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A modified Mickens iteration procedure for nonlinear oscillators

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Abstract

Based on the Mickens procedure, a new iteration scheme is proposed for nonlinear oscillators. Different from the existing Mickens procedures, in the proposed procedure, the algebraic equation governing the frequency of oscillation deduced at each iteration stage is always linear. Taking the Duffing equation as an illustrative example, excellent approximations can be easily obtained. Furthermore, the proposed procedure is extended to obtain the limit cycle of the van der Pol equation. Additionally, a computational disadvantage of the Mickens procedure is eliminated.

1. Introduction

Firstly, consider a nonlinear conservative oscillator described as

$$x'' + f(x) = 0, \quad x(0) = A, \quad x'(0) = 0 \tag{1}$$

where the superscript denotes the differentiation with respect to time t, A is a given constant, $f(x)$ is an odd function and its derivative near $x = 0$ is positive. Eq. (1) can be rewritten as

$$x'' + \omega^2 x = \omega^2 x - f(x) := g(\omega, x) \tag{2}$$

where ω is a priori unknown frequency of the periodic solution $x(t)$ being sought. The original Mickens procedure is given as [1]

$$x_k'' + \omega^2 x_k = g(\omega, x_{k-1}), \quad k = 1, 2, \ldots \tag{3}$$

where the input of starting function is

$$x_0(t) = A\cos(\omega t) \tag{4}$$

This iteration scheme was used to solve many nonlinear oscillating equations [2–4]. Lim et al. [5] proposed a modified iteration scheme

$$x_{k+1}'' + \omega^2 x_{k+1} = g(\omega, x_{k-1}) + g_x(\omega, x_{k-1})(x_k - x_{k-1}), \quad k = 0, 1, 2, \ldots \tag{5}$$

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with the inputs of starting functions as

$$x_{-1}(t) = x_0(t) = A\cos(\omega t) \tag{6}$$

where $g_x(\omega, x) = \partial g(\omega, x)/\partial x$. The modified procedure was also applied to solve many nonlinear oscillators [6–11]. Later, Marinca et al. [12] proposed a new iteration method by combining Mickens' and He's iteration methods. These iteration procedures have been used to solve both nonlinear conservative and nonconservative oscillators. For most conservative oscillators, like some other methods for nonlinear oscillators [13–15], the second-order (even the first-order) approximations can give uniformly accurate solutions.

In all the aforementioned papers about the Mickens procedure, the second-order approximations were obtained at most. In principle, the approximations can be obtained to any desired order. However, as the iteration proceeds, more and more complicated nonlinear algebraic equations in ω have to be solved. Thus, it is necessary to propose some techniques to simplify the implementation of the Mickens iteration procedure (e.g., without solving nonlinear algebraic equation).

2. A new iteration scheme

At the kth iteration stage of Eq. (3), the frequency ω (considered as ω_{k-1}) is calculated anew by demanding that the right hand side contains no terms giving rise to secular terms. But, the Fourier components of x_k are considered as functions of ω_k being determined at the next iteration stage. As a result, the equation in ω_k deduced at the $(k+1)$th iteration stage becomes more and more complicated as k increases. This drawback also exists in Lim's modified iteration scheme (5).

In order to simplify the attained equation in ω_k, we try to make the Fourier components of x_k independent of ω_k. To this end, a new iteration scheme is proposed as

$$x_k'' + \omega_{k-1}^2 x_k = g(\omega_{k-1}, x_{k-1}), \quad k = 1, 2, \ldots \tag{7}$$

subject to initial conditions

$$x_k(0) = A, \quad x_k'(0) = 0 \tag{8}$$

where the input of starting function is

$$x_0(t) = A\cos(\omega_0 t) \tag{9}$$

ω_0 is the zeroth-order approximation for ω. It is obvious that as long as the sequences $\{\omega_k, k = 0, 1, 2, \ldots\}$ and $\{x_k, k = 0, 1, 2, \ldots\}$ are convergent, they must converge to the exact solutions of Eq. (1). For each iteration stage, the right-hand side of Eq. (7) can be expanded in the Fourier series

