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# A novel approach for nonlinearity detection in vibrating systems

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**Abstract**

This paper proposes a novel approach for nonlinearity detection in vibrating systems. The approach is developed based on a new concept recently proposed by the author known as nonlinear output frequency response functions (NOFRFs) and the properties of the NOFRFs for nonlinear systems with multiple degrees of freedom (mdof). The results of numerical simulation studies verify the effectiveness of the approach. Nonlinear components often represent faults in practical mdof systems including beams. The proposed approach therefore has significant potential in the fault diagnosis of practical mdof engineering systems and structures.


## 1. Introduction

In engineering practice, many mechanical and structural systems require more than one set of coordinates to describe the systems' behaviours and, consequently, need a multi-degree-of-freedom (mdof) model to represent the systems. mdof systems can behave nonlinearly simply due to the nonlinear characteristics of one component within the systems. In many practical cases, such a nonlinear component may represent a fault. Typical examples of such a case are beams, as it is well known that beams can behave nonlinearly due to the presence of internal breathing cracks [1,2]. Therefore, locating nonlinear components in mdof systems has considerable significance in fault diagnosis for a wide range of mdof engineering systems and structures, which can, like beams, behave nonlinearly due to the existence of faults.

The detection of faults in mdof structures has been studied by Zhu and Wu [3] where the structure with faults is still considered to be linear and the location and magnitude of the fault are estimated using measured changes in the natural frequencies. Based on a one-dimensional structure model, Sakellariou and Fassois [4,5] have used a stochastic output error vibration-based methodology to detect faults in structures where the faulty elements are modelled as components of cubic stiffness.

The present study is concerned with the development of a novel approach for detecting the nonlinearity in mdof vibrating systems, which can represent a fault in the system. This work is the latest in a series of investigations conducted by the authors on this subject. For simplicity in demonstrating the main ideas, simple one-dimensional mdof systems with only one nonlinear component will be considered, as shown in Fig. 1.

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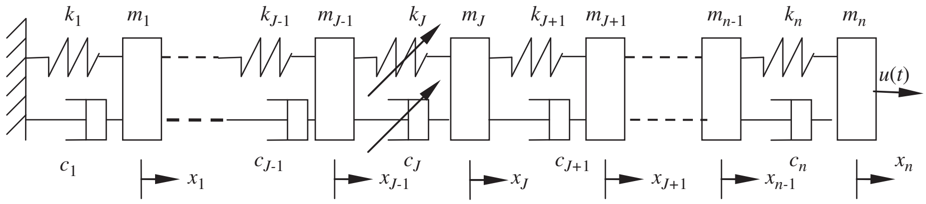


Fig. 1. The modf system considered in the present study.

However, compared with existing methods, the present study addresses the issues of detecting nonlinearity for a much wider class of mdof systems. The basis of this work is a totally new concept the author proposed in paper [6] known as nonlinear output frequency response functions (NOFRFs) together with several important properties of the NOFRFs of mdof nonlinear systems recently revealed by the authors [7,8]. The results can be extended to much more general cases including multidimensional mdof systems with multiple nonlinear components, and therefore have significant practical engineering applications.

## 2. Mdof systems with nonlinear component

The systems considered in the present study are described by a typical multi-degree-of-freedom oscillator as shown in Fig. 1 with the input force $u(t)$ applied at the $n$th mass.

If the characteristics of all the springs and dampers are linear, then this oscillator is an mdof linear system with the motion governing equation

$$M\ddot{x} + C\dot{x} + Kx = F(t), \tag{1}$$

where

$$M = \begin{bmatrix} m_1 & 0 & \cdots & 0 \\ 0 & m_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & m_n \end{bmatrix}$$

is the system mass matrix, and

$$C = \begin{bmatrix} c_1 + c_2 & -c_2 & 0 & \cdots & 0 \\ -c_2 & c_2 + c_3 & -c_3 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & -c_{n-1} & c_{n-1} + c_n & -c_n \\ 0 & \cdots & 0 & -c_n & c_n \end{bmatrix},$$

$$K = \begin{bmatrix} k_1 + k_2 & -k_2 & 0 & \cdots & 0 \\ -k_2 & k_2 + k_3 & -k_3 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & -k_{n-1} & k_{n-1} + k_n & -k_n \\ 0 & \cdots & 0 & -k_n & k_n \end{bmatrix}$$

are the system damping and stiffness matrix, respectively. $x = (x_1, \ldots, x_n)'$ is the displacement vector, and

$$F(t) = (\overbrace{0, \ldots, 0}^{n-1}, u(t))' \tag{2}$$

is the external force vector acting on the oscillator.

In order to show the main ideas, consider the simple case where there is only one nonlinear component in the system and assume that this nonlinear component is located between the $(J-1)$th and $J$th masses with $J \in \{1, \ldots, n\}$ and the 0th mass denotes the wall in Fig. 1. In addition, it is assumed that the nonlinear component can consist of a nonlinear spring and/or a nonlinear damper, and the restoring forces of the nonlinear spring and damper $\mathrm{FS}(\Delta)$ and $\mathrm{FD}(\dot{\Delta})$ are the polynomial functions of the deformation $\Delta = (x_{J-1} - x_J)$ and its derivative $\dot{\Delta}$, respectively, i.e.,

$$\mathrm{FS}(\Delta) = \sum_{l=1}^{P} r_l \Delta^l, \quad \mathrm{FD}(\dot{\Delta}) = \sum_{l=1}^{P} w_l \dot{\Delta}^l,$$

where $P$ is the degree of the polynomials. Thus by denoting

$$\mathrm{NF} = \left( \overbrace{0, \ldots, 0}^{J-2}, -\mathrm{FS}(\Delta) - \mathrm{FD}(\dot{\Delta}), \ \mathrm{FS}(\Delta) + \mathrm{FD}(\dot{\Delta}), \overbrace{0, \ldots, 0}^{n-J} \right)', \tag{3}$$

the mdof nonlinear oscillator considered in the present study can be described as

$$M\ddot{x} + C\dot{x} + Kx = \mathrm{NF} + F(t). \tag{4}$$

Eqs. (3) and (4) are the motion governing equations of mdof systems with one nonlinear component. Obviously, this nonlinear component can make the whole system behave nonlinearly. The basic issue addressed in this study is how to locate the position of this nonlinear component from the input force and the corresponding responses of the masses in the system. If for example, as mentioned in the introduction, the existence of the nonlinear component implies the existence of a fault in the system such as cracks in beams, locating the nonlinear component in system (3, 4) is equivalent to detecting the fault in the system and therefore has significant implications in engineering practices.

