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Harmonic balance approaches to the nonlinear oscillators in which the restoring force is inversely proportional to the dependent variable

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Abstract

The second-order harmonic balance method is used to construct three approximate frequency–amplitude relations for a conservative nonlinear singular oscillator in which the restoring force is inversely proportional to the dependent variable. Two procedures are used to solve the nonlinear differential equation approximately. In the first the differential equation is rewritten in a form that does not contain the y^{-1} expression, while in the second the differential equation is solved directly. The approximate frequency obtained using the second procedure is more accurate than the frequency obtained with the first one and the discrepancy between the approximate frequency and the exact one is lower than 1.28%.

Mickens [1] has recently analyzed the nonlinear differential equation [2]

$$\frac{d^2y}{dt^2} + \frac{1}{y} = 0, \tag{1}$$

with initial conditions

$$y(0) = A, \quad \left(\frac{dy}{dt}\right)_{t=0} = 0. \tag{2}$$

Mickens has also shown that all the motions corresponding to Eq. (1) are periodic [1,3]; the system will oscillate within symmetric bounds $[-A, A]$, and the angular frequency and corresponding periodic solution of the nonlinear oscillator are dependent on the amplitude A. Integration of Eq. (1) gives the first integral

$$\frac{1}{2}\left(\frac{dy}{dt}\right)^2 + \log y = \log A, \tag{3}$$

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where the integration constant was evaluated using the initial conditions of Eq. (2). From Eq. (3), the expression for the exact period, $T_{ex}(A)$, for the nonlinear oscillator given by Eq. (1) taking into account the initial conditions in Eq. (2) is

$$T_{ex}(A) = 4\int_0^A \frac{dy}{\sqrt{2\log(A/y)}}. \tag{4}$$

The transformation $y = Ae^{t^2}$ reduces this equation to the form

$$T_{ex}(A) = 4\sqrt{2}A\int_0^\infty e^{-t^2}\,dt = 4\sqrt{2}A\frac{\sqrt{\pi}}{2} = 2\sqrt{2\pi}A. \tag{5}$$

From Eq. (5), the exact value for the angular frequency is given by the expression

$$\omega_{ex}(A) = \frac{2\pi}{T_{ex}(A)} = \frac{2\pi}{2\sqrt{2\pi}A} = \frac{\sqrt{2\pi}}{2A} = \frac{1.2533141}{A}. \tag{6}$$

It is difficult to solve nonlinear differential equations and, in general, it is often more difficult to obtain an analytic approximation than a numerical one for a given nonlinear oscillatory system [3,4]. There are many approaches for approximating solutions to nonlinear oscillatory systems. The most widely studied approximation methods are the perturbation methods [5]. The simplest and perhaps one of the most useful of these approximation methods is the Lindstedt–Poincaré perturbation method, whereby the solution is analytically expanded in the power series of a small parameter [3]. To overcome this limitation, many new perturbative techniques have been developed. Modified Lindstedt–Poincaré techniques [6–8], the homotopy perturbation method [9–15] or linear delta expansion [16–18] are only some examples of them. A recent detailed review of asymptotic methods for strongly nonlinear oscillators can be found in Ref. [4]. The harmonic balance method is another procedure for determining analytical approximations to the periodic solutions of differential equations by using a truncated Fourier series representation [3,19–25]. This method can be applied to nonlinear oscillatory systems where the nonlinear terms are not small and no perturbation parameter is required.

The main objective of this paper is to approximately solve Eq. (1) by applying the harmonic balance method, and to compare the approximate frequency obtained with the exact one and with another approximate frequency obtained applying the harmonic balance method to the same oscillatory system but rewriting Eq. (1) in a way suggested previously by Mickens [1]. The approximate frequency derived here is more accurate and closer to the exact solution. The error in the resulting frequency is reduced and the maximum relative error is less than 1.3% for all values of A.