$$g(\omega_{k-1}, x_{k-1}) = \sum_{i=1}^{\varphi(k)} a_{k-1,i}(\omega_{k-1})\cos(\mathrm{i}\omega_{k-1} t) \tag{10}$$

where the coefficients $a_{k-1,i}$ are functions of ω_{k-1} and $\varphi(k)$ is a positive integer. The $(k-1)$th-order approximation ω_{k-1} is obtained by eliminating the so-called secular terms, i.e., letting

$$a_{k-1,1}(\omega_{k-1}) = 0, \quad k = 1, 2, \ldots \tag{11}$$

Different from the regular Mickens procedure [1–12], no matter what the integer k is, Eq. (11) is always a linear algebraic equation in ω_{k-1}^2 (considered as an independent unknown). It simplifies the Mickens procedure significantly, as shown later.

3. The Duffing equation

The details as to how to carry out the proposed iteration scheme are illustrated in the following example. The Duffing equation can be described as

$$x'' + x + \varepsilon x^3 = 0, \quad x(0) = A, \quad x'(0) = 0 \tag{12}$$

where the coefficient $\varepsilon>0$ needs not be small. Using iteration procedure (7), we have

$$x_k'' + \omega_{k-1}^2 x_k = \omega_{k-1}^2 x_{k-1} - x_{k-1} - \varepsilon x_{k-1}^3, \quad k = 1, 2, \ldots \tag{13}$$

When $k = 1$, substituting Eq. (9) into Eq. (13), we obtain

$$x_1'' + \omega_0^2 x_1 = \left(\omega_0^2 - 1 - \frac{3}{4}\varepsilon A^2\right) A\cos(\omega_0 t) - \frac{\varepsilon A^3}{4}\cos(3\omega_0 t) \tag{14}$$

Equating the coefficient of $\cos(\omega_0 t)$ to zero yields

$$\omega_0^2 = 1 + \frac{3}{4}\varepsilon A^2 \tag{15}$$

Thus, the equation governing x_1 becomes

$$x_1'' + \omega_0^2 x_1 = -\frac{\varepsilon A^3}{4}\cos(3\omega_0 t) \tag{16}$$

Considering the initial conditions (8), then $x_1(t)$ can be obtained as

$$\begin{aligned} x_1(t) &= \left(A - \frac{\varepsilon A^3}{32\omega_0^2}\right)\cos(\omega_0 t) + \frac{\varepsilon A^3}{32\omega_0^2}\cos(3\omega_0 t) \\ &:= \left(A - \frac{\varepsilon A^3}{32\omega_0^2}\right)\cos(\omega_1 t) + \frac{\varepsilon A^3}{32\omega_0^2}\cos(3\omega_1 t) \end{aligned} \tag{17}$$

Write $c_{1,1} = A - (\varepsilon A^3/32\omega_0^2)$ and $c_{1,3} = \varepsilon A^3/32\omega_0^2$ for simply. Note that $c_{1,1}$ and $c_{1,3}$ are independent of ω_1. Substituting Eq. (17) into Eq. (13), we obtain

$$x_2'' + \omega_1^2 x_2 = \sum_{i=1}^{9} a_{1,k}\cos(\mathrm{i}\omega_1 t) \tag{18}$$

where the coefficients are listed as follows:

$$\begin{aligned} a_{1,1} &= c_{1,1}(\omega_1^2 - 1) - \frac{3\varepsilon}{4}c_{1,1}^3 - \frac{3\varepsilon}{4}c_{1,1}^2 c_{1,3} - \frac{3\varepsilon}{2}c_{1,1}c_{1,3}^2 \\ a_{1,3} &= -\frac{3\varepsilon}{4}c_{1,1}^2 c_{1,3} - \frac{\varepsilon}{4}c_{1,1}^3 - \frac{3\varepsilon}{4}c_{1,3}^3 + c_{1,3}(\omega_1^2 - 1), \\ c_{1,5} &= -\frac{3\varepsilon}{4}c_{1,1}c_{1,3}(c_{1,1} + c_{1,3}), \\ c_{1,7} &= -\frac{3\varepsilon}{4}c_{1,1}c_{1,3}^2, \quad c_{1,9} = -\frac{\varepsilon}{4}c_{1,3}^3, \quad \text{and} \\ a_{1,i} &= 0, \quad i = 2, 4, 6, 8. \end{aligned}$$