## 3. NOFRFs of mdof systems

### 3.1. The basic concept of NOFRFs

The concept of NOFRFs was recently proposed by the author to provide an alternative description for the frequency domain behaviours of nonlinear systems [6].

For nonlinear systems which are stable at zero equilibrium and can be described in a neighbourhood of the equilibrium by a Volterra series [9,10], the output frequency response to a general input is described in Ref. [11] as

$$\begin{cases} X(\mathrm{j}\omega) = \sum\limits_{\bar{n}=1}^{N} X_{\bar{n}}(\mathrm{j}\omega) \quad \text{for } \forall\omega, \\ X_{\bar{n}}(\mathrm{j}\omega) = \dfrac{1/\sqrt{\bar{n}}}{(2\pi)^{\bar{n}-1}} \int_{\omega_1+\cdots+\omega_{\bar{n}}=\omega} H_{\bar{n}}(\mathrm{j}\omega_1, \ldots, \mathrm{j}\omega_{\bar{n}}) \prod\limits_{i=1}^{\bar{n}} U(\mathrm{j}\omega_i)\, \mathrm{d}\sigma_{\bar{n}\omega}, \end{cases} \tag{5}$$

where $X(\mathrm{j}\omega)$ is the spectrum of the system output, $X_{\bar{n}}(\mathrm{j}\omega)$ represents the $n$th-order output frequency response of the system, $U(\mathrm{j}\omega)$ is the spectrum of the system input,

$$H_{\bar{n}}(\mathrm{j}\omega_1, \ldots, \mathrm{j}\omega_{\bar{n}}) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} h_{\bar{n}}(\tau_1, \ldots, \tau_{\bar{n}}) \mathrm{e}^{-(\omega_1\tau_1+\cdots+\omega_{\bar{n}}\tau_{\bar{n}})\mathrm{j}}\, \mathrm{d}\tau_1 \ldots \mathrm{d}\tau_{\bar{n}} \tag{6}$$

and is known as the $\bar{n}$th-order generalised frequency response function (GFRF) [9,12]. The term

$$\int_{\omega_1+\cdots+\omega_{\bar{n}}=\omega} H_{\bar{n}}(\mathrm{j}\omega_1,\ldots,\mathrm{j}\omega_{\bar{n}})\prod_{i=1}^{\bar{n}} U(\mathrm{j}\omega_i)\,\mathrm{d}\sigma_{\bar{n}\omega}$$

in Eq. (5) denotes the integration of

$$H_{\bar{n}}(\mathrm{j}\omega_1,\ldots,\mathrm{j}\omega_{\bar{n}})\prod_{i=1}^{\bar{n}} U(\mathrm{j}\omega_i)$$

over the $\bar{n}$-dimensional hyper-plane $\omega_1+\cdots+\omega_{\bar{n}}=\omega$.

For the analysis of the nonlinear system output spectrum given by Eq. (5), the concept of the NOFRFs was proposed in Ref. [6] as

$$G_{\bar{n}}(\mathrm{j}\omega)=\frac{\int_{\omega_1+\cdots+\omega_{\bar{n}}=\omega} H_{\bar{n}}(\mathrm{j}\omega_1,\ldots,\mathrm{j}\omega_{\bar{n}})\prod_{i=1}^{\bar{n}} U(\mathrm{j}\omega_i)\,\mathrm{d}\sigma_{\bar{n}\omega}}{\int_{\omega_1+\cdots+\omega_{\bar{n}}=\omega}\prod_{i=1}^{\bar{n}} U(\mathrm{j}\omega_i)\,\mathrm{d}\sigma_{\bar{n}\omega}},\quad \bar{n}=1,\ldots,N \tag{7}$$

under the condition that

$$U_{\bar{n}}(\mathrm{j}\omega)=\int_{\omega_1+\cdots+\omega_{\bar{n}}=\omega}\prod_{i=1}^{\bar{n}} U(\mathrm{j}\omega_i)\,\mathrm{d}\sigma_{\bar{n}\omega}\neq 0. \tag{8}$$

It was also revealed in Ref. [6] that the NOFRF concept has several important properties such as

(1) The NOFRFs are insensitive to a change of the input spectrum by a constant gain, that is

$$G_{\bar{n}}(\mathrm{j}\omega)\big|_{U(\mathrm{j}\omega)=\alpha\bar{U}(\mathrm{j}\omega)}=G_{\bar{n}}(\mathrm{j}\omega)\big|_{U(\mathrm{j}\omega)=\bar{U}(\mathrm{j}\omega)}. \tag{9}$$

(2) By introducing the NOFRF concept, the system output spectrum can be described as

$$X(\mathrm{j}\omega)=\sum_{\bar{n}=1}^{N} X_{\bar{n}}(\mathrm{j}\omega)=\sum_{\bar{n}=1}^{N} G_{\bar{n}}(\mathrm{j}\omega)U_{\bar{n}}(\mathrm{j}\omega), \tag{10}$$

which is similar to the description of the output frequency response of linear systems.

It is known from Eq. (7) that the $\bar{n}$th-order NOFRF $G_{\bar{n}}(\mathrm{j}\omega)$ depends not only on the GFRF $H_{\bar{n}}(\mathrm{j}\omega_1,\ldots,\mathrm{j}\omega_{\bar{n}})(\bar{n}=1,\ldots,N)$ but also on the input spectrum $U(\mathrm{j}\omega)$. That is, the NOFRFs generally reflect a combined contribution of both the system and the input to the system output frequency response behaviour. The dynamic properties of nonlinear systems are determined by the GFRFs in the frequency domain [9]. However, the multidimensional nature implies that the GFRFs are difficult for use in nonlinear system analyses. Eq. (7) indicates $G_{\bar{n}}(\mathrm{j}\omega)$ consists of a weighted sum of the $n$th-order GFRF $H_{\bar{n}}(\mathrm{j}\omega_1,\ldots,\mathrm{j}\omega_{\bar{n}})$ over the $n$-dimensional hyper-plane $\omega_1+\cdots+\omega_{\bar{n}}=\omega$ where the weights depend on the input, and hence the NOFRFs can be regarded as a one-dimensional representation of the frequency domain properties of nonlinear systems. This implies that, compared to the GFRFs, the NOFRFs can more easily be used to analyse nonlinear systems in the frequency domain in many practical applications.

