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Additive resonances of a controlled van der Pol–Duffing oscillator

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Abstract

The trivial equilibrium of a controlled van der Pol–Duffing oscillator with nonlinear feedback control may lose its stability via a non-resonant interaction of two Hopf bifurcations when two critical time delays corresponding to two Hopf bifurcations have the same value. Such an interaction results in a non-resonant bifurcation of co-dimension two. In the vicinity of the non-resonant Hopf bifurcations, the presence of a periodic excitation in the controlled oscillator can induce three types of additive resonances in the forced response, when the frequency of the external excitation and the frequencies of the two Hopf bifurcations satisfy a certain relationship.

With the aid of centre manifold theorem and the method of multiple scales, three types of additive resonance responses of the controlled system are investigated by studying the possible solutions and their stability of the four-dimensional ordinary differential equations on the centre manifold. The amplitudes of the free-oscillation terms are found to admit three solutions; two non-trivial solutions and the trivial solution. Of two non-trivial solutions, one is stable and the trivial solution is unstable. A stable non-trivial solution corresponds to a quasi-periodic motion of the original system. It is also found that the frequency-response curves for three cases of additive resonances are an isolated closed curve. It is shown that the forced response of the oscillator may exhibit quasi-periodic motions on a three-dimensional torus consisting of three frequencies; the frequencies of two bifurcating solutions and the frequency of the excitation. Illustrative examples are given to show the quasi-periodic motions.

1. Introduction

In the absence of external excitation, the equilibrium of a controlled nonlinear system involving time delay may lose its stability and give rise to periodic solutions via Hopf bifurcations, when the time delay reaches certain values. In the neighbourhood of the critical point of Hopf bifurcations, an interaction of bifurcating periodic solutions and an external excitation may induce rich dynamic behaviour. The forced behaviour of the non-autonomous system may exhibit non-resonant response, primary resonances, sub-harmonic and super-harmonic resonances, depending on the relationship of the frequency of Hopf bifurcation and the forcing

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frequency. The effect of time delays on the stability and dynamics of time-delayed systems have received considerable interest in the literature [1–8]. However, there has been less effort made in studying an interaction of the external excitation and two bifurcating solutions, which result from non-resonant Hopf bifurcations of the corresponding autonomous systems. The main purpose of the present paper is to study an interaction of the forcing and bifurcating periodic solutions in the vicinity of non-resonant bifurcation of co-dimension two, which appears after the trivial solution of the van der Pol–Duffing oscillator loses its stability via non-resonant Hopf bifurcations.

An externally forced van der Pol–Duffing oscillator under a linear-plus-nonlinear feedback control considered in the present paper is of the form

$$\ddot{x} - (\mu - \beta x^2)\dot{x} + \omega^2 x + \alpha x^3 = e_0 \cos(\Omega_0 t) + px(t-\tau) + q\dot{x}(t-\tau) + k_1 x^3(t-\tau) + k_2 \dot{x}^3(t-\tau) + k_3 \dot{x}(t-\tau)x^2(t-\tau) + k_4 \dot{x}^2(t-\tau)x(t-\tau), \tag{1}$$

where x is the displacement, an overdot indicates the differentiation with respect to time t, ω is the natural frequency, α is the coefficient of the nonlinear term, μ and β are the linear and nonlinear damping coefficients with $\mu>0$, $\beta>0$, e_0 and Ω_0 represent the amplitude and frequency of the external excitation, p and q are the proportional and derivative linear feedback gains of a linear-plus-nonlinear feedback control scheme, k_i $(i = 1, 2, 3, 4)$ are the weakly nonlinear feedback gains, and τ denotes the time delay occurring in the feedback path. Only one time delay is considered here for simplicity.

