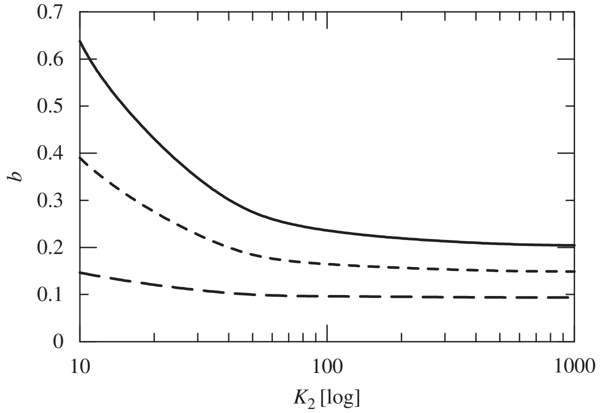

Fig. 16. The influence of the value of the elasticity coefficient of oscillator K_2 and its mass M_2 on the value of coefficient b for $a = 1$, $l = 0.5$: $M_2 = 1$ ———, $M_2 = 0.6$------, $M_2 = 0.2$– – – –.

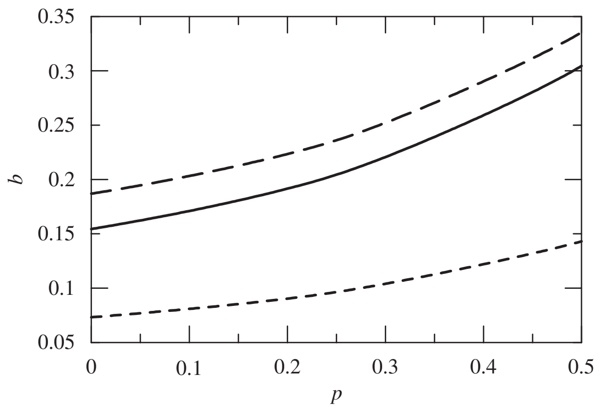

Fig. 17. The influence of the value of the static force loaded beam p on the value of coefficient b at determined relations K_2 and M_2 of the oscillator: $K_2 = 100$ and $M_2 = 0.2$ ———, $K_2 = 100$ and $M_2 = 1$– – – –, $K_2 = 1000$ and $M_2 = 1$------.

of the vibrations of a beam elastically supported in the middle of its length before and after the violation of boundary values K_1 is also presented in this figure.

Analysis of the results presented in Figs. 4–10 leads to the conclusion that an increase in mass M_1 has an influence on the increase in coefficient b in the Mathieu equation (Figs. 4 and 5). The mounting position of the mass on the beam has a significant influence on the value of coefficient b, and if the mass position is closer to the midpoint of the beam, the value of coefficient b is higher (Fig. 5). The central position of mass M_1 causes even a threefold increase in coefficient b (case when mass $M_1 = 1$) in relation to the position of the mass near supports.

According to the research results an increase in the values of the elasticity coefficient of the spring K_1 lowers coefficient b (Fig. 7). The central position of the spring results in the largest decrease in coefficient b, and this decrease is higher at the higher value of the elasticity coefficient of the spring K_1.

Coefficient b (Fig. 9) increases with an increase in static loaded force for selected relations between K_1 and M_1.

Case 2. ($K_1 = 0$ and $M_1 = 0$) Fig. 11.

Analysis of the research results of the influence of the oscillator (K_2 and M_2) and its placement on the beam on the value of coefficient b in Eqs. (17) allows the following conclusions to be drawn: an increase in oscillator mass M_2 leads to an increase in the value of coefficient b, while an increase in the elasticity coefficient K_2 of the oscillator leads to a decrease in coefficient b (Figs. 14 and 16). Analysing the influence of the oscillator placement on the beam (Figs. 12, 13 and 15) it can be stated that, independently of the values K_2 and M_2, the closer oscillator mounting is to the centre of the beam the higher the increase in b.

The coefficient b (Fig. 17) increases at chosen relations between K_2 and M_2 with an increase in the static loaded force.

5. Conclusions

The results of the dynamic stability of a beam with additional discrete elements mounted in a chosen place on its length are presented in this paper. The beam was loaded by a harmonically varying force. The value of coefficient b in the Mathieu equation (17) was assumed as a measure of the possibility of loss in stability.

On the basis of the research results it can be stated that:

- an increase in the concentrated mass M_1 mounted at a chosen position on its length leads to an increase in possibility of a loss in stability of the investigated system (range of unstable solutions is growing);
- an increase in support elasticity (rigidity of spring K_1 to the boundary values at determined load p) stabilises the investigated system;
- the location of the concentrated element application influences the stability of the investigated system. In the case of mass M_1 a position in the centre of the beam is the most disadvantageous while support elasticity K_1 maximally stabilises the system in a central position;
- an increase in oscillator mass M_2 makes the system more unstable;
- an increase in the elasticity of oscillator K_2 stabilises the investigated system;
- the closer oscillator is mounted to the centre of the beam the more unstable the system is (independently on the values K_2 and M_2); and
- an increase in the static force loading the system leads to instability in the system for selected relations between K_1 and M_1 and K_2 and M_2.

