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He's homotopy perturbation method to periodic solutions of nonlinear Jerk equations

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Abstract

In this paper, He's homotopy perturbation method is applied to nonlinear Jerk equations involving the third temporal derivative of displacement. The result reveals that the first-order approximate period is identical to that obtained by the harmonic balance method. However, the high-order analytical approximate periods and periodic solutions are more accurate and in better agreement with the exact results. Thus, He's homotopy perturbation method is very effective for these third-order nonlinear differential equations.

1. Introduction

The nonlinear Jerk equations involving the third temporal derivative of displacement have been widely studied [1–5]. Recently, Gottlieb [5] has focused on the existence of periodic solutions in appropriate parameter regions. He has investigated the lowest-order analytical approximations, via the method of harmonic balance, to periodic solutions to these nonlinear Jerk equations. In this paper, we consider a Jerk equation of the form

$$\dddot{x} = J(x, \dot{x}, \ddot{x}). \tag{1}$$

Consequent to restrictions on the existence of periodic solutions, we only consider the following cubic nonlinear functions desired by Gottlieb: (I) $x\dot{x}\ddot{x}$; (II) $\dot{x}\ddot{x}^2$; (III) $x^2\dot{x}$ and (IV) $\dot{x}^3$, with initial conditions $x(0) = 0$, $\dot{x}(0) = B$ and $\ddot{x}(0) = 0$.

Later, He [6,7] proposed a new perturbation method, the so-called "He's homotopy perturbation method", which does not require a small parameter in the equation, but in which a homotopy with an imbedding parameter $p \in [0,1]$ is constructed, and the imbedding parameter is considered as a "small parameter". It takes full advantage of the traditional perturbation methods and the homotopy techniques and yields a very rapid

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convergence of the solution series. Consequently, this method has been successfully applied to some nonlinear oscillators and nonlinear problems [6–31].

In this paper, our goal is to apply this method to the Jerk equation. In Section 2, He's homotopy perturbation method is applied to the Jerk equation and we obtain the high-order analytic approximate periods and periodic solutions. In Section 3, we compare the analytic approximate periods and periodic solutions with the results obtained by the harmonic balance method, as well as with the exact results. Finally, a brief conclusion is presented in Section 4.

2. General nonlinear Jerk function

The most general Jerk function, which is invariant under time-reversal and space-reversal and only cubic nonlinear as specified above, may be written as

$$\dddot{x} + \alpha x\dot{x}\ddot{x} + \beta \dot{x}\ddot{x}^2 + \delta x^2\dot{x} + \varepsilon \dot{x}^3 + \gamma \dot{x} = 0 \tag{2}$$

where the parameters α, β, δ, ε and γ are constants.

2.1. Jerk function containing displacement times velocity times acceleration, and velocity

For function (I), and a linear term in the velocity is incorporated. The resulting standardized Jerk equation, after rescaling of both x and t, is taken to be

$$\dddot{x} = x\dot{x}\ddot{x} - \dot{x} \tag{3}$$

with initial conditions

$$x(0) = 0,\ \dot{x}(0) = B,\ \ddot{x}(0) = 0. \tag{4}$$

Assume that the angular frequency is ω, with the period T given by

$$T = \frac{2\pi}{\omega}. \tag{5}$$

Gottlieb [5] gave the approximate result of frequency and period

$$\omega = \frac{1}{2}\sqrt{B^2 + 4}, \quad T = \frac{2\pi}{\omega}. \tag{6}$$

Now, we illustrate the solution of this problem given in Eqs. (3) and (4) by He's homotopy perturbation method. We construct the following simple homotopy:

$$\dddot{x} + 1\dot{x} = px\dot{x}\ddot{x}, \quad p \in [0, 1], \tag{7}$$

where p is a homotopy parameter. For $p = 0$, Eq. (7) becomes a linear differential equation and an exact solution can be calculated. For $p = 1$, Eq. (7) becomes the original problem. Now the homotopy parameter p is used to expand the solution $x(t)$ and the square of the unknown angular frequency ω [15,16] as follows:

$$x(t) = x_0(t) + px_1(t) + p^2x_2(t) + p^3x_3(t) + \dots, \tag{8}$$

$$1 = \omega^2 - p\alpha_1 - p^2\alpha_2 - p^3\alpha_3 - \cdots, \tag{9}$$

where $\alpha_i (i = 1, 2, 3, \dots)$ are to be determined.