The corresponding autonomous system for which the external excitation is neglected in Eq. (1), which can be obtained by letting $e_0 = 0$ in Eq. (1), is given by

$$\ddot{x} - \mu\dot{x} + \omega^2 x - px(t-\tau) - q\dot{x}(t-\tau) + \beta x^2\dot{x} + \alpha x^3 - k_1 x^3(t-\tau) - k_2\dot{x}^3(t-\tau) - k_3\dot{x}(t-\tau)x^2(t-\tau) - k_4\dot{x}^2(t-\tau)x(t-\tau) = 0. \tag{2}$$

It was shown that the trivial equilibrium of the autonomous system (2) may lose its stability via a subcritical or a supercritical Hopf bifurcation and regain its stability via a reverse subcritical or a supercritical Hopf bifurcation as the time delay increases [9]. It was found that an interaction of two Hopf bifurcations may occur when the two critical time delays corresponding to two Hopf bifurcations have the same value. In the vicinity of non-resonant Hopf bifurcations, the controlled oscillator modelled by Eq. (2) was found to have the initial equilibrium solution, two periodic solutions and a quasi-periodic solution on a two-dimensional (2D) torus [10].

The presence of an external periodic excitation can induce complicated dynamic behaviour of the controlled oscillator given by Eq. (1), which includes two types of primary resonances, two types of sub-harmonic resonances, two types of super-harmonic resonances, three types of additive resonances, and four types of difference resonances [11]. These resonances result from an interaction of the external excitation and the bifurcating periodic solutions that immediately follow the non-resonant Hopf bifurcations of co-dimension two occurring in the corresponding autonomous system given by Eq. (2).

By following the normal procedure for the reduction of delay differential equations to ordinary differential equations based on semigroups of transformations and the decomposition theory [12–14]; the dynamic behaviour of the solutions of Eq. (1) in the neighbourhood of non-resonant Hopf–Hopf interactions can be interpreted by the solutions and their stability of a set of four ordinary differential equations on the centre manifolds.

For simplicity, it is assumed that an intersection of non-resonant Hopf bifurcations occurs at the point (p_0, q_0, τ_0), where the corresponding characteristic equation of the autonomous system (2) has two pairs of purely imaginary roots $\pm i\delta_{01}$, $\pm i\delta_{02}$, and all other roots have negative real parts. In order to study the dynamics of the controlled oscillator in the neighbourhood of the bifurcation point (p_0, q_0, τ_0), three small perturbation parameters, namely α_1, α_2, and α_3, are introduced in Eq. (1) in terms of $p = p_0 + \alpha_1$, $q = q_0 + \alpha_2$, $\tau = \tau_0 + \alpha_3$. These perturbation parameters can conveniently account for the small variations of the critical linear feedback gains and the critical time delay.

By treating the external excitation in Eq. (1), as an additional perturbation term and performing similar algebraic manipulations to those done in Ref. [10], the four-dimensional (4D) ordinary differential equations governing the local flow on the centre manifold can be expressed in the component form as

$$\begin{aligned}
\dot{z}_1 &= l_{11}z_1 + (\delta_1 + l_{12})z_2 + l_{13}z_3 + l_{14}z_4 + f_{10}(z_1, z_2, z_3, z_4) + e_{10}\cos(\Omega t),\\
\dot{z}_2 &= (-\delta_1 + l_{21})z_1 + l_{22}z_2 + l_{23}z_3 + l_{24}z_4 + f_{20}(z_1, z_2, z_3, z_4) + e_{20}\cos(\Omega t),\\
\dot{z}_3 &= l_{31}z_1 + l_{32}z_2 + l_{33}z_3 + (\delta_2 + l_{34})z_4 + f_{30}(z_1, z_2, z_3, z_4) + e_{30}\cos(\Omega t),\\
\dot{z}_4 &= l_{41}z_1 + l_{42}z_2 + (-\delta_2 + l_{43})z_3 + l_{44}z_4 + f_{40}(z_1, z_2, z_3, z_4) + e_{40}\cos(\Omega t),
\end{aligned} \tag{3}$$

where δ_1 and δ_2 are the normalized frequencies of Hopf bifurcations which have been rescaled in the units of the critical time delay τ_0, $e_{10} = b_{12}e_0$, $e_{20} = b_{22}e_0$, $e_{30} = b_{32}e_0$, $e_{40} = b_{42}e_0$, the other coefficients and polynomial functions of order three $f_{i0}(z_1, z_2, z_3, z_4)$ (where $i = 1, 2, 3, 4$) involved in Eq. (3) are explicitly given in Section 3 of Ref. [10].