In order to approximately solve Eq. (1), Mickens has rewritten this equation in a form that does not contain the y^{-1} expression [1]

$$y\frac{d^2y}{dt^2} + 1 = 0, \quad y(0) = A, \quad \left(\frac{dy}{dt}\right)_{t=0} = 0. \tag{7}$$

It is possible to solve Eq. (7) by applying the harmonic balance method. Following the lowest order harmonic balance method, a reasonable and simple initial approximation satisfying the conditions in Eq. (7) would be

$$y_1(t) = A\cos\omega t. \tag{8}$$

The substitution of Eq. (8) into Eq. (7) gives

$$-A^2\omega^2\cos^2\omega t + 1 = 0, \tag{9}$$

then expanding and simplifying the resulting expression gives

$$-\tfrac{1}{2}A^2\omega^2 + 1 + (\text{higher-order harmonics}) = 0 \tag{10}$$

and the solution for the angular frequency, $\omega_{M1}(A)$, is

$$\omega_{M1}(A) = \frac{\sqrt{2}}{A} = \frac{1.414214}{A} \tag{11}$$

and the percentage error of this approximate frequency in relation to the exact one is

$$\text{percentage error} = \left| \frac{\omega_{ex} - \omega_{M1}}{\omega_{ex}} \right| \times 100 = 12.8\%. \tag{12}$$

Mickens [1] also used the second-order harmonic balance approximation

$$y_2(t) = A_2 \cos \omega t + B_2 \cos^3 \omega t \tag{13}$$

to the periodic solution of Eq. (7). Substitution of Eq. (13) into Eq. (7), simplifying the resulting expression and equating the coefficients of the lowest harmonics to zero give two equations and taking into account that $A = A_2 + B_2$, Mickens [1] obtained $A_2 = 10A/9$ and $B_2 = -A/9$, and the second-order approximate solution (Eq. (13)) to Eq. (7) can be written as follows:

$$y_{M2}(t) = \frac{10}{9} A \cos(\omega_{M2} t) - \frac{1}{9} A \cos(3\omega_{M2} t), \tag{14}$$

where the second-order approximate frequency, $\omega_{M2}(A)$, is given by

$$\omega_{M2}(A) = \frac{\sqrt{162}}{10A} = \frac{1.272792}{A} \tag{15}$$

and the percentage error is

$$\text{percentage error} = \left| \frac{\omega_{ex} - \omega_{M2}}{\omega_{ex}} \right| \times 100 = 1.55\%. \tag{16}$$

As we pointed out previously, the main objective of this paper is to solve Eq. (1) instead of Eq. (7) by applying the harmonic balance method. Substitution of Eq. (8) into Eq. (1) gives

$$-A\omega^2 \cos \omega t + \frac{1}{A \cos \omega t} = 0. \tag{17}$$

In order to apply the first-order harmonic balance method to Eq. (17) we have to expand Eq. (17) and set the coefficient of $\cos \omega t$ (the lowest order harmonic) equal to zero. For this, firstly we expand $1/A \cos \omega t$ as a Fourier series expansion:

$$\frac{1}{A \cos \omega t} = \sum_{n=0}^{\infty} a_{2n+1} \cos[(2n+1)\omega t] = a_1 \cos \omega t + a_3 \cos^3 \omega t + \cdots, \tag{18}$$

where the first term of this expansion can be obtained by means of the following equation:

$$a_1 = \frac{4}{\pi} \int_0^{\pi/2} \frac{1}{A \cos \theta} \cos \theta \, d\theta = \frac{2}{A}. \tag{19}$$

Substituting Eq. (18) into Eq. (17) and taking into account Eq. (19) gives

$$\left[-\omega^2 + \frac{2}{A} \right] \cos \omega t + (\text{higher-order harmonics}) = 0. \tag{20}$$

For the lowest order harmonic to be equal to zero, it is necessary to set the coefficient of $\cos wt$ equal to zero in Eq. (20), then

$$\omega_1(A) = \frac{\sqrt{2}}{A} = \frac{1.4142}{A}. \tag{21}$$

Consequently, in this limit, the low-order harmonic balance method applied to Eq. (1) gives exactly the same results as the low-order harmonic balance method applied to Eq. (7).