Substituting Eqs. (8) and (9) into Eq. (7) and equating the terms with identical powers of p, we obtain a series of linear equations, of which we write only the first four

$$\dddot{x}_0 + \omega^2\dot{x}_0 = 0,\ x_0(0) = 0,\ \dot{x}_0(0) = B,\ \ddot{x}_0(0) = 0, \tag{10}$$

$$\dddot{x}_1 + \omega^2\dot{x}_1 = \alpha_1\dot{x}_0 + x_0\dot{x}_0\ddot{x}_0,\ x_1(0) = 0,\ \dot{x}_1(0) = 0,\ \ddot{x}_1(0) = 0, \tag{11}$$

$$\dddot{x}_2 + \omega^2\dot{x}_2 = \alpha_1\dot{x}_1 + \alpha_2\dot{x}_0 + x_0\dot{x}_1\ddot{x}_0 + x_0\dot{x}_0\ddot{x}_1 + x_1\dot{x}_0\ddot{x}_0,\ x_2(0) = 0,\ \dot{x}_2(0) = 0,\ \ddot{x}_2(0) = 0, \tag{12}$$

$$\dddot{x}_3 + \omega^2 \dot{x}_3 = \alpha_1 \dot{x}_2 + \alpha_2 \dot{x}_1 + \alpha_3 \dot{x}_0 + x_0 \dot{x}_1 \ddot{x}_1 + x_0 \dot{x}_0 \ddot{x}_2 + x_0 \dot{x}_2 \ddot{x}_0 + x_1 \dot{x}_0 \ddot{x}_1 + x_1 \dot{x}_1 \ddot{x}_0 + x_2 \dot{x}_0 \ddot{x}_0,$$

$$x_3(0) = 0,\ \dot{x}_3(0) = 0,\ \ddot{x}_3(0) = 0. \tag{13}$$

The solution of Eq. (10) can be easily obtained:

$$x_0(t) = \frac{B}{\omega} \sin \omega t. \tag{14}$$

Substituting this result into the right-hand side of Eq. (11), we have

$$\dddot{x}_1 + \omega^2 \dot{x}_1 = \left(\alpha_1 B - \frac{B^3}{4}\right) \cos \omega t + \frac{B^3}{4} \cos 3\omega t. \tag{15}$$

No secular terms in $x_1(t)$ require eliminating contributions proportional to $\cos \omega t$ on the right-hand side of Eq. (15): then $\alpha_1 = B^2/4$. According to Eq. (9), we obtain the first-order approximate frequency and period of Eqs. (3) and (4):

$$\omega_1 = \sqrt{1 + \alpha_1} = \frac{1}{2}\sqrt{4 + B^2}, \quad T_1 = \frac{2\pi}{\omega_1}. \tag{16}$$

It is in agreement with Gottlieb's result.

The periodic solution of Eq. (11) can be obtained as

$$x_1(t) = D\left(\sin \omega t - \frac{1}{3} \sin 3\omega t\right), \quad D \triangleq \frac{B^3}{32\omega_1^3}. \tag{17}$$

Likewise, substituting Eqs. (14), (17) into the right-hand side of Eq. (12) gives

$$\dddot{x}_2 + \omega^2 \dot{x}_2 = \left(\alpha_2 B + \alpha_1 D\omega - \frac{B^2 \omega D}{6}\right) \cos \omega t + \left(\frac{5}{4} B^2 \omega D - D\omega\right) \cos 3\omega t - \frac{13}{12} B^2 \omega D \cos 5\omega t. \tag{18}$$

No secular terms in $x_2(t)$ require eliminating contributions proportional to $\cos \omega t$ on the right-hand side of Eq. (18): then $\alpha_2 = -BD\omega_1/12$. According to Eq. (9), we obtain the second-order approximate frequency and period of Eqs. (3) and (4):

$$\omega_2 = \sqrt{1 + \alpha_1 + \alpha_2} = \sqrt{\omega_1^2 + \alpha_2}, \quad T_2 = \frac{2\pi}{\omega_2}. \tag{19}$$

The periodic solution of Eq. (12) can be obtained as

$$x_2(t) = E\left(23 \sin \omega t - 12 \sin 3\omega t + \frac{13}{5} \sin 5\omega t\right), \quad E \triangleq \frac{B^2 D}{288\omega_2^2}. \tag{20}$$

