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A unified solution for longitudinal wave propagation in an elastic rod

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Abstract

This paper proposes a unified analytical solution of the wave equation governing the propagation of the longitudinal stress wave in an elastic rod. Within a single formula derived by using the Laplace transform and inverse transform, the solution covers the contributions of the external excitations, the nonzero initial conditions and the inhomogeneous boundary conditions altogether, including such boundary conditions that the dependent variables at the ends of the rod are restricted with an equation. The proposed formula, particularly suitable for transient problems, could be regarded as an exact interpolation function in the time domain, provided the rod works as a component in a complex system, etc. Four examples are presented to show the applications of the solution.

1. Introduction

The propagation of the longitudinal elastic waves of an elastic rod obeys a one-dimensional wave equation, which can be solved either in the time domain or in the frequency domain. In the time domain, for a given problem, the equation can be solved without much difficulty by using some conventional methods such as the separation of variables, d'Alembert's solution, and Green's function. These methods can be found in textbooks of partial differential equations or structural dynamics (see e.g. Refs. [1–5]). However, all of the above-mentioned methods were used to solve individual problems only, i.e., they provide approaches for solving given problems, rather than unified final solutions. Here, a unified solution would mean that the formula of the solution alone covers all the contributions of the external excitations, the nonzero initial conditions and the inhomogeneous boundary conditions altogether. A unified solution is important in complex engineering problems in which a straight rod works as a member or a part of the whole structure, such as a tall building subjected to longitudinal earthquake, or a pipe filled with water [6,7], or a system consisting of several stepped rods [8].

On the other hand, in the frequency domain, the solution can be obtained by using the Fourier/Laplace transform (see e.g. Refs. [4,5]). As an example, Hull [9] developed a closed-form series solution of a

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longitudinal bar with a viscous damper at one end and fixed at the other. The explicit results were given in the frequency domain only, namely the eigenvalues and eigenvectors only. Cortes and Elejabarrieta [10] solved a similar problem, where a viscoelastic damper replaced the viscous damper, of which the solution was in the form of a complex Fourier series with infinite terms, and the transient response in the time domain was given by the real part of the series. It is believed that, to deal with the wave propagation in complex engineering structures, the spectral element would be one of the most efficient methods among the frequency domain methods [11]. More recently, Krawczuk et al. [12] presented a detailed review and further investigations on the spectral element. Although the spectral element could be regarded as a unified solution in the frequency domain, to describe the propagation of the longitudinal elastic wave in the time domain, the numerical algorithm of inverse fast Fourier transform has to be used, which would inevitably induce various errors.

Nevertheless, to solve problems directly in the time domain is necessary. Particularly, in some specific cases, e.g. impact problems, a transient response of a structure is even important. Furthermore, in the above-mentioned complex structures, two or more rods connect each other at the joints. At a joint, a compatibility condition for the displacements is need for solving the whole system, e.g. the displacements of all the rods connected at the joint are all the same. Such compatibility condition would demand the final form of the variables of all the rods instead of an intermediate result. Therefore, for dealing with such complex engineering structures, it would be desirable to derive a unified final solution for a single rod. To the knowledge of the author, such a unified solution, either classical or generalized, is not yet available. The current investigation would offer such a unified generalized solution in the time domain. Before doing so, we would like to review the conventional methods in the time domain, to see how they deal with the problem, and what difficulties they would encounter in order to obtain a unified solution.

First, we consider the method of separation of variables, which is a widely used method. Using this method, one can obtain a Fourier-series solution associated with Duhamel's integral. Nevertheless, the form of the series depends strongly on the boundary conditions. If the boundary conditions were not homogeneous or not simple (the exact meaning of the simple and complex boundary condition is given in Section 2), it would be rather difficult or impossible to get solutions with this method. Furthermore, the uniform convergence of the series should be ensured so that the solution has the second derivative in classical sense, which is still a problem and often not discussed [13]. From the view of engineering practice, a solution in the form of the Fourier series is not suitable for obtaining the transient response, since too many terms of the series should be calculated in order to get a value at a single time instant (see Eq. (57) in Section 3.3), which is also a fault of some methods in the frequency domain.

Second, d'Alembert's method provides a general solution, which is in the form of the sum of two wave functions. Thus, the task to solve a certain problem is to determine the form of the two wave functions so as to satisfy the initial and boundary conditions. However, in the process to determine the two functions, some algebraic equations or ordinary differential equations may encounter. In this sense, d'Alembert's solution is more an approach to obtain the results than a final solution. In other word, the method gives an unfinished solution only. Even so, it is also widely used by many researchers to solve various kinds of problems, e.g. Shi [14,15], Hu et al. [16], and Li et al. [17]. Although these researchers deal with different problems, the forms of the formulas given by the above literatures are similar to that obtained in this paper (see Eqs. (74) and (80)) more or less.

Third, with Green's function, one can get a generalized solution of some partial differential equations including the wave equation in the time domain (see e.g. Ref. [1]). Based on the fundamental solution, the method can be directly used for non-homogeneous equation with zero initial conditions and homogeneous boundary conditions. For other kinds of initial- and boundary-value problems, an appropriate transform is required to change the problem to an equivalent one with zero initial conditions and homogeneous boundary conditions. Therefore, for a general problem, it would be difficult to write out a final solution with Green's function. Furthermore, being expressed as the integral of a series of Heaviside functions, the solution is not clear enough in physical meaning, and inconvenient for engineering application.

In view of the above discussion, from each of the three methods, one can obtain merely the solving process instead of a final solution. Although with the Fourier transform or the Laplace transform one can solve the problems directly, the methods are mostly used in the frequency domain rather than the time domain, since it is normally impossible to perform the inverse transform analytically. In the current paper, we shall devote ourselves to deriving a unified solution in time domain. We shall organize the rest of the paper as follows:

In Section 2, the governing equation and the definite conditions are written in the form suitable for solving. After that, by means of the Laplace transform the solution is obtained in the frequency domain. In Section 3, after properly arranging the solution in the frequency domain, we perform the inverse transform, and obtain the solution in the time domain. In Section 4, we first express the solution in a recurrence formula and an accumulative formulation, and then we prove that the solution satisfies both the governing equation and the initial and boundary conditions. At the end of the section, we present a relation between the velocity and strain at any time and any point in the rod, particularly at both ends, which provides the possibility for solving the problems with complex boundary conditions. Section 5 presents four examples that are solved using the presented approach, and shows how to deal with different initial and boundary conditions with the approach.

2. The equation and its solution in the frequency domain

2.1. The governing equation

The well-known wave equation governing the longitudinal wave propagation in an elastic rod is

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2} + f(x,t), \quad 0 \leqslant x \leqslant L, \ t \geqslant 0, \tag{1}$$

where t denotes time, x denotes the axial distance along the rod, $u(x,t)$ the axial displacement of the rod. $f(x,t)$ is a known function of x and t, corresponding to the external force per unit length. The wave velocity a is then given by $a = \sqrt{E/\rho}$, in which E and ρ are the elastic modulus and the mass density of the rod, respectively.

It is noticed that the displacement is not unique when both ends of the rod are free, which means that Eq. (1), with displacement as the unknowns, does not have unique solution, and is therefore, not appropriate for accomplishing a unified solution. As contrasted with the displacement, the stress (or strain) and velocity are equally important in engineering practice, particularly in experimental investigations. Therefore, we introduce the normal strain $\varepsilon(x,t) = \partial u/\partial x$ and the velocity $\dot{u}(x,t) = \partial u/\partial t$ in Eq. (1), and arrive at a set of equations

$$\frac{\partial \dot{u}}{\partial x} = \frac{\partial \varepsilon}{\partial t}, \ \frac{\partial \dot{u}}{\partial t} = a^2 \frac{\partial \varepsilon}{\partial x} + f(x,t), \quad 0 \leqslant x \leqslant L, \quad t \geqslant 0 \tag{2a,b}$$

or in the matrix form

$$\frac{\partial}{\partial t} y(x,t) = A \frac{\partial}{\partial x} y(x,t) + \bar{f}(x,t), \quad 0 \leqslant x \leqslant L, \ t \geqslant 0, \tag{3}$$

where

$$\mathbf{y}(x,t) \equiv \begin{Bmatrix} \dot{u}(x,t) \\ a\varepsilon(x,t) \end{Bmatrix}, \quad \mathbf{A} \equiv \begin{bmatrix} 0 & a \\ a & 0 \end{bmatrix}, \quad \bar{\mathbf{f}}(x,t) \equiv \begin{Bmatrix} f(x,t) \\ 0 \end{Bmatrix}. \tag{4}$$

In Eq. (4) and in the following sections, the symbol "≡" is used to define a variable, a function or a matrix when they first be introduced. Clearly, $a\varepsilon(x,t)$ and $\dot{u}(x,t)$ have the same dimensions (in m/s). Even so, in the rest of the paper, we shall still refer to $a\varepsilon(x,t)$ as the strain for simplicity.

As a matter of fact, if u is smooth enough, by substituting $\varepsilon = \partial u/\partial x$, $\dot{u} = \partial u/\partial t$ into Eq. (2b), we may return to Eq. (1).

2.2. The initial and boundary conditions

As usual, the initial conditions associated with Eq. (1) are given by

$$u(x,0) = \bar{u}_0(x), \quad \dot{u}(x,0) = I_{\dot{u}}(x), \tag{5}$$

where $\bar{u}_0(x)$ and $I_{\dot{u}}(x)$ are known functions of x. In the case of Eq. (3), after defining the initial strain as $I_\varepsilon(x) = (\partial/\partial x)\bar{u}_0(x)$, the initial conditions may be expressible as

$$\varepsilon(x,0) = I_\varepsilon(x), \quad \dot{u}(x,0) = I_{\dot{u}}(x). \tag{6a,b}$$

To express the boundary conditions, we introduce two *boundary* functions $\varphi_0(t)$ and $\varphi_L(t)$ defined by

$$\begin{aligned} \varphi_0(t) &= \tfrac{1}{2}[\,1+I_0 \quad 1-I_0\,]\mathbf{y}(0,t) \\ \varphi_L(t) &= \tfrac{1}{2}[\,1+I_L \quad 1-I_L\,]\mathbf{y}(L,t) \end{aligned} \quad t>0, \tag{7}$$

where L is the length of the rod, I_0 and I_L are defined to show what a boundary function means, namely

$$\varphi_0(t) = \begin{cases} \dot{u}(0,t) & \text{iff} \quad I_0 = 1 \\ a\varepsilon(0,t) & \text{iff} \quad I_0 = -1 \end{cases}, \quad \varphi_L(t) = \begin{cases} \dot{u}(L,t) & \text{iff} \quad I_L = 1, \\ a\varepsilon(L,t) & \text{iff} \quad I_L = -1, \end{cases} \quad t>0. \tag{8}$$

For example, we see from Eqs. (7) and (8) that the boundary function $\varphi_0(t)$ may represent either the velocity or the strain at $x=0$. If $\varphi_0(t)$ represents the velocity $\dot{u}(0,t)$, then $I_0 = 1$, otherwise $I_0 = -1$. The same can be said for the boundary function $\varphi_L(t)$.

In particular, if the velocities and the strains are prescribed, the boundary functions as well as I_0 and I_L may have one of the four expressions listed in Table 1, where the functions with a super bar indicate any known function on the corresponding boundary. In the simple cases listed in Table 1, Eq. (7) may reduce to

$$\begin{aligned} \varphi_0(t) &= \tfrac{1}{2}[(1-I_0)(a\bar{\varepsilon}_0(t)) + (1+I_0)\bar{\dot{u}}_0(t)], \\ \varphi_L(t) &= \tfrac{1}{2}[(1-I_L)(a\bar{\varepsilon}_L(t)) + (1+I_L)\bar{\dot{u}}_L(t)]. \end{aligned} \tag{9a,b}$$

There are, however, other kinds of boundary conditions when the variables satisfy some equations such as

$$\begin{aligned} g_0(u(0,t), \dot{u}(0,t), \varepsilon(0,t)) &= 0, \\ g_L(u(L,t), \dot{u}(L,t), \varepsilon(L,t)) &= 0, \end{aligned} \tag{10a,b}$$

where g_0 and g_L are any functions defined at the two ends, respectively.