In order to obtain the next level of harmonic balance, we express the periodic solution to Eq. (1) with the assigned conditions in Eq. (2) in the form of [19–22]

$$y_2(t) = y_1(t) + u(t), \tag{22}$$

where $u(t)$ is the correction part, which is a periodic function of t of period $2\pi/\omega$ and

$$u(0) = 0, \quad \left(\frac{\mathrm{d}u}{\mathrm{d}t}\right)_{t=0} = 0. \tag{23}$$

Substituting Eq. (22) into Eq. (1) gives

$$\frac{\mathrm{d}^2 y_1}{\mathrm{d}t^2} + \frac{\mathrm{d}^2 u}{\mathrm{d}t^2} + \frac{1}{y_1(t) + u(t)} = 0. \tag{24}$$

Wu and Lim [19–21] presented an approach by combining the harmonic balance method and the linearization of nonlinear oscillation equation with respect to displacement increment only, $u(t)$. This harmonic balance approach will be used to approximately solve Eq. (24). Linearizing the governing equation (24) with respect to the correction $u(t)$ at $y_1(t)$ leads to

$$\frac{\mathrm{d}^2 y_1}{\mathrm{d}t^2} + \frac{\mathrm{d}^2 u}{\mathrm{d}t^2} + \frac{1}{y_1(t)} - \frac{u(t)}{y_1^2(t)} = 0 \tag{25}$$

and

$$u(0) = 0, \quad \left(\frac{\mathrm{d}u}{\mathrm{d}t}\right)_{t=0} = 0. \tag{26}$$

To obtain the second approximation to the solution, $u(t)$, in Eq. (22), which must satisfy the initial conditions in Eq. (26), we take into account the second-order harmonic balance approximation in Eq. (13), which can be written as follows:

$$y_2(t) = A_2 \cos \omega t + B_2 \cos^3 \omega t = A \cos \omega t - B_2 \cos \omega t + B_2 \cos^3 \omega t = A \cos \omega t + B_2(\cos^3 \omega t - \cos \omega t), \tag{27}$$

where we have taken into account that $A = A_2 + B_2$. From Eqs. (8), (22) and (27) we can see that $u(t)$ takes the form

$$u(t) = B_2(\cos^3 \omega t - \cos \omega t), \tag{28}$$

where B_2 is a constant to be determined.

Substituting Eqs. (22), (8) and (28) into Eq. (25), expanding the expression in a trigonometric series and setting the coefficients of the terms $\cos \omega t$ and $\cos 3\omega t$ equal to zero, respectively, leads to

$$-(A - B_2)\omega^2 + \frac{2}{A^2}(A + 2B_2) = 0 \tag{29}$$

and

$$\frac{-2A - 8B_2 - 9B_2 A^2 \omega^2}{A^2} = 0. \tag{30}$$

From Eqs. (29) and (30) we can obtain B_2 and ω_2 as follows:

$$B_{\mathrm{WL2}} = \frac{A}{14}(\sqrt{22} - 6) = -0.093542A, \tag{31}$$

$$\omega_{\mathrm{WL2}}(A) = \sqrt{\frac{2A + 8B_{\mathrm{WL2}}}{-9B_{\mathrm{WL2}} A^2}} = \frac{1.2193273}{A}. \tag{32}$$

With this value for B_2, Eq. (27) can be written as

$$y_{\mathrm{WL2}}(t) = \frac{20-\sqrt{22}}{14} A \cos(\omega_{\mathrm{WL2}} t) + \frac{\sqrt{22}-6}{14} A \cos(3\omega_{\mathrm{WL2}} t). \tag{33}$$

The percentage error for the second-order approximate frequency is

$$\text{percentage error} = \left| \frac{\omega_{\mathrm{ex}} - \omega_{\mathrm{WL2}}}{\omega_{\mathrm{ex}}} \right| \times 100 = 2.71\%, \tag{34}$$

which is higher than the percentage error for the second approximate frequency obtained by Mickens when the harmonic balance method is applied to Eq. (7).