Depending on the relationship of the two natural frequencies δ_1 and δ_2 with the forcing frequency Ω, the nonlinear system given by Eq. (3) may exhibit either non-resonant or resonant response. Primary, sub-harmonic and super-harmonic resonances, additive and difference resonances may occur in the forced response. The non-resonant response and primary resonance response of the system have been studied in Ref. [11] using the method of multiple scales [15]. It was shown that the non-resonant response of the forced oscillator may exhibit quasi-periodic motions on a 2D or 3D torus. The resonant response may exhibit either periodic motion or quasi-periodic motions on a 2D torus. The present paper focuses on studying three types of additive resonances, when the forcing frequency is nearly equal to half the sum of the first and second natural frequencies (i.e., $\Omega \cong (\delta_1 + \delta_2)/2$); or the sum of the first frequency and twice the second frequency (i.e., $\Omega \cong \delta_1 + 2\delta_2$); or the sum of the second frequency and twice the first frequency (i.e., $\Omega \cong 2\delta_1 + \delta_2$).

A closed form of solutions to Eq. (3) cannot be found analytically. The approximate solutions to additive resonance response of Eq. (3) will be obtained using the method of multiple scales. The dynamic behaviour of the controlled system in the neighbourhood of the point of non-resonant bifurcations of co-dimension two will be explored by studying the solutions of a set of four averaged equations that determine the amplitudes and phases of the free-oscillation terms in additive resonance response.

It is assumed that the approximate solutions to Eq. (3) in the neighbourhood of the trivial equilibrium are represented by an expansion of the form

$$z_i(t;\varepsilon) = \varepsilon^{1/2} z_{i1}(T_0, T_1, \ldots) + \varepsilon^{3/2} z_{i2}(T_0, T_1, \ldots) + \cdots \quad (i = 1, 2, 3, 4), \tag{4}$$

where ε is a non-dimensional small parameter, and the new multiple independent variables of time are introduced according to $T_k = \varepsilon^k t$, $k = 0,1,2,\ldots$.

Substituting the approximate solutions (4) into Eq. (3) and then balancing the like powers of ε results in the following ordered perturbation equations:

$$\begin{aligned}
\varepsilon^{1/2}:\quad D_0 z_{11} &= \delta_1 z_{21} + e_1\cos(\Omega T_0),\\
D_0 z_{21} &= -\delta_1 z_{11} + e_2\cos(\Omega T_0),\\
D_0 z_{31} &= \delta_2 z_{41} + e_3\cos(\Omega T_0),\\
D_0 z_{41} &= -\delta_2 z_{31} + e_4\cos(\Omega T_0),
\end{aligned} \tag{5}$$

$$\begin{aligned}
\varepsilon^{3/2}:\quad D_0 z_{12} &= g_{11}(z_{j1}) + \delta_1 z_{22} - D_1 z_{11} + f_{11}(z_{j1}),\\
D_0 z_{22} &= g_{21}(z_{j1}) - \delta_1 z_{12} - D_1 z_{21} + f_{21}(z_{j1}),\\
D_0 z_{32} &= g_{31}(z_{j1}) + \delta_2 z_{42} - D_1 z_{31} + f_{31}(z_{j1}),\\
D_0 z_{42} &= g_{41}(z_{j1}) - \delta_2 z_{32} - D_1 z_{41} + f_{41}(z_{j1}),
\end{aligned} \tag{6}$$