Because $c_{1,1}$ and $c_{1,3}$ are independent of ω_1, the equation $a_{1,1}(\omega_1^2) = 0$ in ω_1^2 is linear. Hence, it is easy to obtain

$$\begin{aligned} \omega_1^2 &= 1 + \frac{3\varepsilon}{4}c_{1,1}^2 + \frac{3\varepsilon}{4}c_{1,1}c_{1,3} + \frac{3\varepsilon}{2}c_{1,3}^2 \\ &= \frac{831\varepsilon^3 A^6 + 3408\varepsilon^2 A^4 + 4608\varepsilon A^2 + 2048}{128(3\varepsilon A^2 + 4)^2} \end{aligned} \tag{19}$$

In both iteration schemes (3) and (5), the counterparts of $c_{1,1}$ and $c_{1,3}$ are updated as $c_{1,1} = A - (\varepsilon A^3/32\omega_1^2)$ and $c_{1,3} = \varepsilon A^3/32\omega_1^2$, respectively. As a consequence, the equation in ω_1^2 is nonlinear. Of the three solutions of ω_1^2, the one that is closest to ω_0^2 is chosen.

The exact frequency of the periodic motion of Eq. (12) is given by [16]

$$\omega_e = \frac{\pi\sqrt{1+\varepsilon A^2}}{2}\left(\int_0^{\pi/2}\frac{\mathrm{d}\theta}{\sqrt{1 - m\sin^2\theta}}\right)^{-1}, \quad m = \frac{\varepsilon A^2}{2(1+\varepsilon A^2)} \tag{20}$$

According to Eqs. (15) and (19), we have

$$\lim_{\varepsilon A^2\to+\infty}\frac{\omega_0}{\omega_e}=1.0222,\qquad \lim_{\varepsilon A^2\to+\infty}\frac{\omega_1}{\omega_e}=1.0025 \tag{21}$$

The relative errors are 2.22% and 0.25%, respectively. Thus, both the zeroth and first-order approximations are uniformly accurate for all values of εA^2.

According to Eq. (18), x_2 can be obtained as

$$x_2=\sum_{i=1}^{9}c_{2,i}\cos(\mathrm{i}\omega_1 t) \tag{22}$$

where $c_{2,i}=a_{1,i}/\omega_1^2(1-i^2)$, $i=2,3,\ldots,9$ and $c_{2,1}=A-\sum_{i=2}^{9}c_{2,i}$. The second-order approximation ω_2 can be obtained as (neglecting the trivial coefficients $c_{2,i}$, $i=2,4,6,8$)

$$\begin{aligned}\omega_2^2=1+\frac{4\varepsilon}{3c_{21}}&(2c_{21}c_{25}c_{29}+2c_{21}c_{23}c_{27}+2c_{23}c_{25}c_{27}\\&+2c_{23}c_{27}c_{29}+c_{23}^2c_{25}+c_{21}^2c_{25}+c_{25}^2c_{29}+c_{23}^2c_{29}\\&+c_{21}^3+2c_{21}c_{23}^2+2c_{21}c_{25}^2+2c_{21}c_{27}^2+2c_{21}c_{29}^2)\end{aligned} \tag{23}$$

Using symbolic calculation, we can obtain

$$\lim_{\varepsilon A^2\to+\infty}\frac{\omega_2}{\omega_e}=1.00025 \tag{24}$$

The relative error is only 0.025%. The accuracy can be improved by an order of magnitude for every iteration stage, which means the convergence is excellent.

Lim's first-order approximation for ω (counterpart of ω_1) is [5]

$$\omega_1^L=\frac{\sqrt{32+25\varepsilon A^2+\sqrt{1024+1472\varepsilon A^2+433\varepsilon^2A^4}}}{8} \tag{25}$$

thus

$$\lim_{\varepsilon A^2\to+\infty}\frac{\omega_1^L}{\omega_e}=0.998596 \tag{26}$$

The relative error is -0.14%. Lim's first-order approximation is relatively more accurate than ω_1 but less accurate than ω_2. Importantly, our results can be easily obtained by solving linear algebraic equations. Thus, the proposed procedure can be carried out to any desired order without any additional difficulty. As shown in Table 1, the higher-order approximations are extremely accurate.