In the following text, if at least at one end, the boundary condition is expressible in the form like Eqs. (10a) or (10b), we may say that there is a *complex boundary condition*. Otherwise, we may say that there are *simple boundary conditions*. That is to say, the velocities or strains are known functions at both ends of the rod (as given by Eq. (9a,b)). Later in Section 4, we shall establish other relation between $\dot{u}(x,t)$ and $\varepsilon(x,t)$ at both ends, so that together with Eq. (10a) or (10b), we have enough equations to determine them, and thus the complex boundary condition reduces to the simple one (see also Case 3 in Section 5). As a matter of fact, until Section 5, we need not known what the boundary functions stand for, and what values I_0 and I_L may have. Thus before Section 5, we may regard Eq. (7a,b) as the boundary conditions.

In some cases, the displacements time progress at certain points of the rod would be more important. After obtaining the velocity, we may calculate the displacement by the integral with the initial displacement $\bar{u}_0(x)$:

$$u(x,t) = \int_0^t \dot{u}(x,\tau)\,\mathrm{d}\tau + \bar{u}_0(x). \tag{11}$$

Table 1
the possible combinations of the boundary functions

Cases	$\varphi_0(t)$ (at $x=0$)	I_0	$\varphi_L(t)$ (at $x=L$)	I_L
1	$\varepsilon(0,t) = \bar{\varepsilon}_0(t)$	−1	$\dot{u}(L,t) = \bar{\dot{u}}_L(t)$	1
2	$\varepsilon(0,t) = \bar{\varepsilon}_0(t)$	−1	$\varepsilon(L,t) = \bar{\varepsilon}_L(t)$	−1
3	$\dot{u}(0,t) = \bar{\dot{u}}_0(t)$	1	$\dot{u}(L,t) = \bar{\dot{u}}_L(t)$	1
4	$\dot{u}(0,t) = \bar{\dot{u}}_0(t)$	1	$\varepsilon(L,t) = \bar{\varepsilon}_L(t)$	−1

On some other cases, displacements at certain time instant would be interested. Since we have obtained the strain at any time instant, we may use another integral to achieve this, namely

$$u(x,t) = \int_0^x \varepsilon(y,t)\,\mathrm{d}y + u(0,t) \quad \text{or} \quad u(x,t) = \int_x^L \varepsilon(y,t)\,\mathrm{d}y + u(L,t), \tag{12}$$

where $u(0,t)$ and $u(L,t)$ are the known boundary values at the ends, respectively. With the help of Eqs. (11) and (12), we might say that Eq. (2a,b) is, not in the strict mathematical sense, equivalent to Eq. (1). Considering that numerically calculating an integral will not cause large errors as derivate does, and we could reasonably believe that there are no further difficulties to obtain displacement. We would, therefore, not mention the displacement any more in the following context.

2.3. Solution in the frequency domain

The Laplace transform of both sides of Eq. (2) yields

$$s\mathbf{Y}(x,s) - \mathbf{y}(x,0) = \mathbf{A}\frac{\partial}{\partial x}\mathbf{Y}(x,s) + \bar{\mathbf{F}}(x,s), \tag{13}$$

where

$$\mathbf{Y}(x,s) \equiv \mathsf{L}[\mathbf{y}(x,s)], \quad \bar{\mathbf{F}}(x,s) \equiv \mathsf{L}[\bar{\mathbf{f}}(x,t)], \quad \mathbf{y}(x,0) = \begin{Bmatrix} I_{\dot{u}}(x) \\ aI_{\varepsilon}(x) \end{Bmatrix}. \tag{14}$$

The general solution of Eq. (13) is

$$\mathbf{Y}(x,s) = \mathbf{I}_c\,\mathrm{diag}\{\mathrm{e}^{-sx/a}, \mathrm{e}^{sx/a}\}\mathbf{C}(s) + \tilde{\mathbf{Y}}(x,s), \tag{15}$$

where

$$\tilde{\mathbf{Y}}(x,s) \equiv \begin{Bmatrix} \tilde{U}(x,s) \\ a\tilde{\mathrm{E}}(x,s) \end{Bmatrix}, \quad \mathbf{I}_c = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}, \quad \mathbf{C}(s) \equiv \begin{Bmatrix} C_1(s) \\ C_2(s) \end{Bmatrix}. \tag{16a–c}$$

In Eq. (16), $\tilde{\mathbf{Y}}(x,s)$ is the particular solution of Eq. (13), $C_1(s)$ and $C_2(s)$ are arbitrary functions of subsidiary variable s. The particular solution of Eq. (13) is assumed of the form

$$\tilde{\mathbf{Y}}(x,s) = \mathbf{I}_c\,\mathrm{diag}\{\mathrm{e}^{-sx/a}, \mathrm{e}^{sx/a}\}\mathbf{P}(x,s), \tag{17}$$

where $\mathbf{P}(x,s)$ is an unknown function to be determined. Substituting Eq. (17) into Eq. (13) yields

$$\mathbf{P}'(x,s) = \frac{1}{2a}\mathrm{diag}\{\mathrm{e}^{sx/a}, -\mathrm{e}^{-sx/a}\}\tilde{\mathbf{R}}(x,s), \tag{18}$$

where

$$\tilde{\mathbf{R}}(x,s) \equiv \begin{Bmatrix} \tilde{R}_1(x,s) \\ \tilde{R}_2(x,s) \end{Bmatrix} \equiv \tilde{\mathbf{I}}(x) + \tilde{\mathbf{F}}(x,s), \tag{19}$$

$$\tilde{\mathbf{I}}(x) \equiv \begin{Bmatrix} \tilde{I}_1(x) \\ \tilde{I}_2(x) \end{Bmatrix} \equiv \begin{Bmatrix} -1 \\ 1 \end{Bmatrix} aI_{\varepsilon}(x) + \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} I_{\dot{u}}(x), \quad \tilde{\mathbf{F}}(x,s) \equiv \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} F(x,s). \tag{20a,b}$$

The solution of Eq. (18) is therefore

$$\mathbf{P}(x,s) = \frac{1}{2a}\int_0^x \mathrm{diag}\{\mathrm{e}^{s\xi/a}, -\mathrm{e}^{-s\xi/a}\}\tilde{\mathbf{R}}(\xi,s)\,\mathrm{d}\xi. \tag{21}$$

Inserting Eq. (21) into Eq. (17) yields

$$\tilde{\mathbf{Y}}(x,s) = \frac{1}{2a}\mathbf{I}_c\int_0^x \mathrm{diag}\{\mathrm{e}^{-s(x-\xi)/a}, -\mathrm{e}^{s(x-\xi)/a}\}\tilde{\mathbf{R}}(\xi,s)\,\mathrm{d}\xi. \tag{22}$$

The Laplace transform of the boundary functions given by Eq. (7) gives

$$\begin{aligned}\Phi_0(s) &\equiv \mathsf{L}[\varphi_0(t)] = \tfrac{1}{2}[\,1+I_0 \quad 1-I_0\,]\mathbf{Y}(0,s),\\ \Phi_L(s) &\equiv \mathsf{L}[\varphi_L(t)] = \tfrac{1}{2}[\,1+I_L \quad 1-I_L\,]\mathbf{Y}(L,s).\end{aligned} \tag{23}$$

Taking $x=0$ and L in Eq. (15), respectively, and then substituting the results into Eq. (23), we obtain

$$\begin{Bmatrix}\Phi_0(s)\\ \Phi_L(s)\end{Bmatrix} = \begin{bmatrix} I_0 & 1\\ I_L\mathrm{e}^{-sL/a} & \mathrm{e}^{sL/a}\end{bmatrix}\begin{Bmatrix}C_1(s)\\ C_2(s)\end{Bmatrix} + \begin{Bmatrix}0\\ \tilde{\Phi}_L(s)\end{Bmatrix}, \tag{24}$$

where

$$\tilde{\Phi}_L(s) \equiv \frac{1}{2a}I_L\int_0^L \mathrm{e}^{-s(L-\xi)/a}\tilde{R}_1(\xi,s)\,\mathrm{d}\xi - \frac{1}{2a}\int_0^L \mathrm{e}^{s(L-\xi)/a}\tilde{R}_2(\xi,s)\,\mathrm{d}\xi. \tag{25}$$

Let $j_0 \equiv I_0 I_L$. Solving Eq. (24) gives

$$\mathbf{C}(s) = \frac{1}{(1-j_0\,\mathrm{e}^{-2sL/a})}\mathbf{E}_0(s)(\mathbf{\Phi}(s) - \tilde{\mathbf{\Phi}}(s)), \tag{26}$$

where $(1/(1-j_0\,\mathrm{e}^{-2sL/a}))\mathbf{E}_0(s)$ is the inverse of the matrix on the right-hand side of Eq. (24) and

$$\mathbf{E}(x,s) \equiv \mathrm{diag}\{\mathrm{e}^{-sx/a}, \mathrm{e}^{sx/a}\}E_0(s) = \begin{bmatrix} I_0\,\mathrm{e}^{-sx/a} & -I_0\,\mathrm{e}^{-s(L+x)/a}\\ -j_0\,\mathrm{e}^{-s(2L-x)/a} & \mathrm{e}^{-s(L-x)/a}\end{bmatrix},$$

$$\mathbf{\Phi}(s) \equiv \begin{Bmatrix}\Phi_0(s)\\ \Phi_L(s)\end{Bmatrix}, \quad \tilde{\mathbf{\Phi}}(s) \equiv \begin{Bmatrix}0\\ \tilde{\mathbf{\Phi}}_L(s)\end{Bmatrix}. \tag{27}$$

The solution of Eq. (13) is therefore

$$\mathbf{Y}(x,s) = \frac{1}{(1-j_0\,\mathrm{e}^{-2sL/a})}\mathbf{I}_c\mathbf{E}(x,s)(\mathbf{\Phi}(s) - \tilde{\mathbf{\Phi}}(s)) + \tilde{\mathbf{Y}}(x,s). \tag{28}$$

It is worth noting that Eq. (26) is a relation between $\mathbf{C}(s)$ and the boundary functions (instead of the prescribed functions on the boundary). Eq. (28) gives, therefore, a relation between $\mathbf{Y}(x,s)$ and its boundary value $\mathbf{Y}(0,s)$ and $\mathbf{Y}(L,s)$ expressed by $\mathbf{\Phi}(s)$ and $\tilde{\mathbf{\Phi}}(s)$. The values of $\mathbf{Y}(x,s)$ at both ends are not necessarily known functions before solving. Considering that one of the aims of the paper is to obtain a solution that can be used in the complex engineering structures, these kinds of relations are necessary.