Substituting Eqs. (14), (17) and (20) into the right-hand side of Eq. (13) and eliminating the secular terms in $x_3(t)$, we obtain the third-order approximate frequency and period of Eqs. (3) and (4):

$$\omega_3 = \sqrt{1 + \alpha_1 + \alpha_2 + \alpha_3} = \sqrt{\omega_2^2 + \alpha_3},\ T_3 = \frac{2\pi}{\omega_3}, \tag{21}$$

where

$$\alpha_3 = \frac{1}{B}\left(\frac{1}{12} BD^2 \omega_2^2 - \frac{19}{2} B^2 E\omega_2 - \alpha_2 D\omega_2\right).$$

2.2. Jerk function containing velocity times acceleration-squared, and velocity

For function (II), a linear term in the velocity is incorporated. The resulting standardized Jerk equation, after rescaling of both x and t, is taken to be

$$\dddot{x} = -\dot{x}\ddot{x}^2 - \dot{x} \tag{22}$$

with initial conditions

$$x(0) = 0,\ \dot{x}(0) = B,\ \ddot{x}(0) = 0. \tag{23}$$

Assume that the angular frequency is ω, with the period T given by

$$T = \frac{2\pi}{\omega}. \tag{24}$$

Gottlieb gave the approximate result of frequency and period

$$\omega = \frac{2}{\sqrt{4 - B^2}}, \quad T = \frac{2\pi}{\omega}. \tag{25}$$

We construct the following homotopy:

$$\dddot{x} + 1\dot{x} = -p\dot{x}\ddot{x}^2,\ p \in [0, 1], \tag{26}$$

where p is a homotopy parameter and used to expand the solution $x(t)$ and the square of the unknown angular frequency ω as follows:

$$x(t) = x_0(t) + px_1(t) + p^2x_2(t) + p^3x_3(t) + \cdots, \tag{27}$$

$$1 = \omega^2 - p\alpha_1 - p^2\alpha_2 - p^3\alpha_3 - \cdots, \tag{28}$$

where $\alpha_i (i = 1, 2, 3, \ldots)$ are to be determined.

Substituting Eqs. (27) and (28) into Eq. (26) and equating the terms with identical powers of p, we obtain a series of linear equations, of which we write only the first four

$$\dddot{x}_0 + \omega^2\dot{x}_0 = 0,\ x_0(0) = 0,\ \dot{x}_0(0) = B,\ \ddot{x}_0(0) = 0, \tag{29}$$

$$\dddot{x}_1 + \omega^2\dot{x}_1 = \alpha_1\dot{x}_0 - \dot{x}_0\ddot{x}_0^2,\ x_1(0) = 0,\ \dot{x}_1(0) = 0,\ \ddot{x}_1(0) = 0, \tag{30}$$

$$\dddot{x}_2 + \omega^2\dot{x}_2 = \alpha_1\dot{x}_1 + \alpha_2\dot{x}_0 - \dot{x}_1\ddot{x}_0^2 - 2\dot{x}_0\ddot{x}_0\ddot{x}_1,\ x_2(0) = 0,\ \dot{x}_2(0) = 0,\ \ddot{x}_2(0) = 0, \tag{31}$$

$$\dddot{x}_3 + \omega^2\dot{x}_3 = \alpha_1\dot{x}_2 + \alpha_2\dot{x}_1 + \alpha_3\dot{x}_0 - \dot{x}_0\ddot{x}_1^2 - 2\dot{x}_0\ddot{x}_0\ddot{x}_2 - 2\dot{x}_1\ddot{x}_0\ddot{x}_1 - \dot{x}_2\ddot{x}_0^2,$$

$$x_3(0) = 0,\ \ \dot{x}_3(0) = 0,\ \ \ddot{x}_3(0) = 0. \tag{32}$$

Table 1
Comparison between approximate periods and exact periods for Eqs. (3) and (4)

B	T_e	T_1 (%error)	T_2 (%error)	T_3 (error%)
0.10	6.275347	6.27534602 (−0.0000016)	6.27534683 (−0.00000027)	6.27534684 (−0.00000025)
0.20	6.252016	6.25200305 (−0.000021)	6.25201582 (−0.00000029)	6.25201599 (−0.000000016)
0.50	6.096061	6.09558510 (−0.00078)	6.09602457 (−0.0000598)	6.09605904 (−0.000003215)
1.0	5.626007	5.61985178 (−0.1094)	5.62454086 (−0.002606)	5.62579479 (−0.000377)
2.0	4.491214	4.44288294 (−1.0761)	4.46620532 (−0.5568)	4.48208113 (−0.20335)