where $D_0 = \partial/\partial T_0$, $D_0 = \partial/\partial T_1$, the coefficients of the perturbation linear terms l_{ij} in Eq. (3) have been rescaled in terms of $l_{ij} = \varepsilon \bar{l}_{ij}$ and the overbars in $\bar{l}_{ij}$ have been removed for brevity. The $g_{i1}(z_{i1})$ $(i = 1, 2, 3, 4)$ are linear functions of (z_{j1}) $(j = 1, 2, 3, 4)$ which are given by

$$g_{11}(z_{j1}) = l_{11}z_{11} + l_{12}z_{21} + l_{13}z_{31} + l_{14}z_{41}, \quad g_{21}(z_{j1}) = l_{21}z_{11} + l_{22}z_{21} + l_{23}z_{31} + l_{24}z_{41},$$

$$g_{31}(z_{j1}) = l_{31}z_{11} + l_{32}z_{21} + l_{33}z_{31} + l_{34}z_{41}, \quad g_{41}(z_{j1}) = l_{41}z_{11} + l_{42}z_{21} + l_{43}z_{31} + l_{44}z_{41}.$$

The $f_{i1}(z_{j1})$ denotes nonlinear functions of z_{j1} $(j = 1, 2, 3, 4)$ which have been solved from Eq. (5) and the amplitudes of the excitations in Eq. (3) have been rescaled in terms of $e_{10} = \varepsilon^{1/2}e_1$, $e_{20} = \varepsilon^{1/2}e_2$, $e_{30} = \varepsilon^{1/2}e_3$, ànd $e_{40} = \varepsilon^{1/2}e_4$.

The solutions to Eq. (5) can be written in a general form as

$$\begin{aligned} z_{11} &= r_1 \cos(\delta_1 T_0 + \phi_1) + A_1 \cos(\Omega T_0) + A_2 \sin(\Omega T_0), \\ z_{21} &= -r_1 \sin(\delta_1 T_0 + \phi_1) + B_1 \cos(\Omega T_0) + B_2 \sin(\Omega T_0), \\ z_{31} &= r_2 \cos(\delta_2 T_0 + \phi_2) + A_3 \cos(\Omega T_0) + A_4 \sin(\Omega T_0), \\ z_{41} &= -r_2 \sin(\delta_2 T_0 + \phi_2) + B_3 \cos(\Omega T_0) + B_4 \sin(\Omega T_0), \end{aligned} \tag{7}$$

where r_1, r_2, ϕ_1, ϕ_2 represent, respectively, the amplitudes and phases of the free-oscillation terms, and the coefficients of the particular solutions are given by

$$A_1 = \delta_1 e_2/(\delta_1^2 - \Omega^2), \quad A_2 = -\Omega e_1/(\delta_1^2 - \Omega^2), \quad B_1 = (\Omega A_2 - e)/\delta_1, \quad B_2 = -\Omega A_1/\delta_1,$$

$$A_3 = \delta_2 e_4/(\delta_2^2 - \Omega^2), \quad A_4 = -\Omega e_3/(\delta_2^2 - \Omega^2), \quad B_3 = (\Omega A_4 - e_3)/\delta_2, \quad B_4 = -\Omega A_3/\delta_2.$$

Differentiating the first and third equation of Eq. (6) and then substituting the second and fourth equation into the resultant equations results in

$$D_0^2 z_{12} + \delta_1^2 z_{12} = D_0 g_{11}(z_{j1}) - D_0 D_1 z_{11} + D_0 f_{11}(z_{j1}) + \delta_1 g_{21}(z_{j1}) - \delta_1 D_1 z_{21} + \delta_1 f_{21}(z_{j1}),$$

$$D_0^2 z_{32} + \delta_2^2 z_{32} = D_0 g_{31}(z_{j1}) - D_0 D_1 z_{31} + D_0 f_{31}(z_{j1}) + \delta_2 g_{41}(z_{j1}) - \delta_2 D_1 z_{41} + \delta_2 f_{41}(z_{j1}). \tag{8}$$