3. The inverse transform

3.1. Performing the inverse transform

Usually, the inverse transform of the right-hand side of Eq. (28) could not be analytically archived, except in some simple special cases. To make the inverse transform possible, we first multiply both sides of Eq. (28) by $(1–j_0\mathrm{e}^{-2sL/a})$ and have

$$(1-j_0\,\mathrm{e}^{-2sL/a})\mathbf{Y}(x,s) = \mathbf{Q}(x,s), \tag{29}$$

where

$$\mathbf{Q}(x,s) \equiv \mathbf{I}_c\mathbf{E}(x,s)\mathbf{\Phi}(s) - \mathbf{I}_c\mathbf{E}(x,s)\tilde{\mathbf{\Phi}}(s) + (1-j_0\,\mathrm{e}^{-2sL/a})\tilde{\mathbf{Y}}(x,s). \tag{30}$$

The inverse transform of the left-hand side of Eq. (29) is obtained by making use of the shift rule for the Laplace transform

$$\mathsf{L}^{-1}[(1-j_0\,\mathrm{e}^{-2sL/a})\mathbf{Y}(x,s)] = \mathbf{y}(x,t) - j_0\mathbf{y}(x,t-2L/a)\mathsf{H}(t-2L/a), \tag{31}$$

where $\mathsf{H}(t)$ is the Heaviside function defined by

$$\mathsf{H}(t) = \begin{cases} 1, & t > 0, \\ 0, & t \leqslant 0. \end{cases} \tag{32}$$

The inverse transform of Eq. (29) may lead to

$$\mathbf{y}(x,t) = j_0 \mathbf{y}(x, t-T)\mathsf{H}(t-T) + \mathbf{q}(x,t), \tag{33}$$

where

$$T \equiv 2L/a, \quad \mathbf{q}(x,t) \equiv \mathsf{L}^{-1}[\mathbf{Q}(x,s)]. \tag{34a,b}$$

In the following text, T is referred to as a *phase*, which has the same meaning as used in the problem of water hammer, where a phase expresses the time interval when the elastic wave travels from one end to another and then returns. Eq. (33) reveals that the solution at time instant t is related to the solution at $t–T$. In the first phase, we have

$$\mathbf{y}(x, t-T)\mathsf{H}(t-T) = 0, \quad 0 < t \leqslant T. \tag{35}$$

Eq. (33) reduces to

$$\mathbf{y}(x,t) = \mathbf{q}(x,t), \quad 0 < t \leqslant T. \tag{36}$$

Eq. (33) implies that once having the solutions in $0 < t \leqslant T$, we may obtain the solutions at any time instant with Eq. (33) step by step, on condition that the right-hand side term $\mathbf{q}(x,t)$ given by Eq. (34b) is known. In order to obtain $\mathbf{q}(x,t)$, we have to rearrange Eq. (30), since the second and the third terms of Eq. (30) are not suitable for performing the inverse transform analytically. Recalling Eqs. (19), (22) and (25), we can write Eq. (30) as

$$\mathbf{Q}(x,s) = \mathbf{I}_c \mathbf{Q}_B(x,s) + \frac{1}{2a}\mathbf{I}_c \mathbf{Q}_I(x,s) + \frac{1}{2a}\mathbf{I}_c \mathbf{Q}_F(x,s), \tag{37}$$

where

$$\begin{aligned} Q_I(x,s) \equiv & \int_0^x \begin{Bmatrix} -a\,\mathrm{e}^{-s(x-\xi)/a}\tilde{I}_1(\xi) \\ a j_0\,\mathrm{e}^{-s(2L-x+\xi)/a}\tilde{I}_2(\xi) \end{Bmatrix} \mathrm{d}\xi \\ & + \int_x^L \begin{Bmatrix} -a j_0\,\mathrm{e}^{-s(2L+x-\xi)/a}\tilde{I}_1(\xi) \\ a\,\mathrm{e}^{-s(\xi-x)/a}\tilde{I}_2(\xi) \end{Bmatrix} \mathrm{d}\xi - \int_0^L \begin{Bmatrix} a I_0\,\mathrm{e}^{-s(x+\xi)/a}\tilde{I}_2(\xi) \\ -a I_L\,\mathrm{e}^{-s(2L-x-\xi)/a}\tilde{I}_1(\xi) \end{Bmatrix} \mathrm{d}\xi, \end{aligned} \tag{38}$$

$$\begin{aligned} Q_F(x,s) \equiv & \int_0^x \begin{Bmatrix} \mathrm{e}^{-s(x-\xi)/a}F(\xi,s) \\ j_0\,\mathrm{e}^{-s(2L-x+\xi)/a}F(\xi,s) \end{Bmatrix} \mathrm{d}\xi \\ & + \int_x^L \begin{Bmatrix} j_0\,\mathrm{e}^{-s(2L+x-\xi)/a}F(\xi,s) \\ \mathrm{e}^{-s(\xi-x)/a}F(\xi,s) \end{Bmatrix} \mathrm{d}\xi - \int_0^L \begin{Bmatrix} I_0\,\mathrm{e}^{-s(x+\xi)/a}F(\xi,s) \\ I_L\,\mathrm{e}^{-s(2L-x-\xi)/a}F(\xi,s) \end{Bmatrix} \mathrm{d}\xi, \end{aligned} \tag{39}$$

$$\mathbf{Q}_B \equiv \begin{Bmatrix} I_0\,\mathrm{e}^{-sx/a}\Phi_0 - I_0\,\mathrm{e}^{-s(L+x)/a}\Phi_L \\ -I_0 I_L\,\mathrm{e}^{-s(2L-x)/a}\Phi_0 + \mathrm{e}^{-s(L-x)/a}\Phi_L \end{Bmatrix}. \tag{40}$$

It is very important to notice that each integrand in Eqs. (38) and (39) contain a factor like e^{-sc} with $c \geqslant 0$ inside the interval of the corresponding integral, which makes the explicit results of the inverse Laplace transform possible. Accordingly, we can denote

$$\mathbf{B}(x,t) \equiv \mathsf{L}^{-1}[\mathbf{Q}_B(x,s)], \quad \mathbf{I}(x,t) \equiv \mathsf{L}^{-1}[\mathbf{Q}_I(x,s)], \quad \mathbf{F}(x,t) \equiv \mathsf{L}^{-1}[\mathbf{Q}_F(x,s)] \tag{41a–c}$$

and

$$\mathbf{q}_B(x,t) \equiv \mathbf{I}_c \mathbf{B}(x,t), \quad \mathbf{q}_I(x,t) \equiv \frac{1}{2a}\mathbf{I}_c \mathbf{I}(x,t), \quad \mathbf{q}_F(x,t) \equiv \frac{1}{2a}\mathbf{I}_c \mathbf{F}(x,t) \tag{42a–c}$$

in view of Eqs. (34), (37), (41) and (42) we have

$$\mathbf{q} = \mathsf{L}^{-1}[\mathbf{Q}(x,s)] = \mathbf{q}_B(x,t) + \mathbf{q}_I(x,t) + \mathbf{q}_F(x,t). \tag{43}$$

Obviously, $\mathbf{q}_B(x,t)$ is related to the boundary conditions at both ends, $\mathbf{q}_I(x,t)$ to the initial conditions, and $\mathbf{q}_F(x,t)$ to $f(x,t)$ the external force per unit length. Therefore, we may refer to them as *the boundary term*, *the initial term* and *the forced term*, respectively. Thus, we may express the elements of three terms as

$$\mathbf{q}_B(x,t) = \begin{Bmatrix} \dot{u}_B(x,t) \\ a\varepsilon_B(x,t) \end{Bmatrix}, \quad \mathbf{q}_I(x,t) = \begin{Bmatrix} \dot{u}_I(x,t) \\ a\varepsilon_I(x,t) \end{Bmatrix}, \quad \mathbf{q}_F(x,t) = \begin{Bmatrix} \dot{u}_F(x,t) \\ a\varepsilon_F(x,t) \end{Bmatrix}. \tag{44a–c}$$

Moreover, $j_0\mathbf{y}(x, t-2L/a)$ will be referred to as *the historical term*. In the follow three subsections, we shall find out the expressions of the three terms given by Eqs. (44a–c), one after another.

3.2. Obtaining the boundary term

The inverse Laplace transform of Eq. (41a) may lead to

$$\mathbf{B}(x,t) \equiv \begin{Bmatrix} I_0\varphi_0(t-x/a)\mathsf{H}(t-x/a) - I_0\varphi_L(t-(L+x)/a)\mathsf{H}(t-(L+x)/a) \\ -j_0\varphi_0(t-(2L-x)/a)\mathsf{H}(t-(2L-x)/a) + \varphi_L(t-(L-x)/a)\mathsf{H}(t-(L-x)/a) \end{Bmatrix}. \tag{45}$$

Obviously, the two components of vector $\mathbf{B}$ are in the form similar to the wave functions of d'Alembert's solution, namely,

$$\mathbf{B}(x,t) = \begin{Bmatrix} B_1(t-x/a) \\ B_2(t+x/a) \end{Bmatrix}. \tag{46}$$

Considering that $\lim_{t\to 0^+}\mathsf{H}(t) = 1$, we may define in the current paper that for all the functions of the form $g(t) = f(t)\mathsf{H}(t)$, the initial value of $g(t)$ may be defined as

$$g(0) \equiv \lim_{t\to 0^+} f(t)\mathsf{H}(t) = \lim_{t\to 0^+} f(t) = f(0^+). \tag{47}$$

Eq. (47) implies that $H(0)$, whatever it may be, has no effect on $g(0)$, and is, therefore, not important. In particular, we could define

$$\varphi_0(0) \equiv \lim_{z\to 0^+}\varphi_0(z)\mathsf{H}(z), \quad \varphi_L(0) \equiv \lim_{z\to 0^+}\varphi_L(z)\mathsf{H}(z). \tag{48}$$

Furthermore, Eq. (45) suggests that we could also define

$$\varphi_0(z)|_{z<0} = 0, \quad \varphi_L(z)|_{z<0} = 0 \tag{49}$$

with Eqs. (48) and (49) in mind, we could, for the sake of short description, omit $\mathsf{H}(z)$ and rewrite Eq. (45) as

$$\mathbf{B}(x,t) = \begin{Bmatrix} B_1(t-x/a) \\ B_2(t+x/a) \end{Bmatrix} = \begin{Bmatrix} I_0\varphi_0(t-x/a) - I_0\varphi_L(t-(L+x)/a) \\ -j_0\varphi_0(t-(2L-x)/a) + \varphi_L(t-(L-x)/a) \end{Bmatrix}. \tag{50}$$

It should be pointed out that Eq. (48) gives the initial value of the boundary conditions, instead of the boundary value of the initial conditions. Normally, the two values are not equal to each other.

3.3. Obtaining the forced term

From Eq. (39), we see that all the terms in $\boldsymbol{Q}_F(x,s)$ are of the form

$$\int_\alpha^\beta \mathrm{e}^{-sc}F(\xi,s)\,\mathrm{d}\xi, \quad c\geqslant 0. \tag{51}$$

Thus,

$$\mathsf{L}^{-1}\int_\alpha^\beta \mathrm{e}^{-sc}F(\xi,s)\,\mathrm{d}\xi = \int_\alpha^\beta f(\xi,t-c)\mathsf{H}(t-c)\,\mathrm{d}\xi \quad c\geqslant 0. \tag{52}$$

Since $f(\xi, t-c)\mathsf{H}(t-c) \overset{t\le c}{\equiv} 0$. and a finite single value $f(x,0)$ has no effect on the result of the integral, we can assuming that if the argument $z\geqslant 0$ then

$$f(x,z)=0, \quad 0\leqslant x\leqslant L. \tag{53}$$

Therefore, with Eq. (53) in mind, we can omit the Heaviside function $H(t-c)$ and write for simplicity

$$\int_\alpha^\beta f(\xi,t-c)\mathsf{H}(t-c)\,\mathrm{d}\xi = \int_\alpha^\beta f(\xi,t-c)\,\mathrm{d}\xi. \tag{54}$$

Thus, the inverse transform of $\mathbf{Q}_F(x,s)$ yields

$$\mathbf{F}(x,t)=\mathsf{L}^{-1}[\mathbf{Q}_F(x,s)] = \begin{Bmatrix} f_1(x,t-x/a) \\ f_2(x,t+x/a) \end{Bmatrix}, \tag{55}$$

where

$$\begin{Bmatrix} f_1(x,t-x/a) \\ f_2(x,t+x/a) \end{Bmatrix} \equiv \int_0^x \begin{Bmatrix} f(\xi,t-(x-\xi)/a) \\ j_0 f(\xi,t-(2L-x+\xi)/a) \end{Bmatrix} \mathrm{d}\xi + \int_x^L \begin{Bmatrix} j_0 f(\xi,t-(2L+x-\xi)/a) \\ f(\xi,t-(\xi-x)/a) \end{Bmatrix} \mathrm{d}\xi \\ - \int_0^L \begin{Bmatrix} I_0 f(\xi,t-(x+\xi)/a) \\ I_L f(\xi,t-(2L-x-\xi)/a) \end{Bmatrix} \mathrm{d}\xi. \tag{56}$$

It is not difficult to verify that Eq. (56) remains valid, even if $f(x,t)$ is a centralized force acted on x_p, namely $f(x,t)=\delta(x-x_p)p(t)$.