For $B = 0.1$, the analytic approximate solutions are: $x_0(t) = 0.099875 \sin \omega_1 t,\ \omega_1 = 1.00124922$,

$$x_0(t) + x_1(t) + x_2(t) = 0.0999 \sin \omega_3 t - 1.039 \times 10^{-5} \sin 3\omega_3 t + 2.80358 \times 10^{-9} \sin 5\omega_3 t,\ \omega_3 = 1.00124908.$$

For $B = 1$: $x_0(t) = 0.894427 \sin \omega_1 t,\ \omega_1 = 1.1180339$,

$$x_0(t) + x_1(t) + x_2(t) = 0.918218 \sin \omega_3 t - 0.00820016 \sin 3\omega_3 t + 1.6176 \times 10^{-4} \sin 5\omega_3 t, \omega_3 = 1.1168529.$$

Similarly, we just give the results for approximate periods and corresponding to periodic solutions of Eqs. (22) and (23)

$$x_0(t) = \frac{B}{\omega} \sin \omega t, \tag{33}$$

$$\omega_1 = \sqrt{1+\alpha_1} = \frac{2}{\sqrt{4-B^2}}, \; T_1 = \frac{2\pi}{\omega_1}, \tag{34}$$

$$x_1(t) = D\left(\sin \omega t - \frac{1}{3}\sin 3\omega t\right), \; D \triangleq \frac{B^3}{32\omega_1}, \tag{35}$$

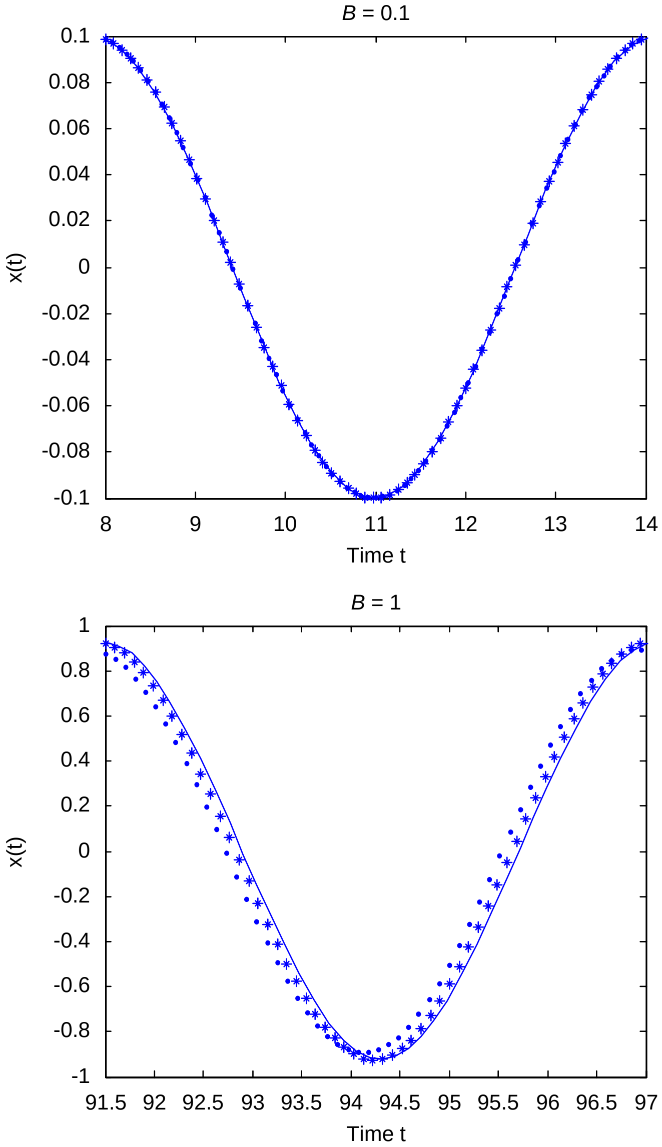

Fig. 1. Comparison between approximate solutions and numerical solution for Eqs. (3) and (4). Numerical: -, first-order approximate solution; ●, third-order approximate solution; *, for $B = 0.1$ and 1.