### 3.2. The NOFRFs of mdof systems

The NOFRF concept can be readily extended to the case of mdof systems. The $\bar{n}$th-order NOFRF associated with the $i$th mass of a mdof nonlinear system was defined as [7,8]

$$G_{(i,\bar{n})}(\mathrm{j}\omega)=\frac{\int_{\omega_1+\cdots+\omega_{\bar{n}}=\omega} H_{(i,\bar{n})}(\mathrm{j}\omega_1,\ldots,\mathrm{j}\omega_{\bar{n}})\prod_{q=1}^{\bar{n}} U(\mathrm{j}\omega_q)\,\mathrm{d}\sigma_{\bar{n}\omega}}{\int_{\omega_1+\cdots+\omega_{\bar{n}}=\omega}\prod_{q=1}^{\bar{n}} U(\mathrm{j}\omega_q)\,\mathrm{d}\sigma_{\bar{n}\omega}}\quad (1\leqslant\bar{n}\leqslant N,\quad 1\leqslant i\leqslant n), \tag{11}$$

where $H_{(i,\bar{n})}(\mathrm{j}\omega_1,\ldots,\mathrm{j}\omega_{\bar{n}})$ represents the $\bar{n}$th-order GFRF associated with the $i$th mass.

In addition to the general properties as given in Eqs. (9) and (10), the NOFRFs of the mdof system (3, 4) also have the following properties [7,8]:

$$\frac{G_{(i,2)}(j\omega)}{G_{(i+1,2)}(j\omega)} = \cdots = \frac{G_{(i,N)}(j\omega)}{G_{(i+1,N)}(j\omega)} = \lambda^{i,i+1}(j\omega) \quad \text{for} \quad 1 \leqslant i \leqslant n-1, \tag{12}$$

$$\frac{G_{(i,1)}(j\omega)}{G_{(i+1,1)}(j\omega)} = \frac{G_{(i,2)}(j\omega)}{G_{(i+1,2)}(j\omega)} \cdots = \frac{G_{(i,N)}(j\omega)}{G_{(i+1,N)}(j\omega)} = \lambda^{i,i+1}(j\omega) \quad \text{for} \quad 1 \leqslant i \leqslant J-2, \tag{13}$$

$$\frac{G_{(i,1)}(j\omega)}{G_{(i+1,1)}(j\omega)} \neq \frac{G_{(i,2)}(j\omega)}{G_{(i+1,2)}(j\omega)} = \cdots = \frac{G_{(i,N)}(j\omega)}{G_{(i+1,N)}(j\omega)} = \lambda^{i,i+1}(j\omega) \quad \text{for} \quad J-1 \leqslant i \leqslant n-1. \tag{14}$$

Eqs. (12)–(14) show the relationships between the NOFRFs associated with two consecutive masses in system (3, 4) and indicate that

(1) the ratios of the $\bar{n}$th-order NOFRFs associated with two consecutive masses are the same for different $\bar{n}$'s if $\bar{n} \geqslant 2$,
(2) if two consecutive masses are all on the left of the system's nonlinear component, then the ratio of the first-order NOFRFs associated with the two masses is the same as the ratio of the higher order NOFRFs associated with the two masses,
(3) if at least one of two consecutive masses is on the right of the system's nonlinear component, then the ratio of the first-order NOFRFs associated with the two masses is different from the ratio of the higher order NOFRFs associated with the two masses.

These properties are a very important development for the recently proposed theories and methods of the NOFRF-based nonlinear system frequency domain analysis [13–20] and are especially useful for the location of the nonlinear component in mdof systems. The properties have been rigorously proved and used for nonlinear mdof system analysis [7,8]. Therefore, only a numerical simulation result for a 6-dof oscillator is provided to demonstrate the validity of these properties.

In the simulation study, $n = 6$, and the parameters of the 6-dof oscillator were chosen as $m_1 = \cdots = m_6 = 1$, $k_1 = \cdots = k_6 = 3.5531 \times 10^4$, $C = \mu K$, $\mu = 0.01$. A nonlinear spring is located between the 3rd and 4th masses in the oscillator, that is $J = 4$. The restoring force of the nonlinear spring is

$$S_4(\Delta) = k_4\Delta + 0.8k_4^2\Delta^2 + 0.4k_4^3\Delta^3. \tag{15}$$

The input considered is a sinusoidal force $u(t) = \sin(\omega_F t)$, where $\omega_F = 2\pi \times 20$.

It is well known that under quite general conditions [9] the response of a nonlinear system to a sinusoidal input is the super-harmonics of the input; and in such cases it can readily show that the even order super-harmonics are contributed by the systems nonlinearity of even orders; the odd order super-harmonics are contributed by the systems nonlinearity of odd orders. Therefore, for the 6-dof oscillator under study, the displacement spectrum of all the masses can be described as

$$\begin{aligned}
&X_i(j\omega_F) = G_{(i,1)}(j\omega_F)U_1(j\omega_F) + G_{(i,3)}(j\omega_F)U_3(j\omega_F) + \cdots + G_{(i,2\bar{n}-1)}(j\omega_F)U_{2\bar{n}-1}(j\omega_F) + \cdots,\\
&X_i(j2\omega_F) = G_{(i,2)}(j2\omega_F)U_2(j2\omega_F) + G_{(i,4)}(j2\omega_F)U_4(j2\omega_F) + \cdots + G_{(i,2\bar{n})}(j2\omega_F)U_{2\bar{n}}(j2\omega_F) + \cdots,\\
&X_i(j3\omega_F) = G_{(i,3)}(j3\omega_F)U_3(j3\omega_F) + \cdots + G_{(i,2\bar{n}-1)}(j3\omega_F)U_{2\bar{n}-1}(j3\omega_F) + \cdots,\\
&X_i(j4\omega_F) = G_{(i,4)}(j4\omega_F)U_4(j4\omega_F) + \cdots + G_{(i,2\bar{n})}(j4\omega_F)U_{2\bar{n}}(j4\omega_F) + \cdots,\\
&\vdots \quad \vdots \quad \vdots\\
&i = 1, \ldots, 6.
\end{aligned}$$