In contrast to the integral of Eq. (56), we would like to look at the results obtained by using the method of separation of variables. For instance, when a rod with zero initial condition and both ends fixed, the latter will results in an integral like

$$u(x,t)=\sum_{n=1}^{\infty}\sin(n\pi x/L)\int_0^t\int_0^L f(\xi,\tau)\sin(n\pi\xi/L)\,\sin(n\pi a(t-\tau)/L)\,\mathrm{d}\xi\,\mathrm{d}\tau. \tag{57}$$

Clearly, once a numerical result is needed, Eq. (57) will be calculated for every n, since the index n appears within the sine functions. This means that at any time instant, one must calculate enough numbers of integrals in order to get satisfying precision, which is time consuming. Furthermore, if the boundary conditions have to transform into a nonzero $f(x,t)$, as does with the Green function and the method of separation of variables, some additional integral(s) will also inevitably appear. However, in the current approach, the integral is needed only when $f(x,t)$ is nonzero, the initial and boundary conditions result in no integral. Moreover, the calculation of the integrals given by Eq. (56) is much simpler than Eq. (57).

3.4. Obtaining the initial term

Eq. (38) shows that all the terms in $\mathbf{Q}_I(x,s)$ are of the form

$$\int_\alpha^\beta \mathrm{e}^{-sc} I(\xi)\,\mathrm{d}\xi, \quad c\geqslant 0. \tag{58}$$

Considering that

$$\mathsf{L}^{-1}[\mathrm{e}^{-sc}] = \delta(t-c), \quad c\geqslant 0, \tag{59}$$

where $\delta(t–c)$ is the Dirac delta function. Therefore, the inverse transform of $\mathbf{Q}_I(x,s)$ yields

$$\mathbf{I}(x,t) \equiv \mathsf{L}^{-1}[\mathbf{Q}_I(x,s)] = \int_0^x \begin{Bmatrix} \tilde{I}_1(\xi)\delta(t-(x-\xi)/a) \\ j_0\tilde{I}_2(\xi)\delta(t-(2L-x+\xi)/a) \end{Bmatrix} \mathrm{d}\xi + \int_x^L \begin{Bmatrix} j_0\tilde{I}_1(\xi)\delta(t-(2L+x-\xi)/a) \\ \tilde{I}_2(\xi)\delta(t-(\xi-x)/a) \end{Bmatrix} \mathrm{d}\xi - \int_0^L \begin{Bmatrix} I_0\tilde{I}_2(\xi)\delta(t-(x+\xi)/a) \\ I_L\tilde{I}_1(\xi)\delta(t-(2L-x-\xi)/a) \end{Bmatrix} \mathrm{d}\xi. \tag{60}$$

Thus, each integral in Eq. (60) could be obtained (see e.g. Ref. [5])

$$\int_\alpha^\beta f(\xi)\delta(\xi-z)\,\mathrm{d}\xi = \begin{cases} 0, & z \notin [\alpha,\beta], \\ f(z), & z \in (\alpha,\beta), \end{cases} \tag{61}$$

where α and β are any real number satisfying $0 \leqslant \alpha < \beta$. Introducing the notation

$$\mathsf{G}(z,\alpha,\beta) \equiv \begin{cases} 0, & z \notin [\alpha,\beta], \\ 1, & z \in (\alpha,\beta). \end{cases} \tag{62}$$

Eq. (61) can be written as

$$\int_\alpha^\beta f(\xi)\delta(\xi-z)\,\mathrm{d}\xi = f(z)\mathsf{G}(z,\alpha,\beta). \tag{63}$$

Note that the integral of Eq. (63) may have different values at $z=\alpha$ and $z=\beta$, and so does the function G given by Eq. (62). Similar to unit step function H, there are several possible definitions available. We will see later that the definite value of G at the two points will make the solution continuous. Moreover, From Eq. (63) we have $\mathsf{G}(z,\alpha,\beta) \overset{\alpha=\beta}{=} 0$. Finally, from Eqs. (60) and (61), and considering that

$$\delta(x-\xi) = \delta(\xi-x), \quad \delta(k\xi) = \frac{1}{k}\delta(\xi) \quad (k>0), \tag{64}$$

we arrive at

$$\mathbf{I}(x,t) = \begin{Bmatrix} I_1(x-at) \\ I_2(x+at) \end{Bmatrix} = a\begin{Bmatrix} \tilde{I}_1(x-at)\mathsf{G}(at,0,x) \\ \tilde{I}_2(at+x)\mathsf{G}(at,0,L-x) \end{Bmatrix} - a\begin{Bmatrix} I_0\tilde{I}_2(at-x)\mathsf{G}(at,x,L+x) \\ I_L\tilde{I}_1(2L-x-at)\mathsf{G}(at,L-x,2L-x) \end{Bmatrix} + a\begin{Bmatrix} j_0\tilde{I}_1(2L+x-at)\mathsf{G}(at,L+x,2L) \\ j_0\tilde{I}_2(at-2L+x)\mathsf{G}(at,2L-x,2L) \end{Bmatrix}. \tag{65}$$

As a mater of fact, the two components of vector $\mathbf{I}(x,t)$ given by Eq. (65) can be written as

$$I_1(x-at) = a\begin{cases} \tilde{I}_1(x-at), & 0<at<x, \\ -I_0\tilde{I}_2(at-x), & x<at<L+x, \\ j_0\tilde{I}_1(2L+x-at), & L+x<at<2L, \end{cases}$$
$$I_2(x+at) = a\begin{cases} \tilde{I}_2(at+x), & 0<at<L-x, \\ -I_L\tilde{I}_1(2L-x-at), & L-x<at<2L-x, \\ j_0\tilde{I}_2(at+x-2L), & 2L-x<at<2L. \end{cases} \tag{66}$$

Eqs. (65) and (66) indicate that like the vector $\mathbf{B}(x,t)$ expressed by Eq. (46), $\mathbf{I}(x,t)$ also has the form similar to d'Alembert's wave functions. Moreover, Eq. (66) shows that $I_1(x–at)$ is discontinuous when $at=x$ or $at=L+x$, and $I_2(x+at)$ is discontinuous when $at=L–x$ or $at=2L$. (see Case 2 of Section 5). To be more exact, the left limit and the right limit at one of the four points are not equal, namely,

$$\lim_{at\to x-0} I_1(x-at) = a\tilde{I}_1(0), \quad \lim_{at\to x+0} I_1(x-at) = -aI_0\tilde{I}_2(0),$$
$$\lim_{at\to L+x-0} I_1(x-at) = -aI_0\tilde{I}_2(L), \quad \lim_{at\to L+x+0} I_1(x-at) = aj_0\tilde{I}_1(L),$$

$$\lim_{at\to L-x-0} I_2(x+at) = a\tilde{I}_2(L), \qquad \lim_{at\to L-x+0} I_2(x+at) = -aI_0\tilde{I}_1(L),$$
$$\lim_{at\to 2L-x-0} I_2(x+at) = -aI_0\tilde{I}_1(0), \qquad \lim_{at\to 2L-x+0} I_2(x+at) = aj_0\tilde{I}_2(0). \tag{67}$$

Therefore like the solution obtained by Green's function method, the proposed solution is generalized rather than a classical solution in a strict sense. Obviously, from Eq. (65), it can be concluded that when $t>T$ the initial term vanishes. Namely,

$$I_1(x-at)|_{t>T} = 0, \quad I_2(at+x)|_{t>T} = 0, \quad \forall x \in [0, L]. \tag{68}$$

Eq. (68) indicates that the initial conditions have their direct effect on the solution only in the first phase (see also Eq. (80) in Section 4.1).

4. Proofs and discussions

So far, the solution of Eq. (3) has been obtained as shown in Eq. (33) of which the right-hand term $\mathbf{q}(x,t)$ has been found by means of the inverse Laplace transform. In this section, the solution will first be expressed in two formulas suitable for application. Then, it will be verified that the solution satisfies the governing equation, the boundary conditions and the initial conditions. Finally, some relations between the velocity and strain are established, which is important for dealing with complex boundary conditions.

4.1. Deriving the recurrence formula and accumulative formula

When $0 \leqslant t \leqslant T$, the historical term $\mathbf{y}(x, t-T)$ vanishes, Eq. (33) results in

$$\mathbf{y}(x,t) = \mathbf{q}_B(x,t) + \mathbf{q}_I(x,t) + \mathbf{q}_F(x,t), \quad 0 \leqslant t \leqslant T. \tag{69}$$

Therefore, when $0 \leqslant t \leqslant T$, we can obtain the solution of Eq. (3) directly. After that, the initial term $\mathbf{q}_I(x,t)$ vanishes, and Eq. (33) becomes

$$\mathbf{y}(x,t) = j_0\mathbf{y}(x, t-T) + \mathbf{q}_B(x,t) + \mathbf{q}_F(x,t), \quad t>T. \tag{70}$$

The first term of the right-hand side of Eq. (70) is determined by Eq. (69), we can, therefore, get the solution in the next time interval $T \leqslant t \leqslant 2T$, the same may be done for $2T \leqslant t \leqslant 3T$ and so on. In particular, the velocity and strain can be expressed as

$$\begin{aligned} \dot{u}(x,t) &= \dot{u}_B(x,t) + \dot{u}_I(x,t) + \dot{u}_F(x,t), && 0 \leqslant t \leqslant T, \\ \dot{u}(x,t) &= \dot{u}(x,t-T) + \dot{u}_B(x,t) + \dot{u}_F(x,t), && t>T \end{aligned} \tag{71}$$

and

$$\begin{aligned} \varepsilon(x,t) &= \varepsilon_B(x,t) + \varepsilon_I(x,t) + \varepsilon_F(x,t), && 0 \leqslant t \leqslant T, \\ \varepsilon(x,t) &= \varepsilon(x,t-T) + \varepsilon_B(x,t) + \varepsilon_F(x,t), && t>T. \end{aligned} \tag{72}$$

Introducing the notation

$$\tau = t - nT, \quad 0 \leqslant \tau \leqslant T, \quad n = 0, 1, 2, \ldots \tag{73}$$

and substituting Eq. (73) into Eqs. (69) and (70) yields

$$\begin{aligned} &\mathbf{y}(x,\tau) = \mathbf{q}(x,\tau), \\ &\mathbf{y}(x,\tau+nT) = j_0\mathbf{y}(x,\tau+(n-1)T) + \mathbf{q}(x,\tau+nT), \quad n = 1, 2, \ldots . \end{aligned} \tag{74a,b}$$

Eqs. (74a,b) are referred to as *the recurrence* formula. It indicates that in the nth phase, the solution is determined by the solution in the prior phase, together with the boundary term and the forced term within the same phase. With the recurrence formula, we can obtain the solution at any time instance step by step. In the text below, unless otherwise stated, we take $0 \leqslant \tau \leqslant T$ and the index $n = 0,1,2,\ldots$.