$$\omega_2 = \sqrt{1+\alpha_1+\alpha_2} = \sqrt{\omega_1^2 - \frac{3}{4}BD\omega_1^3}, \quad T_2 = \frac{2\pi}{\omega_2}, \tag{36}$$

$$x_2(t) = E(25\sin\omega t - 20\sin 3\omega t + 7\sin 5\omega t), \quad E \triangleq \frac{DB^2}{480}, \tag{37}$$

$$\omega_3 = \sqrt{1+\alpha_1+\alpha_2+\alpha_3} = \sqrt{\omega_2^2+\alpha_3}, \quad T_3 = \frac{2\pi}{\omega_3}, \tag{38}$$

where

$$\alpha_3 = \frac{7}{2}D^2\omega_2^4 - \frac{125}{2}EB\omega_2^3.$$

2.3. Jerk function containing velocity-cubed and velocity times displacement-squared

For functions (III) and (IV), after rescaling of both x and t, the corresponding standardized Jerk equation would take the form

$$\dddot{x} = -\dot{x}(\dot{x}^2 + x^2) \tag{39}$$

with the initial conditions

$$x(0) = 0,\ \dot{x}(0) = B,\ \ddot{x}(0) = 0. \tag{40}$$

Assume that the angular frequency is ω, with the period T given by

$$T = \frac{2\pi}{\omega}. \tag{41}$$

Gottlieb gave the approximate result of frequency and period:

$$\omega = \frac{1}{2\sqrt{2}}\sqrt{3B^2 + \sqrt{9B^4 + 16B^2}}, \quad T = \frac{2\pi}{\omega}. \tag{42}$$

We construct the following homotopy:

$$\dddot{x} + 1\dot{x} = p[\dot{x} - \dot{x}(\dot{x}^2 + x^2)],\ p \in [0,1], \tag{43}$$

Table 2
Comparison between approximate periods and exact periods for Eqs. (22) and (23)

B	T_e	T_1 (%error)	T_2 (%error)	T_3 (%error)
0.10	6.2753338	6.27532641 (−0.00001178)	6.27533376 (−0.0000000637)	6.27533377 (−0.0000000478)
0.20	6.251809	6.25169045 (−0.0001896)	6.25180767 (−0.000002127)	6.25180780 (−0.00000192)
0.50	6.088449	6.08366801 (−0.00785)	6.08812873 (−0.000526)	6.08815979 (−0.000475)
1.0	5.527200	5.44139809 (−1.55236)	5.50630772 (−0.37806)	5.50818960 (−0.34394)
1.5	4.690247	4.15593644 (−11.39195)	4.42685472 (−5.615744)	4.44735707 (−5.17861)

For $B = 0.5$, the analytic approximate solutions are: $x_0(t) = 0.4841\sin\omega_1 t$, $\omega_1 = 1.0328$,

$$x_0(t) + x_1(t) + x_2(t) = 0.488\sin\omega_3 t - 0.0013\sin 3\omega_3 t + 1.3789\times10^{-5}\sin 5\omega_3 t,\ \omega_3 = 1.032.$$

For $B = 1$: $x_0(t) = 0.866\sin\omega_1 t$, $\omega_1 = 1.1547$,

$$x_0(t) + x_1(t) + x_2(t) = 0.8945\sin\omega_3 t - 0.0101\sin 3\omega_3 t + 3.9467\times10^{-4}\sin 5\omega_3 t,\ \omega_3 = 1.1407.$$

where p is a homotopy parameter and used to expand the solution $x(t)$ and the square of the unknown angular frequency ω as follows:

$$x(t) = x_0(t) + px_1(t) + p^2x_2(t) + p^3x_3(t) + \cdots, \tag{44}$$

$$1 = \omega^2 - p\alpha_1 - p^2\alpha_2 - p^3\alpha_3 - \cdots, \tag{45}$$

where $\alpha_i (i = 1, 2, 3, \ldots)$ are to be determined.