4. The van der Pol equation

In this section, we extend the proposed procedure to obtain limit cycle of self-excited oscillator exemplified by the famous van der Pol equation

$$x''+x+\varepsilon(x^2-1)x'=0 \tag{27}$$

Table 1
Comparison of the frequency solutions obtained by the proposed procedure with Lim's first-order approximations and the numerical ones

εA^2	ω_0	ω_2	ω_4	ω_9	ω_e	ω_1^L
0.1	1.03682207	1.03671691	1.03671691	1.03671691	1.03671691	1.03671659
1	1.32287566	1.31778315	1.31777607	1.31777606	1.31777606	1.31766980
10	2.91547595	2.86703623	2.86664253	2.86664014	2.86664014	2.86408175
100	8.71779789	8.53558685	8.53360297	8.53358619	8.53358619	8.52219663
10,000	86.6083137	84.7486265	84.7274967	84.7274799	84.7274799	84.6088325

where ε is a constant needing not be small or positive. Eq. (27) possesses one stable limit cycle when $\varepsilon>0$ while an unstable one when $\varepsilon<0$. Denote the frequency of the limit cycle as ω, and rewrite Eq. (27) as

$$x'' + \omega^2 x = \omega^2 x - x - \varepsilon(x^2 - 1)x' := h(\omega, x, x') \tag{28}$$

Because the limit cycle is independent of initial conditions, we can introduce such simple initial conditions as

$$x(0) = \alpha, \quad x'(0) = 0 \tag{29}$$

where α is the amplitude of the limit cycle to be determined. Similar to Eq. (7), we construct an iteration scheme

$$x_k'' + \omega_{k-1}^2 x_k = h(\omega_{k-1}, x_{k-1}, x_{k-1}'), \quad k = 1, 2, \ldots \tag{30}$$

subject to initial conditions as

$$x_k(0) = \alpha_k, \quad x_k'(0) = 0 \tag{31}$$

where the input of starting function is

$$x_0 = \alpha_0 \cos(\omega_0 t) \tag{32}$$

where α_{k-1}, ω_{k-1}, $k = 1, 2, \ldots$ are to be determined at the kth iteration stage.

Different from Eqs. (7) and (8), x_{k-1} contains the unknown α_{k-1}. At each iteration stage, the right-hand side of Eq. (30) can be expressed as

$$h(\omega_{k-1}, x_{k-1}, x_{k-1}') = \sum_{i=1}^{\varphi(k)} [c_{k-1,i}(\omega_{k-1}, \alpha_{k-1}) \cos(\mathrm{i}\omega_{k-1} t) + s_{k-1,i}(\omega_{k-1}, \alpha_{k-1}) \sin(\mathrm{i}\omega_{k-1} t)] \tag{33}$$

where the coefficients $c_{k-1,i}$ and $s_{k-1,i}$ are functions of ω_{k-1} and α_{k-1}, $\varphi(k)$ is a positive integer. The $(k-1)$ th-order approximations ω_{k-1} and α_{k-1} are obtained by solving

$$c_{k-1,1}(\omega_{k-1}, \alpha_{k-1}) = 0, \quad s_{k-1,1}(\omega_{k-1}, \alpha_{k-1}) = 0, \quad k = 1, 2, \ldots \tag{34}$$

Eq. (30) contains the first-order derivative (i.e., x'_{k-1}), which causes ω_{k-1}^2 can no longer be considered as an independent unknown; thus, Eq. (34) is in general a set of nonlinear algebraic equations. Fortunately, they do not become more complex as k increases. Several pairs of real roots for ω_{k-1} and α_{k-1} maybe exist. The pair in which ω_{k-1} is closest to the corresponding value (i.e., ω_{k-2}) attained at the prior iteration stage is chosen. As Fig. 1 shows, very accurate approximation can be obtained by only five iteration stages. Fig. 2 shows the phase planes given by the several approximations x_k or numerical solution. From it, we can intuitively see the rapid