We now try to delimit the historical term $\mathbf{y}(x, \tau + (n-1)T)$ in Eq. (74b). Multiplying both sides of the ith equation in Eqs. (74a,b) by $j_0^i = (\pm 1)^i$, and then summing up the results over i, we have

$$\sum_{i=0}^{n} j_0^i \mathbf{y}(x, \tau + iT) = \sum_{i=1}^{n} j_0^{i+1} \mathbf{y}(x, \tau + (i-1)T) + \sum_{i=0}^{n} j_0^i \mathbf{q}(x, \tau + iT). \tag{75}$$

The left-hand term of Eq. (75) can be written as

$$\sum_{i=0}^{n} j_0^i \mathbf{y}(x, \tau + iT) = j_0^n \mathbf{y}(x, \tau + nT) + \sum_{i=0}^{n-1} j_0^i \mathbf{y}(x, \tau + iT). \tag{76}$$

The first term on the right-handside of Eq. (75) can be written as

$$\sum_{i=1}^{n} j_0^i \mathbf{y}(x, \tau + (i-1)T) \overset{j=i-1}{=} \sum_{j=0}^{n-1} j_0^{j+2} \mathbf{y}(x, \tau + jT) \overset{i=j}{=} \sum_{i=0}^{n-1} j_0^i \mathbf{y}(x, \tau + iT). \tag{77}$$

Note that

$$j_0 = \pm 1, \quad j_0^n = j_0^{-n}, \quad j_0^{n+i} = j_0^{n-i}. \tag{78}$$

Inserting Eqs. (76)–(78) into Eq. (75) gives

$$\mathbf{y}(x, \tau + nT) = \sum_{i=0}^{n} j_0^{n-i} \mathbf{q}(x, \tau + iT). \tag{79}$$

In view of Eq. (68) and $\mathbf{q}_I(x, \tau + iT)|_{i>0} = 0$, Eq. (77) can be rewritten as

$$\mathbf{y}(x, \tau + nT) = j_0^n \mathbf{q}_I(x, \tau) + \sum_{i=0}^{n} j_0^{n-i} [\mathbf{q}_B(x, \tau + iT) + \mathbf{q}_F(x, \tau + iT)]. \tag{80}$$

Eq. (80) is referred to as *the accumulative* formula. The right-hand terms of Eq. (80) are determined by Eq. (42) associated with Eqs. (50), (56) and (65).

It should be mentioned that formulas similar to the recurrence formula and the accumulative formula could be found in several other literatures [11–15]. Nevertheless, in all of these references, the formulas are expressed by means of d'Alembert's wave functions instead of the displacements, and are used only for dealing with certain problems.

4.2. Verifying the satisfaction of the governing equation

Eq. (69) has shown that the solution involves only three terms $\mathbf{q}_I(x, t)$, $\mathbf{q}_B(x, t)$ and $\mathbf{q}_F(x, t)$. Here, we shall prove that the boundary term $\mathbf{q}_B(x, t)$ and the initial term $\mathbf{q}_I(x, t)$ satisfy a homogeneous governing equation, and the forced term $\mathbf{q}_F(x, t)$ alone satisfies the non-homogeneous governing equation.

Considering that

$$\frac{\mathrm{d}}{\mathrm{d}z} \mathsf{H}(z) = \delta(z), \tag{81}$$

we can verify without difficulties that

$$\frac{1}{a} \frac{\partial}{\partial t} \begin{Bmatrix} -B_1(t - x/a) \\ B_2(t + x/a) \end{Bmatrix} = \frac{\partial}{\partial x} \begin{Bmatrix} B_1(t - x/a) \\ B_2(t + x/a) \end{Bmatrix}. \tag{82}$$

Premultiplying both sides of Eq. (82) by $\mathbf{I}_c(\mathbf{I}_c^{-1}\mathbf{A}\mathbf{I}_c)$, and considering that (see Eqs. (4) and (16b))

$$\mathbf{I}_c^{-1}\mathbf{A}\mathbf{I}_c = \begin{bmatrix} -a & 0 \\ 0 & a \end{bmatrix}, \tag{83}$$

we have

$$\frac{\partial}{\partial t} \mathbf{q}_B(x, t) = \mathbf{A} \frac{\partial}{\partial x} \mathbf{q}_B(x, t). \tag{84}$$

Similarly, from

$$\frac{1}{a}\frac{\partial}{\partial t}\left\{\begin{array}{c}-I_1(x-at)\\ I_2(x+at)\end{array}\right\}=\frac{\partial}{\partial x}\left\{\begin{array}{c}I_1(x-at)\\ I_2(x+at)\end{array}\right\}, \tag{85}$$

we may have

$$\frac{\partial}{\partial t}\mathbf{q}_I(x,t)=\mathbf{A}\frac{\partial}{\partial x}\mathbf{q}_I(x,t). \tag{86}$$

Eqs. (84) and (86) indicate that for all $t>0$ and $0\leqslant x\leqslant L$, $\mathbf{q}_B(x,t)$ and $\mathbf{q}_I(x,t)$ satisfy a homogeneous governing equation.

Furthermore, from Eqs. (55) and (56), we have

$$\frac{\partial}{\partial x}\left\{\begin{array}{c}f_1(x,t-x/a)\\ f_2(x,t+x/a)\end{array}\right\}=\frac{1}{a}\frac{\partial}{\partial t}\left\{\begin{array}{c}-f_1(x,t-x/a)\\ f_2(x,t+x/a)\end{array}\right\}+\left\{\begin{array}{c}f(x,t)-j_0f(x,t-2L/a)\\ j_0f(x,t-2L/a)-f(x,t)\end{array}\right\}. \tag{87}$$

Premultiplying both sides of Eq. (87) by $(1/2a)\mathbf{I}_c(\mathbf{I}_c^{-1}\mathbf{A}\mathbf{I}_c)$, and considering Eqs. (55) and (83), we have

$$\frac{\partial}{\partial t}\mathbf{q}_F(x,t)-\mathbf{A}\frac{\partial}{\partial x}\mathbf{q}_F(x,t)=\bar{\mathbf{f}}(x,t)-j_0\bar{\mathbf{f}}(x,t-T). \tag{88}$$

Recalling that $f(x,t)=0,\ t\leqslant 0$, we know that in Eq. (88), $\bar{\mathbf{f}}(x,t-T)=0$, provided $0<t\leqslant T$. Combining Eqs. (84) and (86), we find that, within $0<t\leqslant T$, Eq. (88) reduce to Eq. (74a), which means that the solution $\mathbf{y}(x,t)$ satisfies the governing equation in the first phase.

With Eq. (88) in mind, carrying out the calculation $\partial/\partial t$ Eq. (70)—$\mathbf{A}(\partial/\partial x)$ Eq. (70) yields

$$\frac{\partial}{\partial t}\mathbf{y}(x,t)-\mathbf{A}\frac{\partial}{\partial x}\mathbf{y}(x,t)=\bar{\mathbf{f}}(x,t)+j_0\left[\frac{\partial}{\partial t}\mathbf{y}(x,t-T)-\mathbf{A}\frac{\partial}{\partial x}\mathbf{y}(x,t-T)-\bar{\mathbf{f}}(x,t-T)\right]. \tag{89}$$

It can be concluded from Eq. (89) that for all $t>0$ and $0\leqslant x\leqslant L$, the solution expressed with Eq. (74b) could satisfy the governing equation, provided $\mathbf{y}(x,t-T)$ satisfies the governing Eq. (3). For the case of centralized force namely $f(x,t)=\delta(x-x_p)p(t)$, differentiating Eq. (56) with respect to t and x, we found Eq. (87) also holds true, and so does Eq. (89).

4.3. Verifying the satisfaction of the boundary conditions

In this section, we shall prove that $\mathbf{q}_F(x,t)$ and $\mathbf{q}_I(x,t)$ satisfy homogeneous boundary conditions, and $\mathbf{q}_B(x,t)$ alone satisfies the boundary conditions. Considering Eq. (44c) taking $x=0$ and L in Eq. (56), and then substituting the results into Eq. (42c) gives

$$\mathbf{q}_F(0,t)=\left\{\begin{array}{c}\dot{u}_F(0,t)\\ a\varepsilon_F(0,t)\end{array}\right\}=\frac{1}{2a}\left\{\begin{array}{c}(1-I_0)\\ (1+I_0)\end{array}\right\}F_{B0}(t,I_L),$$
$$\mathbf{q}_F(L,t)=\left\{\begin{array}{c}\dot{u}_F(L,t)\\ a\varepsilon_F(L,t)\end{array}\right\}=\frac{1}{2a}\left\{\begin{array}{c}(1-I_L)\\ -(1+I_L)\end{array}\right\}F_{BL}(t,I_0) \tag{90}$$

in which

$$F_{B0}(t,I_L)\equiv\int_0^L[f(\xi,t-\xi/a)-I_Lf(\xi,t-(2L-\xi)/a)]\,\mathrm{d}\xi,$$
$$F_{BL}(t,I_0)\equiv\int_0^L[f(\xi,t-(L-\xi)/a)-I_0f(\xi,t-(L+\xi)/a)]\,\mathrm{d}\xi. \tag{91}$$

Taking $x=0$ and L in Eq. (60) and inserting into Eqs. (42b) and (44b) yields

$$\mathbf{q}_I(0,t)=\begin{Bmatrix}\dot{u}_I(0,t)\\ a\varepsilon_I(0,t)\end{Bmatrix}=\frac{1}{2}\begin{Bmatrix}(1-I_0)\\ (1+I_0))\end{Bmatrix}I_{B0}(t),$$
$$\mathbf{q}_I(L,t)=\begin{Bmatrix}\dot{u}_I(L,t)\\ a\varepsilon_I(L,t)\end{Bmatrix}=\frac{1}{2}\begin{Bmatrix}(I_L-1)\\ (I_L+1)\end{Bmatrix}I_{BL}(t) \quad (92)$$

in which

$$I_{B0}(t)\equiv\begin{cases}I_{\dot{u}}(at)+aI_\varepsilon(at), & 0<t<L/a,\\ I_L[aI_\varepsilon(2L-at)-I_{\dot{u}}(2L-at)], & L/a<t<2L/a,\end{cases}$$
$$I_{BL}(t)\equiv\begin{cases}aI_\varepsilon(L-at)-I_{\dot{u}}(L-at), & 0<t<L/a,\\ I_0[aI_\varepsilon(at-L)+I_{\dot{u}}(at-L)], & L/a<t<2L/a.\end{cases} \quad (93)$$

Eqs. (90) and (92) imply that if $I_0=1$ then $\dot{u}_F(0,t)=\dot{u}_I(0,t)=0$, and if $I_0=-1$ then. The entire same, if $I_L=1$ then $\dot{u}_F(L,t)=\dot{u}_I(L,t)=0$, and if $I_L=-1$ then $\varepsilon_F(L,t)=\varepsilon_I(L,t)=0$. These mean that, the initial term and the forced term can satisfy a homogeneous boundary condition.