Substituting solutions given by Eq. (7) into the right-hand sides of Eq. (8) yields 44 terms involving trigonometric functions, some of which may produce secular or nearly secular terms in seeking the second-order solutions; z_{12} and z_{32}. In addition to four secular terms that are proportional to $\sin(\delta_1 t + \phi_1)$, $\cos(\delta_1 t + \phi_1)$, $\sin(\delta_2 t + \phi_2)$, and $\cos(\delta_2 t + \phi_2)$, nearly secular terms for additive resonances may appear when $\Omega \cong (\delta_1 + \delta_2)/2$ or $\Omega \cong 2\delta_1 + \delta_2$ or $\Omega \cong \delta_1 + 2\delta_2$. These three cases of additive resonances will be referred to here as Cases I, II, and III, respectively.

The remainder of the present paper proceeds as follows. In the next section, three types of additive resonance responses of the controlled system are analytically studied using the method of multiple scales. In Section 3, illustrative examples are given to show the frequency-response curves and time histories of additive resonance response of the controlled system. Conclusion is given in Section 4.

2. Additive resonances

To account for the nearness of the forcing frequency to the combination of two natural frequencies for three types of additive resonances, three detuning parameters, namely σ_1, σ_2, and σ_3, are introduced as follows:

$$2\Omega = \delta_1 + \delta_2 + \varepsilon\sigma_1, \tag{9}$$

$$\Omega = \delta_1 + 2\delta_2 + \varepsilon\sigma_2, \tag{10}$$

$$\Omega = 2\delta_1 + \delta_2 + \varepsilon\sigma_3. \tag{11}$$

The averaged equations of the amplitudes and phases for three types of additive resonances will be subsequently obtained using the method of multiple scales.

Case I: $2\Omega = \delta_1 + \delta_2 + \varepsilon\sigma_1$. In seeking the second-order solutions for additive resonance Case I from Eq. (8), the secular terms are the terms of trigonometric functions having the arguments $(\delta_1 t + \phi_1)$ and $(\delta_2 t + \phi_2)$, and the nearly secular terms are the trigonometric terms with the arguments $(2\Omega t - \delta_2 t - \phi_2)$ and $(2\Omega t - \delta_1 t - \phi_1)$.

Elimination of these secular or nearly secular terms gives rise to the following averaged equations that determine the amplitudes and phases of the free-oscillation terms in Eq. (7):

$$\begin{aligned}\dot{r}_1 &= -\mu_1 r_1 + s_{11} r_1^3 + s_{12} r_1 r_2^2 + s_{13} r_2 \cos(\gamma_{11}+\gamma_{12}) - s_{33} r_2 \sin(\gamma_{11}+\gamma_{12}),\\ r_1\dot{\gamma}_{11} &= (\tfrac{1}{2}\sigma_1 - \rho_1) r_1 - s_{31} r_1^3 - s_{32} r_1 r_2^2 - s_{13} r_2 \sin(\gamma_{11}+\gamma_{12}) - s_{33} r_2 \cos(\gamma_{11}+\gamma_{12}),\\ \dot{r}_2 &= -\mu_2 r_2 + s_{21} r_1^2 r_2 + s_{22} r_2^3 + s_{23} r_1 \cos(\gamma_{11}+\gamma_{12}) - s_{43} r_1 \sin(\gamma_{11}+\gamma_{12}),\\ r_2\dot{\gamma}_{12} &= (\tfrac{1}{2}\sigma_1 - \rho_2) r_2 - s_{41} r_1^2 r_2 - s_{42} r_2^3 - s_{23} r_1 \sin(\gamma_{11}+\gamma_{12}) - s_{43} r_1 \cos(\gamma_{11}+\gamma_{12}),\end{aligned} \tag{12}$$

where $\gamma_{11} = \frac{1}{2}\sigma_1 T_0 - \phi_{11}$, $\gamma_{12} = \frac{1}{2}\sigma_1 T_0 - \phi_{12}$,

$$s_{13} = \frac{1}{8\delta_1}(4\Omega a_{133} A_1 A_3 b_{12} - 2\Omega a_{134} A_2 A_3 b_{12} + 6\Omega a_{333} A_3^2 b_{12} + \cdots),$$