Substituting Eqs. (44) and (45) into Eq. (43) and equating the terms with identical powers of p, we obtain a series of linear equations, of which we write only the first three

$$\dddot{x}_0 + \omega^2\dot{x}_0 = 0,\ x_0(0) = 0,\ \dot{x}_0(0) = B,\ \ddot{x}_0(0) = 0, \tag{46}$$

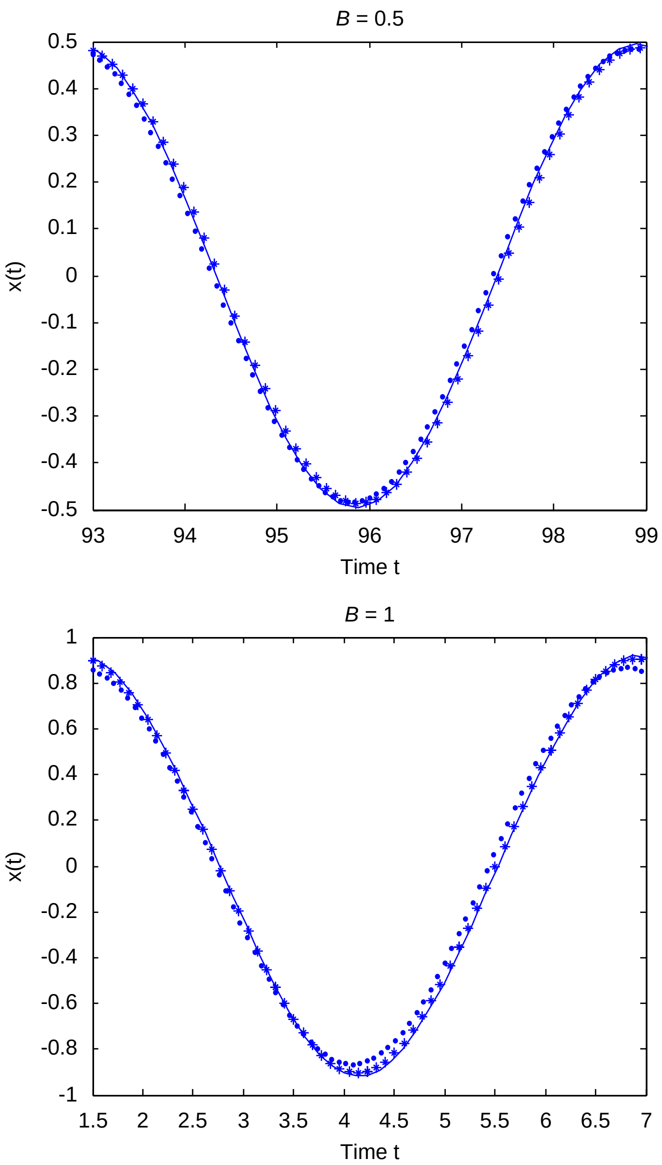

Fig. 2. Comparison between approximate solutions and numerical solution for Eqs. (22) and (23). Numerical: -, first-order approximate solution; ●, third-order approximate solution; *, for $B = 0.5$ and 1.

$$\dddot{x}_1 + \omega^2 \dot{x}_1 = \dot{x}_0 + \alpha_1 \dot{x}_0 - \dot{x}_0^3 - x_0^2 \dot{x}_0,\ x_1(0) = 0,\ \dot{x}_1(0) = 0,\ \ddot{x}_1(0) = 0, \tag{47}$$

$$\dddot{x}_2 + \omega^2 \dot{x}_2 = \dot{x}_1 + \alpha_1 \dot{x}_1 + \alpha_2 \dot{x}_0 - 3\dot{x}_0^2 \dot{x}_1 - 2\dot{x}_0 x_0 x_1 - \dot{x}_1 x_0^2,$$

$$x_2(0) = 0, \cdots \dot{x}_2(0) = 0,\ \ \ddot{x}_2(0) = 0. \tag{48}$$

Similarly, we give the results for approximate periods and corresponding to periodic solutions of Eqs. (39) and (40):

$$x_0(t) = \frac{B}{\omega} \sin \omega t, \tag{49}$$

$$\omega_1 = \frac{1}{2\sqrt{2}} \sqrt{3B^2 + \sqrt{9B^4 + 16B^2}},\ T_1 = \frac{2\pi}{\omega_1}, \tag{50}$$

$$x_1(t) = D\left(\sin \omega t - \frac{1}{3} \sin 3\omega t\right),\ D \triangleq \frac{B^3(1-\omega_1^2)}{32\omega_1^5}, \tag{51}$$

$$\omega_2 = \sqrt{1 + \alpha_1 + \alpha_2} = \sqrt{\omega_1^2 + \alpha_2},\ T_2 = \frac{2\pi}{\omega_2}, \tag{52}$$

where

$$\alpha_2 = \frac{BD}{6\omega_1}(5 + 9\omega_1^2) - \frac{D\omega_1^3}{B}.$$

3. Comparison with the exact results and that obtained by the harmonic balance method approximate results

In this section, we illustrate the accuracy of He's homotopy perturbation method by comparing the approximate results, which are obtained by the harmonic balance method, with the exact numerical results obtained by solving the third-order differential equation with initial conditions using the computational software ODE Workbench [32].