Now we consider the boundary term. Letting $x=0$ and L in Eq. (50), and then inserting the results into Eq. (42a), we have

$$\mathbf{q}_B(0,t)=\begin{Bmatrix}\dot{u}_B(0,t)\\ a\varepsilon_B(0,t)\end{Bmatrix}=\begin{Bmatrix}I_0\\ -I_0\end{Bmatrix}\varphi_0(t)+\begin{Bmatrix}(1-I_0)\\ (1+I_0)\end{Bmatrix}\varphi_L(t-L/a),$$
$$\mathbf{q}_B(L,t)=\begin{Bmatrix}\dot{u}_B(L,t)\\ a\varepsilon_B(L,t)\end{Bmatrix}=\begin{Bmatrix}1\\ 1\end{Bmatrix}\varphi_L(t)+\begin{Bmatrix}I_0(1-I_L)\\ -I_0(1+I_L)\end{Bmatrix}\varphi_0(t-L/a). \quad (94)$$

Eq. (94) implies that if we take boundary function as the velocity at the end $x=0$, then $I_0=1$, Eq. (94) results in

$$\dot{u}_B(0,t)=\varphi_0(t),\quad \dot{u}_I(0,t)=\dot{u}_F(0,t)=0. \quad (95)$$

At the same time, from Eqs. (90) and (92), we have known that $\dot{u}_I(0,t)=\dot{u}_F(0,t)=0$, so that if $0<t\leqslant T$:

$$\dot{u}(0,t)=\dot{u}_B(0,t)=\varphi_0(t). \quad (96)$$

Furthermore, from Eqs. (50), (69) to (72) we have

$$\begin{aligned}&\dot{u}(0,t)=\varphi_0(t), && 0<t\leqslant T,\\ &\dot{u}(0,t)=\varphi_0(t)+j_0(\dot{u}(0,t-T)-\varphi_0(t-T)), && t>T.\end{aligned} \quad (97a,b)$$

Eqs. (97a, b) imply that for all $t>0$, $\dot{u}(0,t)=\varphi_0(t)$. In addition, if $\varphi_0(t)$ is known on the boundary (see Table 1), then the solution satisfies the boundary condition at $x=0$. The same can be said about other cases, namely if $x=0$ and $I_0=-1$, then

$$\begin{aligned}&a\varepsilon(0,t)=\varphi_0(t), && 0<t\leqslant T,\\ &a\varepsilon(0,t)=\varphi_0(t)+j_0(a\varepsilon(0,t-T)-\varphi_0(t-T)), && t>T.\end{aligned} \quad (98)$$

If $x=L$ and $I_L=1$ then

$$\begin{aligned}&\dot{u}(L,t)=\varphi_L(t), && 0<t\leqslant T,\\ &\dot{u}(L,t)=\varphi_L(t)+I_0(\dot{u}(L,t-T)-\varphi_L(t-T)), && t>T.\end{aligned} \quad (99)$$

if $x=L$ and $I_L=-1$, then

$$\begin{aligned}&a\varepsilon(L,t)=\varphi_L(t), && 0<t\leqslant T,\\ &a\varepsilon(L,t)=\varphi_L(t)+I_0(-a\varepsilon(L,t-T)+\varphi_L(t-T)), && t>T.\end{aligned} \quad (100)$$

It is worth mentioning that the proof in this section does not depend on whether the boundary velocity (or strain) is prescribed or not. What we have proved is that on both of the ends, the boundary values of the solution are equal to a boundary function $\varphi_0(t)$ or $\varphi_L(t)$. Thus, if we have obtained the boundary functions, either from prescribed functions or through the solution of other equations on the boundaries (see Eqs. (11) and (12) and Case 3), we will ensure that the boundary conditions can then be satisfied.

4.4. Verifying the satisfaction of the initial condition

Here, we shall show that $\mathbf{q}_I(x,t)$ satisfies the initial condition, and that $\mathbf{q}_B(x,t)$ and $\mathbf{q}_F(x,t)$ satisfy zero initial condition. First, from Eq. (65) we have

$$\mathbf{q}_I(x,0)=\frac{1}{2}\mathbf{I}_c\left\{\begin{array}{c}\tilde{I}_1(x-at)\\ \tilde{I}_2(x+at)\end{array}\right\}_{t=0}=\frac{1}{2}\mathbf{I}_c\left\{\begin{array}{c}\tilde{I}_1(x)\\ \tilde{I}_2(x)\end{array}\right\}=\left\{\begin{array}{c}I_{\dot{u}}(x)\\ aI_\varepsilon(x)\end{array}\right\}. \tag{101}$$

Eq. (101) indicates that, for all $x\in[0,L]$, $\mathbf{q}_I(x,t)$ satisfies the initial conditions.
Next, letting $t=0$ in Eq. (50) gives

$$\mathbf{B}(x,0)=\left\{\begin{array}{c}I_0\varphi_0(-x/a)\mathrm{H}(-x/a)\\ \varphi_L(-(L-x)/a)\mathrm{H}(-(L-x)/a)\end{array}\right\}=\left\{\begin{array}{c}0\\ 0\end{array}\right\}. \tag{102}$$

Thus,

$$\mathbf{q}_B(x,0)=\mathbf{I}_c\mathbf{B}(x,0)=\mathbf{0},\quad 0\leqslant x\leqslant L. \tag{103}$$

Finally, taking $t=0$ in Eq. (56) yields

$$\left\{\begin{array}{c}f_1(x,t-x/a)\\ f_2(x,t+x/a)\end{array}\right\}_{t=0}=\int_0^x\left\{\begin{array}{c}f(\xi,-(x-\xi)/a)\\ 0\end{array}\right\}\mathrm{d}\xi+\int_x^L\left\{\begin{array}{c}0\\ f(\xi,-(\xi-x)/a)\end{array}\right\}\mathrm{d}\xi. \tag{104}$$

According to Eqs. (52) and (53), we have $f(\xi,-(x-\xi)/a)\overset{x\geq\xi}{=}0$ and $f(\xi,-(\xi-x)/a)\overset{x\leq\xi}{=}0$, so that

$$\int_0^x f(\xi,-(x-\xi)/a)\,\mathrm{d}\xi=0,\quad \int_x^L f(\xi,-(\xi-x)/a)\,\mathrm{d}\xi=0. \tag{105}$$

Thus,

$$\mathbf{q}_F(x,0)=\frac{1}{2a}\mathbf{I}_c\left\{\begin{array}{c}f_1(x,t-x/a)\\ f_2(x,t+x/a)\end{array}\right\}_{t=0}=0. \tag{106}$$

Eqs. (101), (103), and (106) indicate

$$\mathbf{y}(x,0)=\mathbf{q}_I(x,0). \tag{107}$$

Eq. (107) means that the solution satisfies the initial condition.

4.5. The relations between the velocity and strain

Eq. (74a,b) can be rewritten as

$$\left\{\begin{array}{c}\dot{u}(x,\tau+nT)\\ a\varepsilon(x,\tau+nT)\end{array}\right\}=\left\{\begin{array}{c}\dot{u}(x,\tau+(n-1)T)\\ a\varepsilon(x,\tau+(n-1)T)\end{array}\right\}+\mathbf{I}_c\left\{\begin{array}{c}\bar{q}_1(x,\tau+nT-x/a)\\ \bar{q}_2(x,\tau+nT+x/a)\end{array}\right\}, \tag{108}$$

where

$$\begin{Bmatrix} \bar{q}_1(x,\tau+nT-x/a) \\ \bar{q}_2(x,\tau+nT+x/a) \end{Bmatrix} \equiv \begin{Bmatrix} B_1(\tau+nT-x/a) \\ B_2(\tau+nT+x/a) \end{Bmatrix} + \frac{1}{2a}\begin{Bmatrix} I_1(x-a(\tau+nT)) \\ I_2(x+a(\tau+nT)) \end{Bmatrix} + \frac{1}{2a}\begin{Bmatrix} f_1(x,\tau+nT-x/a) \\ f_2(x,\tau+nT+x/a) \end{Bmatrix}. \tag{109}$$

From Eq. (108), it is easy to find that

$$\begin{aligned} \dot{u}(x,\tau+nT)+a\varepsilon(x,\tau+nT) &= \dot{u}(x,\tau+(n-1)T)+a\varepsilon(x,\tau+(n-1)T)+2\bar{q}_2(x,\tau+nT+L/a), \\ \dot{u}(x,\tau+nT)-a\varepsilon(x,\tau+nT) &= \dot{u}(x,\tau+(n-1)T)-a\varepsilon(x,\tau+(n-1)T)+2\bar{q}_1(x,\tau+nT+L/a). \end{aligned} \tag{110}$$

This means that the sum and the difference of the velocity and strain satisfy another kind of recurrence formula. On the other hand, from Eq. (109), the accumulative formula Eq. (80) can be rewritten as

$$\mathbf{y}(x,\tau+nT) = \mathbf{I}_c \begin{Bmatrix} F_1(x,\tau,n) \\ F_2(x,\tau,n) \end{Bmatrix}, \tag{111}$$

where

$$\begin{aligned} F_1(x,\tau,n) &\equiv \sum_{i=0}^{n} j_0^{n-i}\bar{q}_1(x,\tau+iT-x/a), \\ F_2(x,\tau,n) &\equiv \sum_{i=0}^{n} j_0^{n-i}\bar{q}_2(x,\tau+iT+x/a). \end{aligned} \tag{112a,b}$$

Thus, from Eqs. (4) and (16b), Eq. (111) can be written as

$$\begin{Bmatrix} \dot{u}(x,\tau+nT) \\ a\varepsilon(x,\tau+nT) \end{Bmatrix} = \begin{Bmatrix} F_1(x,\tau,n)+F_2(x,\tau,n) \\ -F_1(x,\tau,n)+F_2(x,\tau,n) \end{Bmatrix}. \tag{113}$$

From Eq. (113), it is easy to obtain the following linear algebraic relations between the velocity and strain

$$\begin{aligned} \dot{u}(x,\tau+nT)+a\varepsilon(x,\tau+nT) &= 2F_2(x,\tau,n), \\ \dot{u}(x,\tau+nT)-a\varepsilon(x,\tau+nT) &= 2F_1(x,\tau,n). \end{aligned} \tag{114}$$

In particular, on the boundary we have

$$\begin{aligned} \dot{u}(0,\tau+nT)+a\varepsilon(0,\tau+nT) &= 2F_2(0,\tau,n), \\ \dot{u}(0,\tau+nT)-a\varepsilon(0,\tau+nT) &= 2F_1(0,\tau,n) \end{aligned} \tag{115a,b}$$

and

$$\begin{aligned} \dot{u}(L,\tau+nT)+a\varepsilon(L,\tau+nT) &= 2F_2(L,\tau,n), \\ \dot{u}(L,\tau+nT)-a\varepsilon(L,\tau+nT) &= 2F_1(L,\tau,n). \end{aligned} \tag{116a,b}$$

Eqs. (115a,b) and (116a,b) are very important in dealing with complex boundary conditions (see Case 3), for they give the relations of the two variables in each phase. For example, if at $x = 0$ the boundary condition is given by such an equation as Eq. (11), we may choose one of the variables as the boundary function and eliminating the other with the help of Eqs. (115a,b) or (116a,b), so that Eq. (11) becomes an equation with only one unknown function. Solving the equation with one unknown either analytically or numerically, we may obtain the boundary function that can be taken as the known function on the boundary to get further solution of Eq. (2).

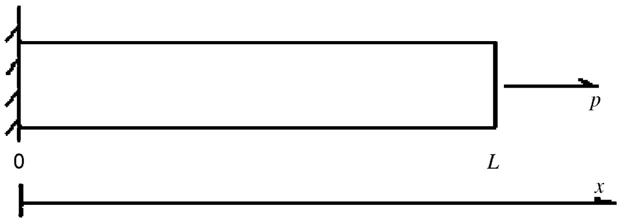

Fig. 1. Rod subjected by an external excitation (Case 1).

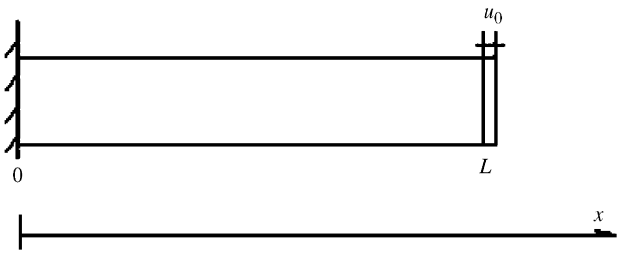

Fig. 2. Rod with initial strain (Case 2).