$$s_{23} = \frac{1}{8\delta_2}(4\Omega a_{113} A_1 A_3 b_{32} - 2\Omega a_{123} A_2 A_3 b_{32} + 2\Omega a_{133} A_3^2 b_{32} + \cdots),$$

$$s_{33} = \frac{1}{8\delta_1}(2\Omega a_{134} A_1 A_2 b_{12} + 4\Omega a_{133} A_2 A_3 b_{12} + 2\Omega a_{334} A_3^2 b_{12} + \cdots),$$

$$s_{43} = \frac{1}{8\delta_2}(-2\Omega a_{123} A_1 A_3 b_{32} - 4\Omega a_{113} A_2 A_3 b_{32} - 2\Omega a_{233} A_3^2 b_{32} + \cdots)$$

and the other coefficients, namely μ_1, ρ_1, s_{11}, s_{12}, s_{31}, s_{32}, μ_2, ρ_2, s_{21}, s_{22}, s_{41}, s_{42} have the same expressions as those obtained for non-resonant response in Section 4 of Ref. [11]. For the sake of brevity, they are not reproduced in the present paper.

Elimination of the trigonometric terms in Eq. (12) gives rise to the so-called frequency-response equations:

$$\begin{aligned} r_1^2(-\mu_1 + s_{11} r_1^2 + s_{12} r_2^2)^2 + r_1^2(\tfrac{1}{2}\sigma_1 - \rho_1 - s_{31} r_1^2 - s_{32} r_2^2)^2 - (s_{13}^2 + s_{33}^2) r_2^2 = 0,\\ r_2^2(-\mu_2 + s_{21} r_1^2 + s_{22} r_2^2)^2 + r_2^2(\tfrac{1}{2}\sigma_1 - \rho_2 - s_{41} r_1^2 - s_{42} r_2^2)^2 - (s_{23}^2 + s_{43}^2) r_1^2 = 0.\end{aligned} \tag{13}$$

The coefficients in Eq. (13) can be numerically obtained for a specific system with a given set of system parameters. Then Eq. (13) can be numerically solved using the Newton–Raphson procedure. The stability of the steady-state solutions to Eq. (13) can be examined by computing the eigenvalues of the coefficient matrix of characteristic equations, which are derived from Eq. (12) in terms of small disturbances to the steady-state solutions.

As the averaged Eq. (12) involves the coupled terms $r_1\dot{\gamma}_{11}$ and $r_2\dot{\gamma}_{11}$, the perturbation equations will not contain the perturbations $\Delta\dot{\gamma}_{11}$ and $\Delta\dot{\gamma}_{12}$ for the trivial solution and hence the stability of the trivial solution cannot be studied by directly perturbing Eq. (12). To overcome this difficulty, normalization method is used by introducing the transformation $p_{11} = r_1 \cos\gamma_{11}$, $q_{11} = r_1 \sin\gamma_{11}$, $p_{12} = r_2\cos\gamma_{12}$, $q_{12} = r_2 \sin\gamma_{12}$, into Eq. (12). Performing trigonometric manipulations leads to the following modulation equations in the Cartesian form:

$$\begin{aligned}\dot{p}_{11} = &-\mu_1 p_{11} + (\rho_1 - \tfrac{1}{2}\sigma_1) q_{11} + s_{13} p_{12} - s_{33} q_{12} + s_{11} p_{11}(p_{11}^2 + q_{11}^2) + s_{12} p_{11}(p_{12}^2 + q_{12}^2)\\ &+ s_{31} q_{11}(p_{11}^2 + q_{11}^2) + s_{32} q_{11}(p_{12}^2 + q_{12}^2),\end{aligned}$$