Consequently, in this specific case, properties (12)–(14) of the NOFRFs of mdof system (3, 4) imply

$$\frac{G_{(i,2)}(j2\omega_F)}{G_{(i+1,2)}(j2\omega_F)} = \frac{G_{(i,4)}(j2\omega_F)}{G_{(i+1,4)}(j2\omega_F)} = \cdots = \frac{G_{(i,2\bar{n})}(j2\omega_F)}{G_{(i+1,2\bar{n})}(j2\omega_F)} = \lambda^{i,i+1}(j2\omega_F) \quad \text{for} \quad 1 \leqslant i \leqslant 5, \tag{16}$$

$$\frac{G_{(i,1)}(\mathrm{j}\omega_F)}{G_{(i+1,1)}(\mathrm{j}\omega_F)} = \frac{G_{(i,3)}(\mathrm{j}\omega_F)}{G_{(i+1,3)}(\mathrm{j}\omega_F)} = \cdots = \frac{G_{(i,2\bar{n}-1)}(\mathrm{j}\omega_F)}{G_{(i+1,2\bar{n}-1)}(\mathrm{j}\omega_F)} = \lambda^{i,i+1}(\mathrm{j}\omega_F) \quad \text{for} \quad 1 \leqslant i \leqslant 2, \tag{17}$$

$$\frac{G_{(i,1)}(\mathrm{j}\omega_F)}{G_{(i+1,1)}(\mathrm{j}\omega_F)} \neq \frac{G_{(i,3)}(\mathrm{j}\omega_F)}{G_{(i+1,3)}(\mathrm{j}\omega_F)} = \cdots = \frac{G_{(i,2\bar{n}-1)}(\mathrm{j}\omega_F)}{G_{(i+1,2\bar{n}-1)}(\mathrm{j}\omega_F)} = \lambda^{i,i+1}(\mathrm{j}\omega_F) \quad \text{for} \quad 3 \leqslant i \leqslant 5. \tag{18}$$

Table 1 shows the comparison between the evaluated $G_{(i,2)}(\mathrm{j}2\omega_F)/G_{(i+1,2)}(\mathrm{j}2\omega_F)$ and $G_{(i,4)}(\mathrm{j}2\omega_F)/G_{(i+1,4)}(\mathrm{j}2\omega_F)$ for $i = 1, \ldots, 5$. Table 2 shows the comparison between the evaluated $G_{(i,1)}(\mathrm{j}\omega_F)/G_{(i+1,1)}(\mathrm{j}\omega_F)$ and $G_{(i,3)}(\mathrm{j}\omega_F)/G_{(i+1,3)}(\mathrm{j}\omega_F)$ for $i = 1, \ldots, 5$. These results were obtained by using the system input, the simulated system response data, and the algorithm developed in Ref. [6] for determining the NOFRFs of nonlinear systems. Obviously, the results in Tables 1 and 2 demonstrate the validity of Eqs. (16)–(18)—the properties of the NOFRFs of mdof systems in the specific case of the considered 6-dof oscillator.

## 4. A novel approach for locating nonlinear component in mdof systems

It is easily seen from the example above that properties (12)–(14) of the NOFRFs of mdof systems can be directly used to locate the nonlinear component in system (3, 4). This is a direct method and involves determining the NOFRFs associated with all the masses in the system, evaluating and comparing the ratios of the NOFRFs associated with two consecutive masses, and finally determining $J$ in Eqs. (13) and (14) from the evaluated NOFRFs' ratios so as to locate the position of the nonlinear component. However this method requires implementing the algorithm in Ref. [6] to determine the NOFRFs associated with all masses in system (3, 4) up to an appropriate order $N$, which is normally unknown in practice and therefore has to be chosen as a sufficiently large number to ensure an effective implementation of the algorithm. Obviously, this problem has to be addressed in practical engineering applications.

In order to solve this problem, the authors recently proposed a dual harmonic input method for locating the nonlinear component of system (3, 4) [7]. As opposed to the direct method where at least $N$ tests are needed to generate the data for locating the nonlinear component, this dual harmonic input method only requires two tests on the system. The problem with this method is that only sinusoidal force inputs can be used to excite an inspected system to generate the system responses for locating the nonlinear component. This can be a limitation to the practical applications of the method.

In this section, a totally new approach is proposed for locating nonlinear component in system (3, 4) to resolve the problems with the available direct and dual harmonic input methods. The basis of this novel

Table 1
The evaluated values of $G_{(i,2)}(\mathrm{j}2\omega_F)/G_{(i+1,2)}(\mathrm{j}2\omega_F)$ and $G_{(i,4)}(\mathrm{j}2\omega_F)/G_{(i+1,4)}(\mathrm{j}2\omega_F)$

| | $G_{(i,2)}(\mathrm{j}2\omega_F)/G_{(i+1,2)}(\mathrm{j}2\omega_F)$ | $G_{(i,4)}(\mathrm{j}2\omega_F)/G_{(i+1,4)}(\mathrm{j}2\omega_F)$ |
|---|---|---|
| $i = 1$ | −0.5078 + 0.1764j | −0.5078 + 0.1765j |
| $i = 2$ | 0.4258−0.0818j | 0.4259−0.0818j |
| $i = 3$ | −1.5199−0.1875j | −1.5199−0.1875j |
| $i = 4$ | 0.9563 + 1.2569j | 0.9563 + 1.2565j |
| $i = 5$ | 0.7572 + 0.6107j | 0.7571 + 0.6108j |

Table 2
The evaluated values of $G_{(i,1)}(\mathrm{j}\omega_F)/G_{(i+1,1)}(\mathrm{j}\omega_F)$ and $G_{(i,3)}(\mathrm{j}\omega_F)/G_{(i+1,3)}(\mathrm{j}\omega_F)$

| | $G_{(i,1)}(\mathrm{j}\omega_F)/G_{(i+1,1)}(\mathrm{j}\omega_F)$ | $G_{(i,3)}(\mathrm{j}\omega_F)/G_{(i+1,3)}(\mathrm{j}\omega_F)$ |
|---|---|---|
| $i = 1$ | −0.5396 + 0.0639j | −0.5396 + 0.0639j |
| $i = 2$ | 0.4207−0.0271j | 0.4207−0.0271j |
| $i = 3$ | 0.6901−0.1197j | −0.7883−1.7427j |
| $i = 4$ | 0.80858−0.2388j | 0.6969 + 0.5123j |
| $i = 5$ | 0.8179−0.3654j | 0.8277 + 0.2166j |

approach is a function of frequency proposed for the measurement of the nonlinear relationship between the responses of two masses.