Substitution of Eq. (22) into Eq. (1) does not give the same result as substitution of Eq. (8) into Eq. (7) and application of the second-order harmonic balance method to Eq. (7) gives a more accurate frequency than application of Wu and Lim's approach to Eq. (1). These questions have been analyzed in detail in Refs. [23,25] for other oscillators analyzed by the first-order harmonic balance method and one would wait to obtain better results when the harmonic balance method is applied to Eq. (1) instead of to Eq. (7). This would be due to the fact that when we substitute Eq. (13) into Eq. (7) we obtain an equation that includes only three even powers of $\cos\omega t$: 1 ($\cos^0\omega t$), $\cos^2\omega t$ and $\cos^4\omega t$ and then there are only three contributions to the coefficient of the first term 1 ($\cos^0\omega t$), from 1 ($\cos^0\omega t$), $\cos^2\omega t$ and $\cos^4\omega t$, and two contributions to the coefficient of the second harmonic $\cos(2\omega t)$, from $\cos^2\omega t$ and $\cos^4\omega t$, Therefore, substituting Eq. (13) into Eq. (7) produces only three terms, 1, $\cos(2\omega t)$ and $\cos(4\omega t)$. However, Eq. (17) includes all odd powers of $\cos\omega t$, which are $\cos^{2n+1}\omega t$ with $n = 0, 1, 2, \ldots, \infty$, and then there are infinite contributions to the coefficient of the first harmonic $\cos\omega t$, that is, 1 from $\cos\omega t$, 3/4 from $\cos^3\omega t$, 5/8 from $\cos^5\omega t, \ldots, 2^{-2n}\binom{2n+1}{n}$ from $\cos^{2n+1}\omega t$, and so on. Therefore, substituting Eq. (27) into Eq. (1) produces the infinite set of higher harmonics, $\cos\omega t$, $\cos 3\omega t, \ldots, \cos[(2n+1)\omega t]$, and so on, and the second-order angular frequency in Eq. (32) would have to be more accurate than the frequency given in Eq. (15). But we obtained the opposite result. The reason is that we have not applied the exact second-harmonic balance method to Eq. (1), but a linearized approximation to this method.

In order to verify this affirmation, we consider a new approach to obtain higher-order approximations using the harmonic balance method. Instead of considering the assumption in Eq. (25), first we use the following series expansion:

$$\frac{1}{y_1(t)+u(t)} = \frac{1}{y_1(t)[1+y_1^{-1}(t)u(t)]} = \sum_{n=0}^{\infty}(-1)^n \frac{u^n(t)}{y_1^{n+1}(t)} \tag{35}$$

and substituting Eq. (35) into Eq. (1) gives

$$\frac{\mathrm{d}^2 y_1}{\mathrm{d}t^2} + \frac{\mathrm{d}^2 u}{\mathrm{d}t^2} + \sum_{n=0}^{\infty} \frac{(-1)^n u^n(t)}{y_1^{n+1}(t)} = 0. \tag{36}$$

To obtain the second approximation to the solution, $u(t)$, in Eq. (22), which must satisfy the initial conditions in Eq. (23), takes the form

$$u(t) = B_2(\cos 3\omega t - \cos\omega t) = 4B_2(\cos^3\omega t - \cos\omega t), \tag{37}$$

where c_2 is a constant to be determined.