Table 3
Comparison between approximate periods and exact periods for Eqs. (39) and (40)

B	T_e	T_1 (%error)	T_2 (%error)
0.10	25.359725	27.06599846 (6.728281)	24.51793815 (−3.319384)
0.20	17.495410	18.43863244 (5.391257)	17.09797854 (−2.2716327)
0.50	10.210761	10.46108252 (2.451546)	10.14660337 (−0.628333)
1.0	2π	2π (0.000000)	2π (0.000000)
2.0	3.508793	3.457325999 (−1.46680)	3.50968304 (0.002536598)
5.0	1.468638	1.43852659 (−2.050295)	1.46795542 (−0.00464771)
10	0.739762	0.72391989 (−2.1415443)	0.73929940 (−0.00625336)
20	0.370580	0.36255873 (−2.1645178)	0.37033273 (−0.00667253)

The analytic approximate solutions for $B = 0.5$: $x_0(t) = 0.8324664967 \sin \omega_1 t$, $\omega_1 = 0.60062477$,

$$x_0(t) + x_1(t) = 0.8708027 \sin \omega_2 t - 0.0106486 \sin 3\omega_2 t,\ \omega_2 = 0.61924.$$

For $B = 5$: $x_0(t) = 1.144743 \sin \omega_1 t$, $\omega_1 = 4.36779225$,

$$x_0(t) + x_1(t) = 1.09806725 \sin \omega_2 t + 0.014807 \sin 3\omega_2 t,\ \omega_2 = 4.28022896.$$

3.1. Jerk function containing displacement times velocity times acceleration, and velocity

Gottlieb obtained the approximate period given in Eq. (6) in agreement with the first-order approximation (16). Table 1 compares the approximate periods T_1, T_2, T_3 corresponding to the exact period T_e. The relative errors are defined as $(T - T_e)/T_e \times 100$. Fig. 1 shows that the approximate solutions and numerical solutions for $B = 0.1$ and 1. The first-order approximate solution is $x(t) = x_0(t)$, the second-order approximate solution is $x(t) = x_0(t) + x_1(t)$ and the third-order approximate solution is $x(t) = x_0(t) + x_1(t) + x_2(t)$.

From Table 1, we find that the third-order approximate period T_3 is more accurate than T_1 obtained by the harmonic balance method. As the amplitude B of velocity increases, the relative errors increase. Even when $B = 2$, the relative error of third-order approximate period obtained by He's method is less than 0.2034%, the