### 4.1. The measurement of the nonlinear relationship between the responses of two masses

From the definition of the NOFRFs of mdof systems, it is known that the output spectrum of the $i$th mass of mdof system (3, 4) can be written as

$$\begin{aligned} X_i(\mathrm{j}\omega) &= \sum_{\bar{n}=1}^{N} G_{(i,\bar{n})}(\mathrm{j}\omega)U_{\bar{n}}(\mathrm{j}\omega) = \sum_{\bar{n}=1}^{N} G_{(i,\bar{n})}(\mathrm{j}\omega)U_{\bar{n}}(\mathrm{j}\omega) + \lambda^{i,k}(\omega)X_k(\mathrm{j}\omega) - \lambda^{i,k}(\omega)X_k(\mathrm{j}\omega) \\ &= \sum_{\bar{n}=1}^{N} G_{(i,\bar{n})}(\mathrm{j}\omega)U_{\bar{n}}(\mathrm{j}\omega) + \lambda^{i,k}(\omega)X_k(\mathrm{j}\omega) - \lambda^{i,k}(\omega)\sum_{\bar{n}=1}^{N} G_{(k,\bar{n})}(\mathrm{j}\omega)U_{\bar{n}}(\mathrm{j}\omega) \\ &= \sum_{\bar{n}=1}^{N}[G_{(i,\bar{n})}(\mathrm{j}\omega) - \lambda^{i,k}(\omega)G_{(k,\bar{n})}(\mathrm{j}\omega)]U_{\bar{n}}(\mathrm{j}\omega) + \lambda^{i,k}(\omega)X_k(\mathrm{j}\omega), \end{aligned} \tag{19}$$

where $k - i \geqslant 1$,

$$\lambda^{i,k}(\mathrm{j}\omega) = \prod_{\bar{i}=i}^{k-1} \lambda^{\bar{i},\bar{i}+1}(\mathrm{j}\omega). \tag{20}$$

Rewrite Eq. (19) as

$$\begin{aligned} X_i(\mathrm{j}\omega) = [G_{(i,1)}(\mathrm{j}\omega) - \lambda^{i,k}(\omega)G_{(k,1)}(\mathrm{j}\omega)]U_1(\mathrm{j}\omega) + \sum_{\bar{n}=2}^{N}[G_{(i,\bar{n})}(\mathrm{j}\omega) \\ - \lambda^{i,k}(\omega)G_{(k,\bar{n})}(\mathrm{j}\omega)]U_{\bar{n}}(\mathrm{j}\omega) + \lambda^{i,k}(\omega)X_k(\mathrm{j}\omega). \end{aligned} \tag{21}$$

From property (12) of system (3, 4), it is known that [7,8]

$$\lambda^{i,k}(\mathrm{j}\omega) = \frac{G_{(i,\bar{n})}(\mathrm{j}\omega)}{G_{(k,\bar{n})}(\mathrm{j}\omega)} \tag{22}$$

for $\bar{n} \geqslant 2$. Substituting Eq. (22) into the second term on the right-hand side of Eq. (21) for $\lambda^{i,k}(\mathrm{j}\omega)$ yields

$$\begin{aligned} X_i(\mathrm{j}\omega) &= [G_{(i,1)}(\mathrm{j}\omega) - \lambda^{i,k}(\omega)G_{(k,1)}(\mathrm{j}\omega)]U_1(\mathrm{j}\omega) + \lambda^{i,k}(\mathrm{j}\omega)X_k(\mathrm{j}\omega) \\ &= E^{i,k}(\mathrm{j}\omega)U_1(\mathrm{j}\omega) + \lambda^{i,k}(\mathrm{j}\omega)X_k(\mathrm{j}\omega), \end{aligned} \tag{23}$$

where

$$E^{i,k}(\mathrm{j}\omega) = [G_{(i,1)}(\mathrm{j}\omega) - \lambda^{i,k}(\mathrm{j}\omega)G_{(k,1)}(\mathrm{j}\omega)]. \tag{24}$$

It is known from Eq. (23) that if $E^{i,k}(\mathrm{j}\omega) = 0$,

$$X_i(\mathrm{j}\omega) = \lambda^{i,k}(\mathrm{j}\omega)X_k(\mathrm{j}\omega).$$

This indicates the relationship between the frequency responses of the $i$th mass and $k$th masses is linear. Therefore, the function of frequency $E^{i,k}(\mathrm{j}\omega)$ as defined in Eq. (24) can be used to indicate the extent to which the relationship between the responses of the $i$th mass and $k$th mass in system (3, 4) is nonlinear.

From properties (12)–(14) of system (3, 4) and the definition of $E^{i,k}(\mathrm{j}\omega)$, it is easy to deduce the properties of $E^{i,k}(\mathrm{j}\omega)$, which link the value of $E^{i,k}(\mathrm{j}\omega)$ to the location of the system nonlinear component, as follows:

(1) If $E^{i,k}(\mathrm{j}\omega) = 0$, there is no nonlinear component between the $i$th and $k$th masses, and if there is a nonlinear component in the system this component is located on the right of the $k$th mass.
(2) If $E^{i,k}(\mathrm{j}\omega) \neq 0$, there is a nonlinear component in the system and this nonlinear component is located on the left of the $k$th mass.

The definition and two properties of $E^{i,k}(\mathrm{j}\omega)$ are the basis of the novel approach for locating nonlinear component in mdof system (3, 4).

### 4.2. The approach

The basic idea of this new approach is to evaluate $E^{i,i+1}(\mathrm{j}\omega)$ for $i=1,\ldots,n-1$ and use the results along with the two properties of $E^{i,k}(\mathrm{j}\omega)$ to determine the position of the nonlinear component in the system.