Substituting Eqs. (8), (22) and (37) into Eq. (36) gives

$$-\omega^2(A - B_2)\cos\omega t - 9\omega^2 B_2 \cos 3\omega t + \sum_{n=0}^{\infty} \frac{4^n B_2^n (1-\cos^2\omega t)^n}{A^{n+1}\cos\omega t} = 0. \tag{38}$$

The formula that allows us to obtain $(1-\cos^2\omega t)^n$ is

$$(1-\cos^2\omega t)^n = \sum_{k=0}^{n}\binom{n}{k}(-\cos^2\omega t)^k = \sum_{k=0}^{n}\frac{(-1)^k n!}{k!(n-k)!}\cos^{2k}\omega t. \tag{39}$$

Substituting Eq. (39) into Eq. (38) gives

$$-\omega^2(A-B_2)\cos\omega t+9\omega^2B_2\cos 3\omega t+\sum_{n=0}^{\infty}\frac{4^n n! B_2^n}{A^{n+1}}\sum_{k=0}^{n}\frac{(-1)^k}{k!(n-k)!}\cos^{2k-1}\omega t=0. \tag{40}$$

It is possible to state the following Fourier series expansion:

$$\cos^{2k-1}\omega t=\sum_{j=0}^{\infty}b_{2j+1}^{(k)}\cos[(2j+1)\omega t]=b_1^{(k)}\cos\omega t+b_3^{(k)}\cos 3\omega t+\cdots, \tag{41}$$

where

$$b_{2j+1}^{(k)}=\frac{4}{\pi}\int_0^{1/2}\cos^{2k-1}\theta\cos[(2j+1)\theta]\,\mathrm{d}\theta=\frac{2(k-1)!\Gamma(k+\frac{1}{2})}{\sqrt{\pi}(k-j-1)!(j+k)!} \tag{42}$$

and where $\Gamma(z)$ is the Euler gamma function [26].

Substituting Eqs. (41) and (42) into Eq. (40), we obtain

$$\begin{aligned}&-\omega^2(A-B_2)\cos\omega t-9\omega^2B_2\cos 3\omega t\\&+\sum_{n=0}^{\infty}\frac{4^n n! B_2^n}{A^{n+1}}\sum_{k=0}^{n}\frac{(-1)^k}{k!(n-k)!}\sum_{j=0}^{k}\frac{2(k-1)!\Gamma(k+\frac{1}{2})}{\sqrt{\pi}(k-j-1)!(j+k)!}\cos[(2j+1)\omega t]=0\end{aligned} \tag{43}$$

and setting the coefficients of the resulting items $\cos\omega t$ $(j=0)$ and $\cos\omega t$ $(j=1)$ equal to zero, respectively, yields

$$-\omega^2(A-B_2)+\sum_{n=0}^{\infty}\frac{4^n n! B_2^n}{A^{n+1}}\sum_{k=0}^{n}\frac{(-1)^k 2\Gamma(k+\frac{1}{2})}{\sqrt{\pi}(k!)^2(n-k)!}=0, \tag{44}$$

$$-9\omega^2B_2+\sum_{n=0}^{\infty}\frac{4^n n! B_2^n}{A^{n+1}}\sum_{k=0}^{n}\frac{(-1)^k 2(k-1)\Gamma(k+\frac{1}{2})}{\sqrt{\pi}k!(k+1)!(n-k)!}=0, \tag{45}$$

which can be written as follows:

$$-\omega^2(A-B_2)+\frac{2}{A}\sqrt{\frac{A}{A-4B_2}}=0, \tag{46}$$

$$-9\omega^2B_2+\frac{2A+2B_2-2\sqrt{A^2-4AB_2}}{B_2\sqrt{A^2-4AB_2}}=0. \tag{47}$$