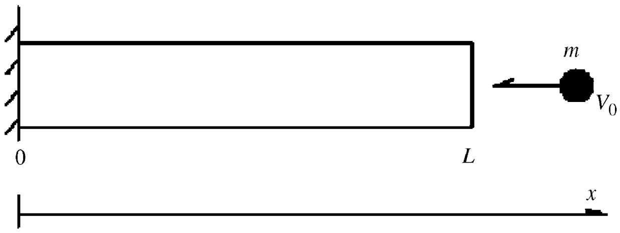

Fig. 3. Rod acted by an impact object with initial velocity v_0 (Case 3).

5. Examples

In this section, four examples are presented for using the proposed approach to solve some complicated problems (see Figs. 1–3). The first example describes a problem with Dirichlet boundary conditions. The second example deals with an initial strain. The third example is a complicated one that presents elastic waves due to an impact. The three examples could also be solved by using other methods that can be found in textbooks. In addition, the forth example is presented to show that, in some simple cases, the proposed method can arrive at the same analytical solution as by using the method of separation of variables.

5.1. Case 1. Simple boundary condition

In this case (see Fig. 1), the rod has zero initial condition and homogenous equation, one of the end is fixed and the other end subject to a prescribed force $p(t)$, which can be obtained from the derivative of the known displacement function as described by the traditional Dirichlet boundary conditions, so that

$$\mathbf{q}(x,t) = \mathbf{I}_c\mathbf{B}(x,t), \quad \mathbf{I}(x,t) = \mathbf{F}(x,t) = 0. \tag{117}$$

The boundary functions at $x = 0$ and $x = L$ are taken as the velocity and the strain, respectively. Accordingly,

$$I_0 = 1,\ I_L = -1,\ j_0 = I_0 I_L = -1. \tag{118}$$

Eq. (33) is therefore of the form

$$\mathbf{y}(x,t) = -\mathbf{y}(x,t-T)\mathsf{H}(t-T) + \mathbf{I}_c\mathbf{B}(x,t). \tag{119}$$

The boundary functions are

$$\varphi_0(t) = 0, \quad \varphi_L(t) = a\varepsilon(L,t) = ap(t)/EA, \tag{120}$$

where A is the cross-sectional area of the rod.

From Eq. (50),

$$\mathbf{B}(x,t) = \begin{Bmatrix} B_1(t-x/a) \\ B_2(t+x/a) \end{Bmatrix} = \begin{Bmatrix} -\varphi_L(t-(L+x)/a) \\ \varphi_L(t-(L-x)/a) \end{Bmatrix} = \frac{a}{EA}\begin{Bmatrix} -p(t-(L+x)/a) \\ p(t-(L-x)/a) \end{Bmatrix}. \tag{121}$$

With the recurrence formula, the solution can be expressed as

$$\begin{Bmatrix} \dot{u}(x,t) \\ a\varepsilon(x,t) \end{Bmatrix} = -\begin{Bmatrix} \dot{u}(x,t-T) \\ a\varepsilon(x,t-T) \end{Bmatrix}\mathsf{H}(t-T) + \frac{a}{EA}\begin{Bmatrix} -p(t-(L+x)/a) + p(t-(L-x)/a) \\ p(t-(L+x)/a) + p(t-(L-x)/a) \end{Bmatrix}. \tag{122}$$

With the accumulative formula Eq. (80), the solution is

$$\begin{Bmatrix} \dot{u}(x,\tau+nT) \\ a\varepsilon(x,\tau+nT) \end{Bmatrix} = \frac{aE}{A}\sum_{i=0}^{n}(-1)^{n-i}\begin{Bmatrix} -p(\tau+iT-(L+x)/a) + p(\tau+iT-(L-x)/a) \\ p(\tau+iT-(L+x)/a) + p(\tau+iT-(L-x)/a) \end{Bmatrix}. \tag{123}$$

5.2. Case 2. With initial strain

Let us consider the rod shown in Fig. 2. There is a constant initial strain $\varepsilon_0 = u_0/L$ throughout the length of the rod, and at time $t = 0$, the initial strain vanishes suddenly. Since the governing equation in this case is homogenous, we have $\mathbf{q}_F(x,t) = 0$. The boundary conditions are

$$\bar{\dot{u}}_0(t) = 0, \quad \bar{\varepsilon}_L(t) = 0, \quad t > 0. \tag{124}$$

The initial conditions are

$$I_{\dot{u}}(x) = 0, \quad I_\varepsilon(x) = \varepsilon_0 = u_0/L. \tag{125}$$

Considering that $\mathbf{q}_F(x,t) = 0$, Eqs. (69) and (70) reduces to

$$\mathbf{y}(x,t) = \begin{cases} \mathbf{q}_I(x,t) + \mathbf{q}_B(x,t), & 0 \leqslant t \leqslant T, \\ -\mathbf{y}(x,t-T)\mathsf{H}(t-T) + \mathbf{q}_B(x,t), & t > T. \end{cases} \tag{126}$$

In this case, the boundary functions are (see Eq. (124))

$$\begin{aligned} \varphi_0(t) &= \dot{u}(0,t) = 0, \quad I_0 = 1, \\ \varphi_L(t) &= \varepsilon(L,t) = 0, \quad I_L = -1. \end{aligned} \tag{127}$$

It follows that $\mathbf{q}_B(x,t) = 0$. From Eq. (20a)

$$\begin{Bmatrix} \tilde{I}_1(x) \\ \tilde{I}_2(x) \end{Bmatrix} = \begin{Bmatrix} -a\varepsilon_0 \\ a\varepsilon_0 \end{Bmatrix}. \tag{128}$$

Eq. (66) becomes

$$I_1(x-at) = \begin{cases} -a\varepsilon_0, & 0 < at \leqslant L+x \\ a\varepsilon_0, & L+x < at \leqslant 2L \end{cases}, \quad I_2(x+at) = \begin{cases} a\varepsilon_0, & 0 < at \leqslant L-x, \\ -a\varepsilon_0, & L-x < at \leqslant 2L. \end{cases} \tag{129}$$

Consequently, Eq. (126) reduces to

$$\mathbf{y}(x,t) = \mathbf{q}_I(x,t), \quad 0 \leqslant t \leqslant T. \tag{130}$$

The final results are

$$\begin{Bmatrix} \dot{u}(x,t) \\ a\varepsilon(x,t) \end{Bmatrix} = \mathbf{q}_I(x,t) = \frac{a\varepsilon_0}{2}\mathbf{I}_c \begin{Bmatrix} -\mathsf{G}(at,0,L+x) + \mathsf{G}(at,L+x,2L) \\ \mathsf{G}(at,0,L-x) - \mathsf{G}(at,L-x,2L) \end{Bmatrix}. \quad (131)$$

In particular, at both ends, when $0 \leqslant t \leqslant T$, we have

$$\dot{u}(0,t) = 0, \quad a\varepsilon(0,t) = a\varepsilon_0 \begin{cases} 1, & 0 \leqslant t \leqslant L/a, \\ -1, & L/a < t \leqslant 2L/a, \end{cases}$$
$$a\varepsilon(L,t) = 0, \quad \dot{u}(L,t) = -a\varepsilon_0. \quad (132)$$

When $t > T$, $\mathbf{q}_I(x,t)$ vanishes, Eq. (70) become

$$\mathbf{y}(x,t) = -\mathbf{y}(x, t-T), \quad t > T. \quad (133)$$

Specifying $L = 2\,\text{m}$, $a = 4000\,\text{m/s}$ and $\varepsilon_0 = 1/1000$, then $T = 2L/a = 1\,\text{s}/1000$. The velocities and the strains at different positions are shown in Figs. 4 and 5.

5.3. Case 3. Impact at one end

In this case (plotted in Fig. 3), an impact object with an initial velocity v_0 acts on the rod. During the impact, the object moves with the rod until $t = T_p$. The initial conditions in this case are

$$\varepsilon(x,0) = I_\varepsilon(x) = 0, \quad \dot{u}(x,0) = I_{\dot{u}}(x) = 0, \quad 0 \leqslant x \leqslant L. \quad (134)$$

One of the boundary conditions is $\dot{u}(0,t) = 0$, and the other is given by

$$\varepsilon(L,t) = \begin{cases} -m\ddot{u}(L,t)/EA, & 0 < t < T_p, \\ 0, & t \geqslant T_p, \end{cases} \quad \dot{u}(L,0) = v_0, \quad (135)$$

where v_0 is regarded as the initial value of the boundary velocity. m is the mass of the impact object.

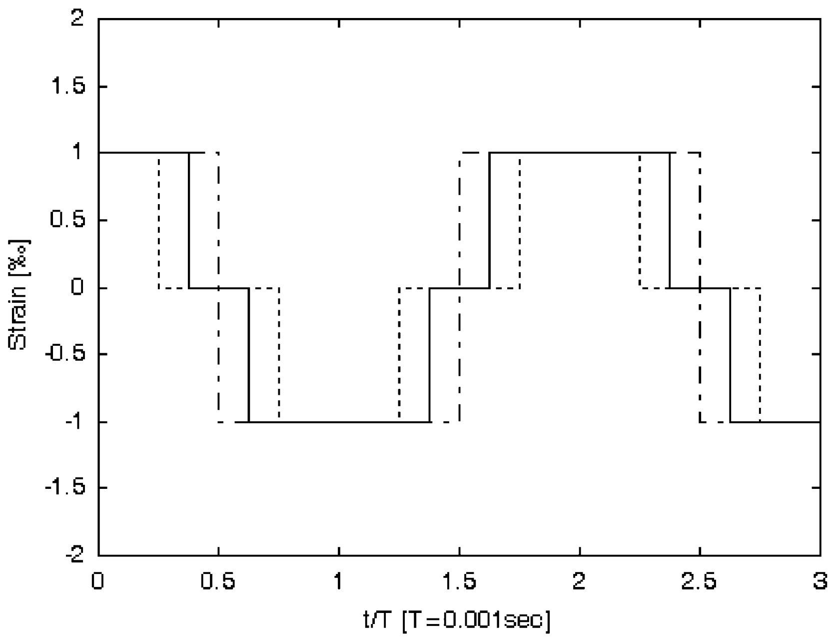

Fig. 4. The calculated strains at different positions. $-\cdot-\cdot-$ at $x = 0.0$, —— at $x = L/4$, - - - - - at $x = L/2$.

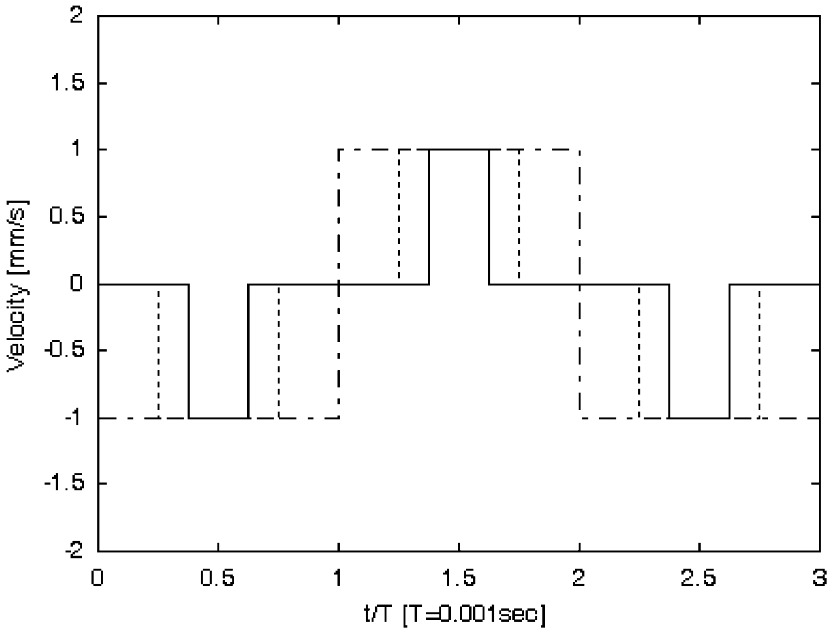

Fig. 5. The calculated velocities at different positions. – – – – – · at $x = 0.0$, —— at $x = L/4$, - - - - - at $x = L/2$.