Eq. (23) implies that if system (3, 4) is excited by two force inputs, respectively, with the spectrum of the two inputs denoted by $U^{(q)}(\mathrm{j}\omega)$, $q=1,2$, which are only different in strength, i.e., $U^{(q)}(\mathrm{j}\omega)=\alpha_q U^*(\mathrm{j}\omega)$, $q=1,2$, where $\alpha_1\neq 0$, $\alpha_2\neq 0$, and $\alpha_1\neq\alpha_2$, then the corresponding output spectra of the $i$th and $k$th masses denoted by $X_i^{(q)}(\mathrm{j}\omega)$ and $X_k^{(q)}(\mathrm{j}\omega)$, $q=1,2$, can be related by the following equations:

$$\begin{cases} X_i^{(1)}(\mathrm{j}\omega)=E^{i,k}(\mathrm{j}\omega)\alpha_1 U^*(\mathrm{j}\omega)+\lambda^{i,k}(\mathrm{j}\omega)X_k^{(1)}(\mathrm{j}\omega), \\ X_i^{(2)}(\mathrm{j}\omega)=E^{i,k}(\mathrm{j}\omega)\alpha_2 U^*(\mathrm{j}\omega)+\lambda^{i,k}(\mathrm{j}\omega)X_k^{(2)}(\mathrm{j}\omega). \end{cases} \tag{25}$$

From Eq. (25), $E^{i,k}(\mathrm{j}\omega)$ can be readily determined from $U^*(\mathrm{j}\omega)$ and $X_i^{(q)}(\mathrm{j}\omega)$, $X_k^{(q)}(\mathrm{j}\omega)$, $q=1$, 2, as

$$E^{i,k}(\mathrm{j}\omega)=[1\ \ 0]\begin{bmatrix}\alpha_1 U^*(\mathrm{j}\omega), & X_k^{(1)}(\mathrm{j}\omega)\\ \alpha_2 U^*(\mathrm{j}\omega), & X_k^{(2)}(\mathrm{j}\omega)\end{bmatrix}^{-1}\begin{bmatrix}X_i^{(1)}(\mathrm{j}\omega)\\ X_i^{(2)}(\mathrm{j}\omega)\end{bmatrix}. \tag{26}$$

From the expression of $E^{i,k}(\mathrm{j}\omega)$ given by Eq. (26), the new approach for locating the nonlinear component in system (3, 4) is proposed as follows:

(1) Excite the system twice as described above, measure the responses, and calculate the spectrum for all the masses in the system to obtain $X_i^{(q)}(\mathrm{j}\omega)$, $i=1,\ldots,n$, and $q=1,2$.
(2) Determine $E^{i,i+1}(\mathrm{j}\omega)$ using (26) for $i=1,\ldots,n-1$.
(3) Evaluate $\overline{E}^{i,i+1}$ for $i=1,\ldots,n-1$ as

$$\overline{E}^{i,i+1}=\frac{\int_{\omega_1}^{\omega_2}\left|E^{i,i+1}(\mathrm{j}\omega)\right|\mathrm{d}\omega}{\max_{i\in\{1,\ldots,n-1\}}\left[\int_{\omega_1}^{\omega_2}\left|E^{i,i+1}(\mathrm{j}\omega)\right|\mathrm{d}\omega\right]}, \tag{27}$$

where $[\omega_1,\omega_2]$ is a frequency band within the frequency range of the input spectrum $U^*(\mathrm{j}\omega)$ and $\overline{E}^{0,1}(\mathrm{j}\omega)=0$.
(4) Examine $\overline{E}^{i,i+1}$ for $i=1,\ldots,n-1$. If an $\hat{i}$ can be found such that

$$\overline{E}^{i,i+1}\approx 0 \quad \text{for} \quad i=1,2,\ldots,(\hat{i}-1) \quad \text{but} \quad \overline{E}^{i,i+1}\text{not}\approx 0 \quad \text{for} \quad i=\hat{i},\ldots,(n-1)$$

then it can be concluded that the nonlinear component of system (3, 4) is located between the $\hat{i}$th mass and the $(\hat{i}+1)$th mass, i.e., $J=\hat{i}+1$.

It can be seen that this approach basically exploits the properties of $E^{i,k}(\mathrm{j}\omega)$ and uses $\overline{E}^{i,i+1}$ which is the normalised integration of $\left|E^{i,i+1}(\mathrm{j}\omega)\right|$ over the frequency band $[\omega_1,\omega_2]$ for $i=0,1,\ldots,n-1$ to locate the position of the nonlinear component in the system. The introduction and use of $\overline{E}^{i,i+1}$ is to improve the robustness of the proposed approach against possible numerical and measurement errors, which are unavoidable in engineering practices. A detailed analysis of the effects of these errors on the performance of the proposed approach is an issue that will be addressed in a further study where different models will be used to represent the characteristics of the errors. In the present study, the effectiveness of the approach will be verified via numerical simulation studies in the following section.

Clearly, the new approach does not need to know or assume the highest order '$N$' of the system nonlinearity, and requires only double testing on the inspected system. Therefore, the approach has the same advantage as that of the dual harmonic input method. In addition, the new approach allows any form of force inputs to be used to excite the system under inspection. This implies that the new approach is suitable for a much wider range of engineering applications.

## 5. Simulation studies

In order to verify the effectiveness of the proposed new approach, simulation case studies were conducted. The results obtained from two case studies, where a damped nonlinear 10-dof system was considered, are presented below.

### 5.1. Case study 1

In the first case, the linear characteristic parameters of the 10-dof system are:

$$m_1 = \cdots = m_{10} = 1, \quad k_1 = \cdots = k_5 = k_{10} = 3.6 \times 10^4, \quad k_6 = k_7 = k_8 = 0.8k_1,$$
$$k_9 = 0.9 \times k_1, \quad \mu = 0.01, \quad C = \mu K$$

and the characteristic of the fourth spring of the system is nonlinear with the parameters:

$$r_{(4,1)} = k_1, \quad r_{(4,2)} = 0.8k_1^2, \quad r_{(4,3)} = 0.4k_1^3, \quad r_{(4,l)} = 0 \quad \text{for} \quad l \geqslant 4.$$

From the procedure in Section 4.2, the new approach was implemented as follows:

(1) Two sinusoidal force inputs

$$u^{(q)}(t) = \alpha_q \sin(\omega_F t), \quad q = 1, 2,$$

where $\omega_F = 2\pi \times 20$, $\alpha_1 = 1$, $\alpha_2 = 1.5$, were applied on the 10th mass to excite the system, respectively, to generate two sets of output responses on the 10 masses. The spectra of the two sets of output responses at the driving frequency $\omega_F$ were determined and the results are denoted by $X_i^{(q)}(j\omega_F)$, $i = 1, \ldots, 10$, $q = 1,2$.