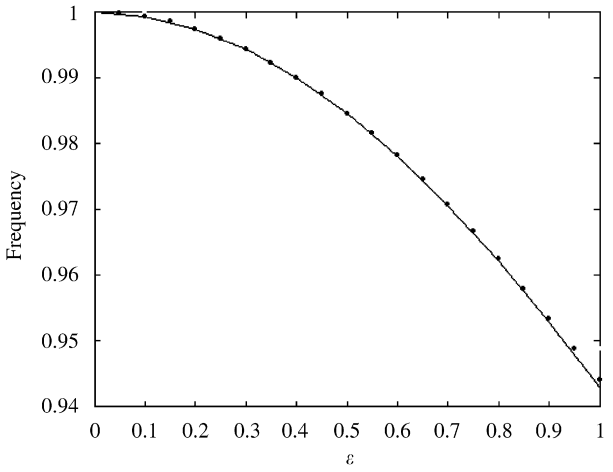

Fig. 1. Comparison of the fourth-order approximations (ω_4) for frequency of the limit cycle of the van der Pol equation obtained by iteration (30) with the numerical solutions, where dots denote ω_4 and solid line denotes numerical solutions.

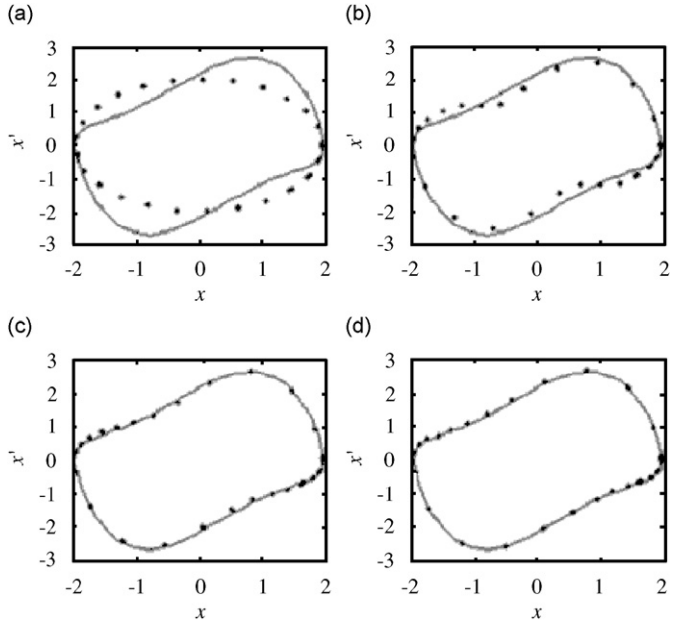

Fig. 2. The convergence of the approximations to the numerical solution of the limit cycle of the van der Pol equation with $\varepsilon = 1$. Dots are given by x_{k-1} obtained by Eq. (30) with (a) $k = 1$; (b) $k = 3$; (c) $k = 5$; (d) $k = 7$, and solid lines represent the numerical solution.

convergence of the approximations to the numerical solution as the iteration proceeds. It is worth noting that this proposed procedure is also effective for Eq. (27) with negative ε. We should also note that when $|\varepsilon|$ increases, say more than 1, iteration scheme (30) does not converge. This is probably because the difference between the input of the starting function and the exact solution is too large.

5. Relationship to the harmonic balance method

In general, the highest harmonic of the kth-order approximation $x_k(\tau)$ increases with k. For many periodic oscillations, the amplitudes of higher harmonics are just small quantities compared with those of lower harmonics. Thus, it is often enough to obtain high accurate solutions by retaining only several lower harmonics. For both the Duffing and the van der Pol equations, there are $\varphi(k) = 3^k$ harmonics in $g(\omega_{k-1}, x_{k-1})$ and $h(\omega_{k-1}, x_{k-1}, x'_{k-1})$, respectively. Accordingly, the number of harmonics in x_k exponentially increases with k. This causes a big computational disadvantage for seeking higher-order approximations. In fact, this problem exists in some other Mickens procedures too [1–12]. In this section, a technique is proposed to eliminate this shortcoming. Further, it reveals the relationship between the presented technique and the harmonic balance (HB) method.