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Appendix A

After substituting Eqs. (7a,b) into Eqs. (6a–i) one obtains the system of nine homogenous equations for the unknown constants C_{ik} and Z, what can be written in matrix form as

$$\mathbf{A}(\omega)\mathbf{C} = 0, \tag{A.1}$$

where $\mathbf{A}(\omega) = [a_{pq}]$, ($p$, $q = 1,2\ldots9$) and $\mathbf{C} = [C_{11}\ldots C_{14},\ C_{21}\ldots C_{24},\ Z]^{\mathrm{T}}$

For a non-trivial solution of Eq. (A.1), the determinant of the matrix $\mathbf{A}(\omega)$ is set equal to zero, yielding the frequency equation:

$$\det\ \mathbf{A}(\omega) = 0, \tag{A.2}$$

where the non-zero elements a_{pq} of matrix $\mathbf{A}(\omega)$ are given as follows:

$$a_{11} = 1, \quad a_{13} = 1,$$

$$a_{21} = \alpha_1^2, \quad a_{23} = -\beta_1^2,$$

$$a_{31} = \alpha_1 \sinh(\alpha_1^*), \quad a_{23} = \alpha_1 \cosh(\alpha_1^*), \quad a_{33} = -\alpha_1 \sin(\beta_1^*), \quad a_{34} = \alpha_1 \cos(\beta_1^*), \quad a_{36} = -\alpha_2, \quad a_{38} = -\beta_2,$$

$$a_{41} = \cosh(\alpha_1^*), \quad a_{42} = \sinh(\alpha_1^*), \quad a_{43} = \cos(\beta_1^*), \quad a_{44} = \sin(\beta_1^*), \quad a_{45} = -1, \quad a_{47} = -1,$$

$$a_{51} = E_1^*\alpha_1^2 \cosh(\alpha_1^*), \quad a_{52} = E_1^*\alpha_1^2 \sinh(\alpha_1^*) \quad a_{53} = -E_1^*\beta_1^2 \cos(\beta_1^*), \quad a_{54} = -E_1^*\beta_1^2 \sin(\beta_1^*),$$

$$a_{55} = -E_2^*\alpha_2^2, \quad a_{57} = E_2^*\beta_2^2, \tag{A.3}$$

$$a_{65} = \cosh(\alpha_2^*), \quad a_{66} = \sinh(\alpha_2^*), \quad a_{67} = \cos(\beta_2^*), \quad a_{68} = \sin(\beta_2^*),$$

$$a_{75} = E_2^*\alpha_2^2 \cosh(\alpha_2^*), \quad a_{76} = E_2^*\alpha_2^2 \sinh(\alpha_2^*), \quad a_{77} = -E_2^*\beta_2^2 \cos(\beta_2^*), \quad a_{78} = -E_2^*\beta_2^2 \sin(\beta_2^*),$$

$$a_{81} = E_1^*\alpha_1^3 \sinh(\alpha_1^*) - (k_1 - \omega^2 m_1)\cosh(\alpha_1^*), \quad a_{82} = E_1^*\alpha_1^3 \cosh(\alpha_1^*) - (k_1 - \omega^2 m_1)\sinh(\alpha_1^*),$$

$$a_{83} = E_1^*\beta_1^3 \sin(\beta_1^*) - (k_1 - \omega^2 m_1)\cos(\beta_1^*), \quad a_{84} = -E_1^*\beta_1^3 \cos(\beta_1^*) - (k_1 - \omega^2 m_1)\sin(\beta_1^*),$$

$$a_{86} = -E_2^*\alpha_2^3, \quad a_{88} = E_2^*\beta_2^3, \quad a_{89} = \omega^2 m_2,$$

$$a_{91} = -k_2 \cosh(\alpha_1^*), \quad a_{92} = -k_2 \sinh(\alpha_1^*), \quad a_{93} = -k_2 \cos(\beta_1^*), \quad a_{94} = -k_2 \sin(\beta_1^*), \quad a_{99} = k_2 - \omega^2 m_2$$

and

$$\alpha_i^* = \alpha_i l_i, \quad \beta_1^* = \beta_i l_i, \quad E_i^* = E_i J_i, \quad i = 1,2.$$

Appendix B

For the nth and mth eigenfunctions, Eqs. (5a,b) take the forms:

$$E_i^* W_{in}^{\mathrm{IV}}(x_i) + P_0 W_{in}^{\mathrm{II}}(x_i) - \rho_i^*\omega_n^2 W_{in}(x_i) = 0, \tag{B.1}$$

$$E_i^* W_{im}^{\mathrm{IV}}(x_i) + P_0 W_{im}^{\mathrm{II}}(x_i) - \rho_i^*\omega_m^2 W_{im}(x_i) = 0, \tag{B.2}$$

where $i = 1,2$, $\rho*_i = \rho_i A_i$.