(2) Eq. (26) was then used to determine $E^{i,i+1}(j\omega_F)$ for $i = 1, \ldots, 9$ as follows:

$$E^{i,i+1}(j\omega_F) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} \alpha_1 U^*(j\omega_F), & X_{i+1}^{(1)}(j\omega_F) \\ \alpha_2 U^*(j\omega_F), & X_{i+1}^{(2)}(j\omega_F) \end{bmatrix}^{-1} \begin{bmatrix} X_i^{(1)}(j\omega_F) \\ X_i^{(2)}(j\omega_F) \end{bmatrix},$$

where $U^*(j\omega_F)$ denotes the spectrum of $u^*(t) = \sin(\omega_F t)$ at $\omega = \omega_F$.

(3) $\omega_1$ and $\omega_2$ were chosen as $\omega_1 = \omega_2 = \omega_F$ and $\overline{E}^{i,i+1}$ $i = 1, \ldots, 9$ were determined as

$$\overline{E}^{i,i+1} = \frac{\int_{\omega_1}^{\omega_2} \left| E^{i,i+1}(j\omega) \right| d\omega}{\max_{i \in \{1,\ldots,n-1\}} \left[ \int_{\omega_1}^{\omega_2} \left| E^{i,i+1}(j\omega) \right| d\omega \right]} = \frac{\left| E^{i,i+1}(j\omega_F) \right|}{\max_{i \in \{1,\ldots,9\}} \left| E^{i,i+1}(j\omega_F) \right|}.$$

The results obtained are given in Table 3 and illustrated in Fig. 2.

(4) From Table 3 or Fig. 2, it can be found that $\hat{i} = 3$. Therefore, the nonlinear component of the system is located between the $\hat{i} = 3$rd mass and $\hat{i} + 1 = 4$th mass, i.e., $J = \hat{i} + 1 = 4$.

Obviously this conclusion reached by using the proposed approach is correct.

Table 3
$\overline{E}^{i,i+1}$ $i = 1, \ldots, 9$ evaluated for Case 1 of the simulation studies

| $\overline{E}^{1,2}$ | $\overline{E}^{2,3}$ | $\overline{E}^{3,4}$ | $\overline{E}^{4,5}$ | $\overline{E}^{5,6}$ | $\overline{E}^{6,7}$ | $\overline{E}^{7,8}$ | $\overline{E}^{8,9}$ | $\overline{E}^{9,10}$ |
|---|---|---|---|---|---|---|---|---|
| 0.0019 | 0.0027 | 0.53 | 0.30 | 0.32 | 0.53 | 0.82 | 1.00 | 0.97 |

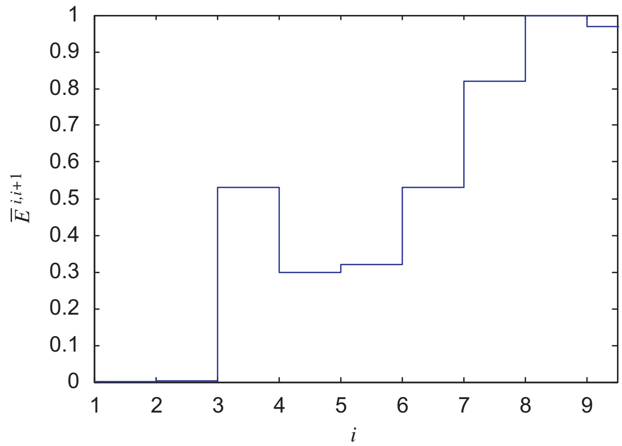


Fig. 2. An illustration of $\overline{E}^{i,i+1}$, $i = 1, \ldots, 9$, evaluated for Case 1 of the simulation studies.

### 5.2. Case study 2

In the second case, the linear characteristic parameters of the 10-dof system are the same as in Case 1. The characteristic of the sixth spring of the system is nonlinear with the parameters:

$$r_{(6,1)} = k_6, \quad r_{(6,2)} = 4k_1^2, \quad r_{(6,3)} = 0.4k_1^3, \quad r_{(6,l)} = 0 \quad \text{for} \quad l \geqslant 4.$$

From the procedure in Section 4.2, the new approach was implemented as follows:

(1) Two pulsed force inputs

$$u^{(q)}(t) = \begin{cases} \alpha_q, & t \in [0, 0.01], \\ 0 & \text{otherwise}, \end{cases} \quad q = 1.2,$$

where $\alpha_1 = 1$, $\alpha_2 = 1.5$, were applied on the 10th mass to excite the system, respectively, to generate two sets of output responses on the ten masses. The spectra of the two sets of output responses were determined and the results are denoted by $X_i^{(q)}(\mathrm{j}\omega)$, $i = 1, \ldots, 10$, and $q = 1, 2$.

(2) Eq. (26) was then used to determine $E^{i,i+1}(\mathrm{j}\omega)$ for $i = 1, \ldots, 9$ as follows:

$$E^{i,i+1}(\mathrm{j}\omega) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} \alpha_1 U^*(\mathrm{j}\omega), & X_{i+1}^{(1)}(\mathrm{j}\omega) \\ \alpha_2 U^*(\mathrm{j}\omega), & X_{i+1}^{(2)}(\mathrm{j}\omega) \end{bmatrix}^{-1} \begin{bmatrix} X_i^{(1)}(\mathrm{j}\omega) \\ X_i^{(2)}(\mathrm{j}\omega) \end{bmatrix}$$

over the frequency band $\omega \in [0, 2\pi \times 100]$ where $U^*(\mathrm{j}\omega)$ denotes the spectrum of the pulsed force input

$$u^*(t) = \begin{cases} 1, & t \in [0, 0.01], \\ 0 & \text{otherwise}. \end{cases}$$

(3) $\omega_1$ and $\omega_2$ were chosen as $\omega_1 = 0$, $\omega_2 = 2\pi \times 100$ and $\overline{E}^{i,i+1}$ $i = 1, \ldots, 9$ were evaluated as

$$\overline{E}^{i,i+1} = \frac{\int_0^{200\pi} \left| E^{i,i+1}(\mathrm{j}\omega) \right| \mathrm{d}\omega}{\max_{i \in \{1,\ldots,9\}} \left[ \int_0^{200\pi} \left| E^{i,i+1}(\mathrm{j}\omega) \right| \mathrm{d}\omega \right]}.$$

The results obtained are shown in Table 4 and illustrated in Fig. 3.