In this case, the boundary functions are taken as the velocity at both ends:

$$\varphi_0(t) = \dot{u}(0,t) = 0, \quad \varphi_L(t) = \dot{u}(L,t),$$
$$I_0 = 1, \quad I_L = 1, \quad j_0 = 1. \tag{136}$$

Since the external force and the initial condition are both equal to zero:

$$\mathbf{q}_F(x, \tau + iT) = 0, \quad \mathbf{q}_I(x,\tau) = 0 \quad \text{for } 0 \leqslant \tau \leqslant T, \quad i = 0, 1, 2, \ldots . \tag{137}$$

Thus, the accumulative formula given by Eq. (80) becomes

$$\mathbf{y}(x, \tau + nT) = \sum_{i=0}^{n} \mathbf{q}_B(x, \tau + iT). \tag{138}$$

From Eq. (115b), considering that

$$\varphi_L(\tau + (i-1)T) \overset{i=0}{=} \varphi_L(\tau - T) \overset{\tau \leq T}{=} 0, \tag{139}$$

we have the relation

$$a\varepsilon(L, \tau + nT) = \dot{u}(L, \tau + nT) - 2F_1(\tau + nT), \tag{140}$$

where in the current case (see Eq. (112a))

$$F_1(L, \tau, n) = -\sum_{i=0}^{n-1} \dot{u}(L, \tau + iT). \tag{141}$$

Thus, during the impact, we have the following differential equation

$$\ddot{u}(L, \tau + nT) + \alpha\dot{u}(L, \tau + nT) = 2\alpha F_1(L, \tau, n), \quad n = 0, 1, \ldots \text{ and } 0 \leqslant \tau + nT \leqslant T_p, \tag{142}$$

where

$$\alpha \equiv \frac{EA}{am}. \tag{143}$$

Here we have, in practice, obtained a series of equations, of which the solutions in each phase are

$$\dot{u}(L,\tau+nT)=C_n\,\mathrm{e}^{-\alpha\tau}+2\alpha\,\mathrm{e}^{-\alpha\tau}\int_0^\tau F_1(L,\xi,n)\mathrm{e}^{\alpha\xi}\,\mathrm{d}\xi, \tag{144}$$

where C_n is a constant. In the first phase $n=0$, with the initial condition $\dot{u}(L,0)=v_0$, we have $C_0=-v_0$. When $n>0$, the following equation is used to determine C_n in order that the velocity might be continuous:

$$\dot{u}(L,\tau+nT)|_{\tau=0}=\dot{u}(L,\tau+(n-1)T)|_{\tau=T}. \tag{145}$$

Substituting Eq. (145) into Eq. (144) yields

$$C_n=\dot{u}(L,\tau+(n-1)T)|_{\tau=T}. \tag{146}$$

Here, we have shown that with the presented approach, the process to determine C_n is much simpler than that by using d'Alembert's solution. Since Eq. (145) is a general expression with respect to C_n with $n>0$, we can therefore write the entire solution of boundary velocity directly within the impact as

$$\dot{u}(L,\tau+nT)=\dot{u}(L,\tau+(n-1)T)|_{\tau=T}\,\mathrm{e}^{-\alpha\tau}-2\alpha\,\mathrm{e}^{-\alpha\tau}\int_0^\tau\sum_{i=0}^{n-1}\dot{u}(L,\tau+iT)\mathrm{e}^{\alpha\xi}\,\mathrm{d}\xi. \tag{147}$$

Specially, the results in the first three phases are

$$\begin{aligned}
&\dot{u}(L,\tau)=-v_0\,\mathrm{e}^{-\alpha\tau},\quad 0\leqslant\tau<T,\\
&\dot{u}(L,\tau+T)=-v_0\,\mathrm{e}^{-\alpha\tau}(\mathrm{e}^{-\alpha T}-2\alpha\tau),\quad T\leqslant\tau<2T,\\
&\dot{u}(L,\tau+2T)=-v_0\{\mathrm{e}^{-\alpha(\tau+T)}(\mathrm{e}^{-\alpha T}-2\alpha T)-2\,\mathrm{e}^{-\alpha\tau}[(\alpha\tau)(1+\mathrm{e}^{-\alpha T})-(\alpha\tau)^2]\},\quad 2T\leqslant\tau<3T.
\end{aligned} \tag{148}$$

Up to now, the boundary velocity has been a known function of time. The solution can be obtained directly with Eq. (148), as shown in Case 1, since the problem has been the same as those with simple boundary conditions.

5.4. Case 4. A simple example that can return to the classical solution

In this section, we consider a simple example that can be found in textbooks (e.g. Ref. [19]).

Example: A bar, fixed at both ends, has an initial displacement $u(x,0)=0$, and has an initial velocity $\dot{u}(x,0)=v_0\,\sin(n\pi x/L)$, in which v_0 is a constant and n is an integer. By using separation of the variables on in the interval $x\in[0,L]$, the analytical solution for displacement is given by [19]

$$u(x,t)=\frac{Lv_0}{n\pi a}\sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{n\pi at}{L}\right). \tag{149}$$

We now solve it with the proposed method. In this special case $\mathbf{F}=0$, and from Eq. (8) we have

$$\begin{aligned}
&I_0=I_L=1,\quad j_0=1,\quad \dot{u}(0,t)=\dot{u}(L,t)=0,\\
&\varphi_0(t)=\varphi_L(t)=0,\quad \mathbf{B}=0\quad \forall t\geqslant 0.
\end{aligned} \tag{150}$$

Thus, from Eq. (42a) and Eq. (42c) we have $\mathbf{q}_B(x,t)=0,\quad \mathbf{q}_F(x,t)=0$, then

$$\mathbf{q}(x,t)=\mathbf{q}_I(x,t)=\frac{1}{2a}\mathbf{I}_c\mathbf{I}(x,t). \tag{151}$$

Eq. (6a,b) becomes now

$$I_{\dot{u}}(x)=\dot{u}(x,0)=v_0\,\sin(n\pi x/L),\quad I_\varepsilon(x)=\varepsilon(x,0)=0. \tag{151$'$}$$

Substituting (151′) into Eq. (20a,b) and Eq. (66), we arrive at

$$\tilde{I}_1(x)=\tilde{I}_2(x)=v_0\,\sin(n\pi x/L) \tag{152}$$

and

$$I_1(x-at) = av_0 \sin\left(\frac{n\pi}{L}(x-at)\right),$$
$$I_2(x+at) = av_0 \sin\left(\frac{n\pi}{L}(x+at)\right). \tag{153}$$

From Eq. (65),

$$\mathbf{I}(x,t) = \left\{ \begin{array}{c} I_1(x-at) \\ I_2(x+at) \end{array} \right\} = I_1(x-at) = av_0 \left\{ \begin{array}{c} \sin\left(\frac{n\pi}{L}(x-at)\right) \\ \sin\left(\frac{n\pi}{L}(x+at)\right) \end{array} \right\}. \tag{154}$$

Considering Eq. (16b), Eq. (42b) becomes

$$\mathbf{q}(x,t) = \mathbf{q}_I(x,t) \equiv v\mathbf{I}_c\mathbf{I}(x,t) = v_0 \left\{ \begin{array}{c} \sin\left(\frac{n\pi}{L}(x-at)\right) + \sin\left(\frac{n\pi}{L}(x+at)\right) \\ -\sin\left(\frac{n\pi}{L}(x-at)\right) + \sin\left(\frac{n\pi}{L}(x+at)\right) \end{array} \right\}. \tag{155}$$

Seeing that $\mathbf{q}_I(x,t) = 0$ for $(t > 2T)$, Eqs. (74a,b) become

$$\mathbf{y}(x,\tau) = \mathbf{q}_I(x,\tau),$$
$$\mathbf{y}(x,\tau+nT) = \mathbf{y}(x,\tau+(n-1)T), \quad n = 1,2,\ldots. \tag{156}$$

From Eqs. (155) and (156), we conclude that

$$\dot{u}(x,t) = v_0 \sin\left(\frac{n\pi}{L}(x-at)\right) + v_0 \sin\left(\frac{n\pi}{L}(x+at)\right), \quad \forall t > 0. \tag{157}$$

The integral of the velocity over [0,t] results in the displacement

$$\begin{aligned} u(x,t) &= \int_0^t \dot{u}(x,t)\,\mathrm{d}t, \\ &\overset{u(x,0)=0}{=} -\frac{Lv_0}{n\pi a}\left(\cos\left(\frac{n\pi}{L}(x+at)\right) - \cos\left(\frac{n\pi}{L}(x-at)\right)\right) \\ &= \frac{Lv_0}{n\pi a}\sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{n\pi at}{L}\right). \end{aligned} \tag{158}$$

Eq. (158) presents the same results as Eq. (149).

6. Conclusions

In the paper, we have obtained an analytical solution of the one-dimensional wave equation governing the propagation of the longitudinal elastic waves in a rod in the time domain, and verified that the solution satisfies the governing equations the initial conditions and the boundary conditions. The method to perform the inverse Laplace transform is somewhat different from conventional approaches, which plays an important role in obtaining the solution. According to the previous discussions, it can be concluded that the presented solution has the following four major advantages.

First, it is a unified solution. The solution consists of four terms, namely the historical term, the forced term, the initial term and the boundary term. It is important that the meaning of the four terms is physically clear. Unlike the method of separation of variables, d'Alembert's solution, and Green's function, there are no needs to carry out any transformation for the dependent variables. Moreover, in contrast to d'Alembert's solution, although the proposed solution has a form similar to the wave functions, we deal with the velocity or strain directly instead of the indirect d'Alembert's wave functions.

Second, the solution is in a final form expressed with the recurrence formula and accumulative formula. One can use one of the two formulas to obtain the solution directly without any additional skillful mathematic derivation needed. More precisely, at any time instant the solution consists of finite terms, which implies that to obtain an exact solution of transient problem, only a few terms need to be calculated. That is to say, no truncation error occurs, as any series solution would encounter. Furthermore, the solution in a final form makes the computer programming to calculate the systems much simpler.

Third, the unified solution makes it possible to solve the problems with complex boundary conditions directly; such as at one end the velocity and strain are restricted with an equation. This is achieved by the two boundary functions that are introduced to express the boundary values of the dependent variables at both ends. Based on this, the form of the solution is independent of what the boundary functions represent (velocities or strains), even regardless of whether they are prescribed or not. Since some additional relations between the boundary values have been established in this paper, we have enough equations to determine the boundary functions.

Fourth, the final solution paves the way for the analysis of complex structural systems directly in the time domain. In such complex structural systems, e.g. the system consisting of several uniform rods, or a pipe carrying fluid, a rod could be regarded as a component or an element (in the sense of the finite element method). In these cases, the proposed solution provides an exact interpolation function, since the velocity and strain within the rod are expressed by the boundary functions.

Although the present unified solution is merely for such rod with uniform cross section, it could also be used to certain kind of non-uniform rods for they can be reduced to the same equation through transform of variables (see e.g. Ref. [18]). It is more important that the approach proposed in the present paper could also be used for other problems of which the governing equation is one-dimension wave equation without any difficulties.

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