Without loss of generality, we consider the Duffing equation firstly. Giving a positive integer N, when $\varphi(k) > N$, neglecting all the harmonics higher than Nth in $g(\omega_{k-1}, x_{k-1})$, and we denote summation of the first N harmonics in $g(\omega_{k-1}, x_{k-1})$ as

$$g_N(\omega_{k-1}, x_{k-1}) = \sum_{n=1}^{N} a_{k-1,i} \cos(\mathrm{i}\omega_{k-1} t) \tag{35}$$

Substituting Eq. (35) into the right-hand side of Eq. (7), using the same procedures described above, the approximations with N harmonics can be obtained. Importantly, when k is relatively large, say $\varphi(k) > N$, the computational effort for every iteration stage remains the same.

Interestingly, the approximation $x_k(t)$ containing N harmonics obtained by the presented technique converges to HB solution. The proof is given in the following.

Denote the approximation $x_k(t)$ containing N harmonics as $y_k(t)$, that

$$\lim_{k\to+\infty}(y_k - y_{k-1}) = 0 \tag{36}$$

According to the definitions of y_k and $g_N(\omega_{k-1}, y_{k-1})$, we have

$$y_k'' + \omega_{k-1}^2 y_k = g_N(\omega_{k-1}, y_{k-1}) \tag{37}$$

Let

$$R_N(\omega_{k-1}, y_{k-1}, y_k) = y_k'' + \omega_{k-1}^2 y_k - g(\omega_{k-1}, y_{k-1}) := \sum_{i=1}^{\phi(N)} C_{k,i}\cos(\mathrm{i}\omega_{k-1}t) \tag{38}$$

where $\phi(N)$ is an integer dependent of N. For the Duffing equation, $\phi(N) = 3N$. Considering Eq. (37), we have $C_{k,i} = 0$ for $1 \leqslant i \leqslant N$. Additionally, according to Eq. (2), R_N can be rewritten as

$$\begin{aligned} R_N(\omega_{k-1}, y_{k-1}, y_k) &= y_k'' + \omega_{k-1}^2(y_k - y_{k-1}) + f(y_{k-1}) \\ &= (y_k'' - y_{k-1}'') + \omega_{k-1}^2(y_k - y_{k-1}) + y_{k-1}'' + f(y_{k-1}) \end{aligned} \tag{39}$$

That $\{y_k,\ k = 0, 1, 2, \ldots\}$ converges means $\lim_{k\to\infty}(y_k - y_{k-1}) = 0$. Letting $k \to +\infty$, we can obtain

$$\begin{aligned} \lim_{k\to+\infty}[y_{k-1}'' + f(y_{k-1})] &:= \sum_{i=1}^{\phi(N)}(\lim_{k\to+\infty} D_{k,i})\cos(\mathrm{i}\omega_{k-1}t) \\ &= \lim_{k\to+\infty}[R_N(\omega_{k-1}, y_{k-1}, y_k)] - \lim_{k\to\infty}(y_k'' - y_{k-1}'') - \lim_{k\to\infty}(y_k - y_{k-1}) \\ &= \sum_{i=1}^{\phi(N)}(\lim_{k\to+\infty} C_{k,i})\cos(\mathrm{i}\omega_{k-1}t) = \sum_{i=N+1}^{\phi(N)}(\lim_{k\to+\infty} C_{k,i})\cos(\mathrm{i}\omega_{k-1}t) \end{aligned} \tag{40}$$

According to the HB method, obviously, as long as the sequence $\{y_k(t)\}$ converges it must converge to one HB solution of Eq. (1).

Table 2 shows the comparison of the approximations obtained by retaining three harmonics (i.e., $N = 3$) and the HB3 results. Rapid convergence of the attained approximations to the HB results can be observed. Additionally, several Fourier components $D_{k,i}$ in the right-hand side of Eq. (40) are plotted in Fig. 3. We can see that $D_{k,i}$ converge to 0 when $i = 5, 7$ (corresponding to $N = 5, 7$) while to constants when $i = 7, 9$ (corresponding to $N = 5, 7$). Similar to Eq. (35), neglecting the higher harmonics in $h(\omega_{k-1}, x_{k-1}, x_{k-1}')$, approximate solution with N harmonics for the limit cycle of the van der Pol equation can also be obtained. With $N = 5$ or 7, the Fourier components of the residues of the van der Pol equation are shown in Fig. 4. Figs. 3 and 4 imply that the convergent solutions obtained by the proposed technique comply with the harmonic balancing principle.