Table 4
$\overline{E}^{i,i+1}$ $i = 1, \ldots, 9$ evaluated for Case 2 of the simulation studies

| $\overline{E}^{1,2}$ | $\overline{E}^{2,3}$ | $\overline{E}^{3,4}$ | $\overline{E}^{4,5}$ | $\overline{E}^{5,6}$ | $\overline{E}^{6,7}$ | $\overline{E}^{7,8}$ | $\overline{E}^{8,9}$ | $\overline{E}^{9,10}$ |
|---|---|---|---|---|---|---|---|---|
| $6.1\times10^{-8}$ | $2.7\times10^{-7}$ | $2.0\times10^{-6}$ | $9.4\times10^{-6}$ | 1.00 | 0.93 | 0.40 | 0.18 | 0.08 |

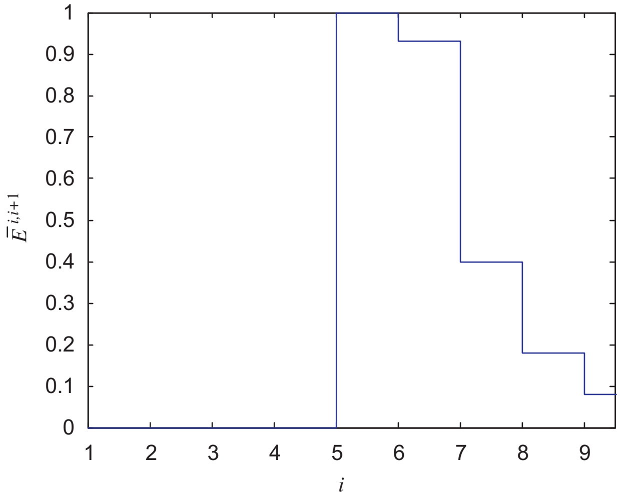


Fig. 3. An illustration of $\overline{E}^{i,i+1}$, $i = 1, \ldots, 9$, evaluated for Case 2 of the simulation studies.

(4) From Table 4 or Fig. 3, it can be found that $\hat{i} = 5$. Therefore, the nonlinear component of the system is located between the $\hat{i} = 5$th mass and $\hat{i} + 1 = 6$th mass, i.e., $J = \hat{i} + 1 = 6$.

Obviously, the conclusion reached by using the proposed approach is again correct.

### *5.3. Discussion*

As demonstrated by the simulation studies, the proposed approach basically evaluates the value of a $E^{i,i+1}(\mathrm{j}\omega)$ related index $\overline{E}^{i,i+1}$ over $i = 1, \ldots, n-1$ and locates the position of the nonlinear component in system (3, 4) via determining an $i \in \{1, \ldots, n-1\}$ where the value of the index has a significant increase from a value near zero. Since a zero value for the proposed function of frequency $E^{i,i+1}(\mathrm{j}\omega)$ indicates that the relationship between the responses of the $i$th mass and $(i+1)$th mass is linear, and a zero value of index $\overline{E}^{i,i+1}$ shares the same physical interpretation, the proposed approach essentially checks whether the relationship between the responses of two consecutive masses is linear or not and determines the nonlinear component position from the result. The proposed approach requires double testing on inspected structures to achieve the objective. This is because only deterministic force inputs are used as the excitation signal. If stochastic input forces can be used, the concept of coherence can be exploited and this can produce a simpler algorithm. More details of these will be reported in a future publication.

It is worth pointing out that a straightforward opinion about the behaviours of system (3, 4) would be that the relationship between the responses of mass $J$ and mass $J+1$ is nonlinear because there is a nonlinear spring and/or damper between the two masses. The relationship between the responses of any other two consecutive masses is linear because the spring and damper located between the masses are linear components. However, this possible intuitive judgement is incorrect. The two case studies clearly indicate that the relationship between the responses of two masses can also be nonlinear even when there is no nonlinear component located between the two masses. This shows that the proposed approach is really a significant technique for the location of system nonlinear component in mdof systems or structures. If there are more than one nonlinear components in system (3, 4), the situation will become more complicated. However, the basic principles of the proposed approach can be extended to locate nonlinear components in these more complicated cases.

The only restriction of the proposed approach and its principles in engineering practice is the assumption that a Volterra series model can represent the behaviours of the mdof system. This condition can be satisfied when the amplitudes of the excitation forces are within a certain range of limit, which, from the engineering viewpoints, should be the case in most practical circumstances.

## 6. Conclusions

In this paper, a novel approach has been proposed for nonlinearity detection in vibrating systems with multiple degrees of freedom (mdof). The new approach is developed based on the concept of nonlinear output frequency response functions (NOFRFs) and the properties of the NOFRFs of mdof nonlinear systems, and only requires testing on inspected systems or structures twice with the applied input forces differing in strength in the two tests. The approach determines the position of the nonlinear component in an mdof system directly from the applied input forces and the corresponding responses of the masses in the system. Simulation studies on a 10-dof oscillator have verified the effectiveness of the new approach.

In many practical mdof systems/structures such as beams, nonlinear components often represent faults. Therefore, the proposed approach has significant potential in fault diagnosis of practical mdof engineering systems and structures.

The present study is the latest in a series of research studies conducted by the authors on this subject. Because compared with previous results, no knowledge about the highest order of the system nonlinearity is required, and any form of input force excitations can be used, the new approach can be used in a much wider range of practical applications.

The aim of further research work will be to extend the results to more general cases such as multidimensional mdof systems with multiple nonlinear components to address problems associated with the fault diagnosis of more complicated engineering systems/structures.

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