After multiplying Eq. (B.1) by $W_{im}(x_i)$ and Eq. (B.2) by $W_{in}(x_i)$ one obtains

$$E_i^* W_{in}^{\mathrm{IV}}(x_i) W_{im}(x_i) + P_0 W_{in}^{\mathrm{II}}(x_i) W_{im}(x_i) - \rho_i^*\omega_n^2 W_{in}(x_i) W_{im}(x_i) = 0, \tag{B.3}$$

$$E_i^* W_{im}^{\mathrm{IV}}(x_i) W_{in}(x_i) + P_0 W_{im}^{\mathrm{II}}(x_i) W_{in}(x_i) - \rho_i^*\omega_m^2 W_{im}(x_i) W_{in}(x_i) = 0. \tag{B.4}$$

Integrating those equations along the length l_i, what gives

$$E_i^* \int_0^{l_i} W_{in}^{IV}(x_i) W_{im}(x_i)\,dx_i + P_0 \int_0^{l_i} W_{in}^{II}(x_i) W_{im}(x_i)\,dx_i - \rho_i^* \omega_n^2 \int_0^{l_i} W_{in}(x_i) W_{im}(x_i)\,dx_i = 0, \tag{B.5}$$

$$E_i^* \int_0^{l_i} W_{im}^{IV}(x_i) W_{in}(x_i)\,dx_i + P_0 \int_0^{l_i} W_{im}^{II}(x_i) W_{in}(x_i)\,dx_i - \rho_i^* \omega_m^2 \int_0^{l_i} W_{im}(x_i) W_{in}(x_i)\,dx_i = 0, \tag{B.6}$$

and subtracting the obtained Eqs. (B.5) and (B.6) with consideration of the boundary conditions (6a–d) and (6f–h), results in:

$$\begin{aligned} &W_{1m}(l_1)[E_1^* W_{1n}'''(l_1) - E_2^* W_{2n}'''(0)] - W_{1n}(l_1)[E_1^* W_{1m}'''(l_1) - E_2^* W_{2m}'''(0)] \\ &+ (\omega_m^2 - \omega_n^2) \sum_{i=1}^{2} \rho_i^* \int_0^{l_i} W_{in}(x_i) W_{im}(x_i)\,dx_i = 0. \end{aligned} \tag{B.7}$$

Combining Eqs. (6e) and (6i) leads to

$$E_1^* W_1'''(l_1) E_2^* W_2'''(0) = \left[\left(m_2 \frac{k_2}{m_2\omega^2 - k_2} - m_1\right)\omega^2 + k_1\right] W_1(l_1). \tag{B.8}$$

Taking into account Eq. (B.8), Eq. (B.7) can be rewritten in the form:

$$\begin{aligned} &W_{1m}(l_1) W_{1n}(l_1)\left[\left(m_2 \frac{k_2}{m_2\omega_n^2 - k_2} - m_1\right)\omega_n^2 + k_1\right] - W_{1n}(l_1) W_{1m}(l_1)\left[\left(m_2 \frac{k_2}{m_2\omega_m^2 - k_2} - m_1\right)\omega_m^2 + k_1\right] \\ &+ (\omega_m^2 - \omega_n^2) \sum_{i=1}^{2} \rho_i^* \int_0^{l_i} W_{in}(x_i) W_{im}(x_i)\,dx_i = 0. \end{aligned} \tag{B.9}$$

After simplification Eq. (B.9) takes the form:

$$\begin{aligned} &W_{1m}(l_1) W_{1n}(l_1)\left(m_1 + m_2\left(\frac{k_2}{m_2\omega_m^2 - k_2} - \frac{k_2}{m_2\omega_n^2 - k_2}\right)\right)(\omega_m^2 - \omega_n^2) \\ &+ (\omega_m^2 - \omega_n^2) \sum_{i=1}^{2} \rho_i^* \int_0^{l_i} W_{in}(x_i) W_{im}(x_i)\,dx_i = 0. \end{aligned} \tag{B.10}$$

Because $\omega_m \neq \omega_n$ for $m \neq n$, the orthogonality condition can be finally expressed as follows:

$$\sum_{i=1}^{2} \rho_i^* \int_0^{l_i} W_{in}(x_i) W_{im}(x_i)\,dx_i + W_{1m}(l_1) W_{1n}(l_1)\left(m_1 + m_2 \frac{k_2^2}{(k_2 - m_2\omega_m^2)(k_2 - m_2\omega_n^2)}\right) = \begin{cases} 0 & \text{for} \quad m \neq n, \\ \gamma_n^2 & \text{for} \quad m = n. \end{cases} \tag{B.11}$$

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