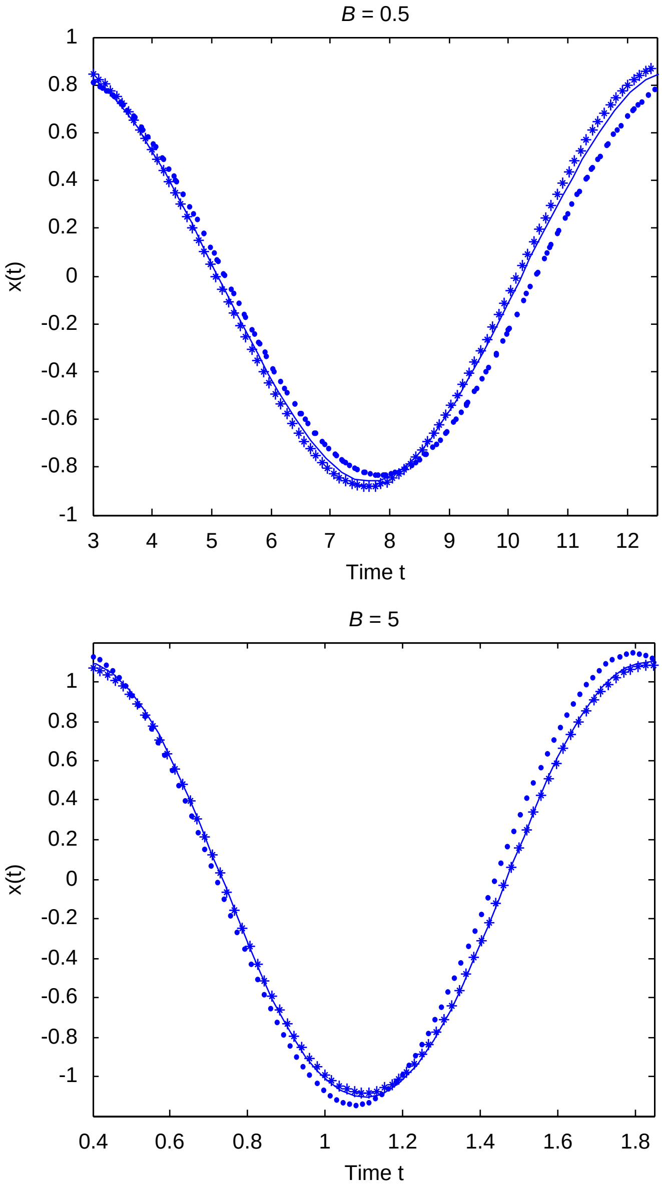

Fig. 3. Comparison between approximate solutions and numerical solution for Eqs. (39) and (40). Numerical: -, first-order approximate solution; ●, second-order approximate solution; *, for $B = 0.5$ and 5.

former result obtained by the harmonic balance method is 1.08%. It is also evident that the third-order approximate solution is more accurate than the former result from Fig. 1.

3.2. Jerk function containing velocity times acceleration-squared, and velocity

Gottlieb obtained the approximate period given in Eq. (25) in agreement with first-order approximation (34). Table 2 compares the approximate periods T_1, T_2, T_3 corresponding to the exact period T_e. The relative errors are defined as $(T - T_e)/T_e \times 100$. Fig. 2 shows the approximate solutions and numerical solutions for $B = 0.5$ and 1. The first-order approximate solution is $x(t) = x_0(t)$, the second-order approximate solution is $x(t) = x_0(t) + x_1(t)$ and the third-order approximate solution is $x(t) = x_0(t) + x_1(t) + x_2(t)$.

From Table 2 and Fig. 2, the third-order approximate results are very accurate.

Particularly for $B > 0.5$, it is also evident that the high-order approximate results are more accurate than the results obtained by the harmonic balance method. In this case, it should be noted that as $B \to 0$, we have T_1, T_2, $T_3 \to 2\pi$.

3.3. Jerk function containing velocity-cubed and velocity times displacement-squared

Gottlieb obtained the approximate frequency and period given in Eq. (42) in agreement with first-order approximation (50). In Section 2.3, we only present the first- and second-order approximate results for conciseness. Table 3 compares the approximate periods T_1, T_2 with the exact period T_e. The relative errors are defined as $(T - T_e)/T_e \times 100$. Fig. 3 shows the approximate solutions and numerical solutions for $B = 0.5$ and 5. The first-order approximate solution is $x(t) = x_0(t)$ and the second-order approximate solution is $x(t) = x_0(t) + x_1(t)$.

From Table 3 and Fig. 3, we find that the second-order approximate results are more accurate than the first-order approximations. It is apparent that if $B = 1$ and $\omega_1 = \omega_2 = 1$, the approximate solution is satisfied exactly. $x(t) = \sin t$ is an exact solution of Eqs. (39) and (40), as can be obtained to solve directly. The relative errors of approximate periods are larger when B is apart from $B = 1$, and as $B \to 0$, we have $T \to \infty$, as $B \to \infty$, we have $T \to 0$.

4. Discussion

In this paper, the Jerk equations involving the third temporal derivative of displacement are analyzed by He's homotopy perturbation method. For a number of different types of appropriate functions (I) $x\dot{x}\ddot{x}$, (II) $\dot{x}\ddot{x}^2$, (III) $x^2\dot{x}$ and (IV) $\dot{x}^3$, the results for periods and periodic solutions were compared with the results obtained by the harmonic balance method and numerical computations. It is found that the first-order approximation is identical to the result obtained by the harmonic balance method. However, the high-order approximations are more accurate and in better agreement with the exact results. Thus, He's homotopy perturbation method is very effective for these third-order nonlinear differential equations.

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