6. Conclusions

A modified Mickens iteration procedure for certain nonlinear conservative oscillators is proposed. Different form the existing procedures, in the presented procedure, no nonlinear algebraic needs be solved. This makes the implementation of the presented procedure rather simple, and hence higher-order approximation can be

Table 2
Comparison of the frequency solutions obtained with $N = 3$ with the HB3 results

$\varepsilon = 1,\ A = 10$	ω_{k-1}	$c_{k-1,1}$	$c_{k-1,3}$
$k = 1$	8.71779788708135	10	0
$k = 3$	8.54968068501462	9.57718396163524	0.42281603836476
$k = 5$	8.54961530209536	9.57700454402057	0.42299545597943
$k = 10$	8.54961528773423	9.57700450461056	0.42299549538944
HB3 solution	8.54961528773423	9.57700450461056	0.42299549538944

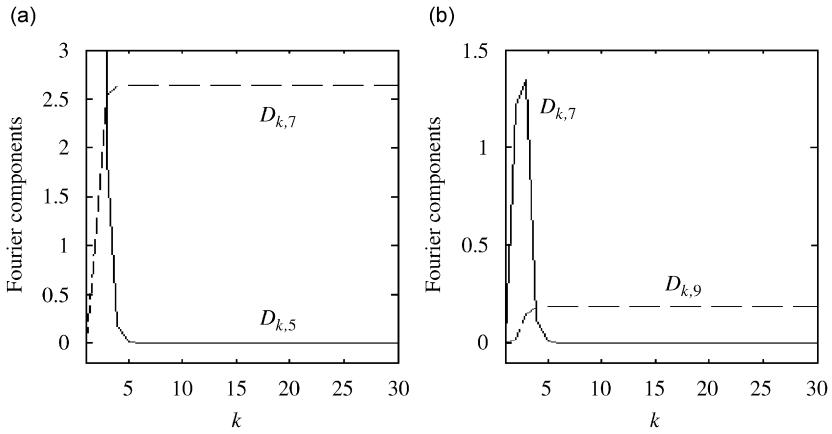

Fig. 3. Fourier components of the residues of the Duffing equation versus k with $N = 5$ for (a) and $N = 7$ for (b), where $\varepsilon = 1$ and $A = 10$.

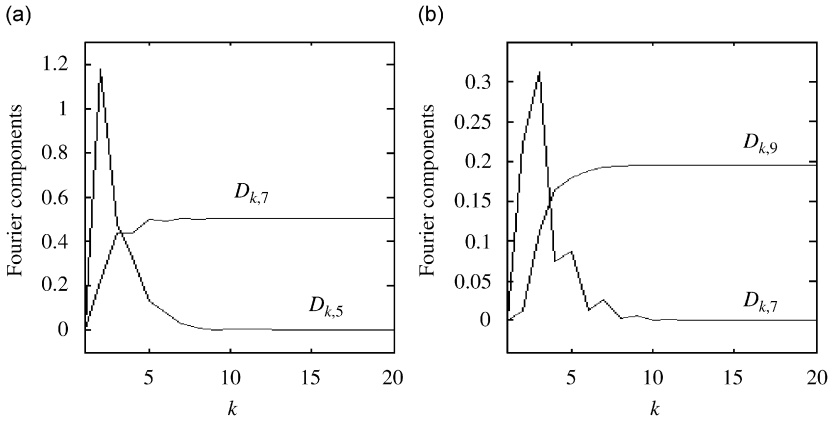

Fig. 4. Fourier components of the residues of the van der Pol equation versus k with $N = 5$ for (a) and $N = 7$ for (b), where $\varepsilon = 1$.

obtained to any desired accuracy. Additionally, the present procedure is extended to obtain the limit cycle of the van der Pol equation.

In the Mickens procedure, a computational disadvantage is that the number of harmonics increases rapidly. For this issue, a technique by truncating the harmonics in the higher-order approximations is suggested. Interestingly, the approximations obtained by this technique converge to HB solutions.

Acknowledgments

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