In Eq. (47) the following relations have to be taken into account:

$$\sum_{k=0}^{n}\frac{(-1)^k 2\Gamma(k+\frac{1}{2})}{\sqrt{\pi}(k!)^2(n-k)!}=\frac{2\Gamma(k+\frac{1}{2})}{\sqrt{\pi}(n!)^2}, \tag{48}$$

$$\sum_{n=0}^{\infty}\frac{4^n n! B_2^n}{A^{n+1}}\frac{2\Gamma(k+\frac{1}{2})}{\sqrt{\pi}(n!)^2}=\frac{2}{A}\sqrt{\frac{A}{A-4B_2}}, \tag{49}$$

while in Eq. (48) the following expressions have been considered:

$$\sum_{k=0}^{n}\frac{(-1)^k 2(k-1)\Gamma(k+\frac{1}{2})}{\sqrt{\pi}k!(n-k)!(1+k)!}=-\frac{2[\Gamma(n+\frac{1}{2})n!+2(n-1)!\Gamma(n+\frac{3}{2})]}{\sqrt{\pi}(n+1)!n!(n-1)!}, \tag{50}$$

$$-\sum_{n=0}^{\infty}\frac{(-1)^n 4^n n! c_2^n}{A^{n+1}}\frac{2[\Gamma(n+\frac{1}{2})n!+2(n-1)!\Gamma(n+\frac{3}{2})]}{\sqrt{\pi}(n+1)!n!(n-1)!}=\frac{-2A+2B_2+2\sqrt{A^2-4AB_2}}{B_2\sqrt{A^2-4AB_2}}. \tag{51}$$

These results have been obtained using *Mathematica*®.

From Eqs. (46) and (47) and once again by using *Mathematica*® we can obtain ω_2 and B_2 as follows:

$$B_2 = -\frac{A}{50}\left[(2159+225\sqrt{106})^{1/3} - 6 - \frac{89}{(2159+225\sqrt{106})^{1/3}}\right] = 0.10158074A, \tag{52}$$

$$\omega_2(A) = \sqrt{\frac{2}{A(A-B_2)}\sqrt{\frac{A}{A-4AB_2}}} = \frac{1.237330058}{A}. \tag{53}$$

With this value for B_2, Eq. (22) can be written as

$$y_2(t) = 1.101581A\cos(\omega_2 t) - 0.101581A\cos(3\omega_2 t). \tag{54}$$

The percentage error for the second-order approximate frequency is

$$\text{percentage error} = \left|\frac{\omega_{\mathrm{ex}} - \omega_2}{\omega_{\mathrm{ex}}}\right| \times 100 = 1.275\%. \tag{55}$$

As we can see, this error is lower than the error obtained by Mickens (1.55%) and we can conclude this is the percentage error obtained when the second-order harmonic balance method is exactly applied to Eq. (1).

The second-order harmonic balance method was used to obtain three approximate frequencies for a nonlinear singular oscillator. The first approximate frequency, ω_{M2}, was obtained by rewriting the nonlinear differential equation in a form that does not contain the y^{-1} term, while the second and the third ones, ω_{WL2} and ω_2, were obtained by solving the nonlinear differential equation containing the y^{-1} term. The second-order approximate frequency ω_{WL2} is obtained by using the approach by Wu and Lim [19–21]. This approach can be described as a linearisation of the harmonic balance method, and works pretty well for the chosen problems. Because the harmonic balance method does not eliminate the secular terms systematically, it is difficult to obtain second- and higher-order approximate solutions by the harmonic balance method. But this approach eliminates this difficulty and may be applied to other nonlinear oscillators. We can conclude that Eqs. (52) and (53) are valid for the complete range of oscillation amplitude, including the limiting cases of amplitude approaching zero and infinity. Excellent agreement of the approximate frequencies with the exact value was demonstrated, and discussed, and the discrepancy between the third approximate frequency, ω_2, and the exact value never exceeds 1.28%. The approximate frequency, ω_2, derived here is the best frequency that can be obtained using the first-order harmonic balance method, and the maximum relative error was reduced as compared with the approximate frequencies ω_{M2} and ω_{WL2}. Finally, we discussed the reason as to why the accuracy of the approximate frequency, ω_2, is better than that of the frequency ω_{M2} obtained by Mickens. This reason is related to the number of harmonics that application of the second-order harmonic balance method produces for each differential equation solved.

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