$$\begin{aligned}\dot{q}_{11} = &-\mu_1 q_{11} - (\rho_1 - \tfrac{1}{2}\sigma_1) p_{11} - s_{33} p_{12} - s_{13} q_{12} + s_{11} q_{11}(p_{11}^2 + q_{11}^2) + s_{12} q_{11}(p_{12}^2 + q_{12}^2)\\ &- s_{31} p_{11}(p_{11}^2 + q_{11}^2) - s_{32} p_{11}(p_{12}^2 + q_{12}^2),\end{aligned}$$

$$\begin{aligned}\dot{p}_{12} = &-\mu_2 p_{12} + (\rho_2 - \tfrac{1}{2}\sigma_1) q_{12} + s_{23} p_{11} - s_{43} q_{11} + s_{21} p_{12}(p_{11}^2 + q_{11}^2) + s_{22} p_{12}(p_{12}^2 + q_{12}^2)\\ &+ s_{41} q_{12}(p_{11}^2 + q_{11}^2) + s_{42} q_{12}(p_{12}^2 + q_{12}^2),\end{aligned}$$

$$\dot{q}_{12} = -\mu_2 q_{12} - (\rho_2 - \tfrac{1}{2}\sigma_1)p_{12} - s_{43}p_{11} - s_{23}q_{11} + s_{21}q_{12}(p_{11}^2 + q_{11}^2) + s_{22}q_{12}(p_{12}^2 + q_{12}^2) - s_{41}p_{12}(p_{11}^2 + q_{11}^2) - s_{42}p_{12}(p_{12}^2 + q_{12}^2). \tag{14}$$

The stability of the steady-state solutions is determined by the eigenvalues of the corresponding Jacobian matrix of Eq. (14). The resultant characteristic equations for both trivial and non-trivial solutions depend in a complicated manner on the system and forcing parameters. Specific results are therefore at best obtained numerically.

Case II: $\Omega = \delta_1 + 2\delta_2 + \varepsilon\sigma_2$. Secular and nearly secular terms for additive resonance Case II are terms that are proportional to the trigonometric terms with the arguments of $(\delta_1 t + \phi_1)$, $(\delta_2 t + \phi_2)$, $(\Omega t - 2\delta_2 t - 2\phi_2)$ and $(\Omega t - \delta_1 t - \delta_2 t - \phi_1 - \phi_2)$. Eliminating these secular and nearly secular terms in seeking the second-order solutions yields the averaged equations for additive resonance Case II:

$$\begin{aligned}
\dot{r}_1 &= -\mu_1 r_1 + s_{11}r_1^3 + s_{12}r_1 r_2^2 + s_{14}r_2^2\cos(\gamma_{21} + 2\gamma_{22}) - s_{34}r_2^2\sin(\gamma_{21} + 2\gamma_{22}),\\
r_1\dot{\gamma}_{21} &= (\tfrac{1}{3}\sigma_2 - \rho_1)r_1 - s_{31}r_1^3 - s_{32}r_1 r_2^2 - s_{14}r_2^2\sin(\gamma_{21} + 2\gamma_{22}) - s_{34}r_2^2\cos(\gamma_{21} + 2\gamma_{22}),\\
\dot{r}_2 &= -\mu_2 r_2 + s_{21}r_1^2 r_2 + s_{22}r_2^3 + s_{24}r_1 r_2\cos(\gamma_{21} + 2\gamma_{22}) - s_{44}r_1 r_2\sin(\gamma_{21} + 2\gamma_{22}),\\
r_2\dot{\gamma}_{22} &= (\tfrac{1}{3}\sigma_2 - \rho_2)r_2 - s_{41}r_1^2 r_2 - s_{42}r_2^3 - s_{24}r_1 r_2\sin(\gamma_{21} + 2\gamma_{22}) - s_{44}r_1 r_2\cos(\gamma_{21} + 2\gamma_{22}),
\end{aligned} \tag{15}$$

where $\gamma_{21} = \frac{1}{3}\sigma_2 T_0 - \phi_{21}$, $\gamma_{22} = \frac{1}{3}\sigma_2 T_0 - \phi_{22}$,

$$\begin{aligned}
s_{14} &= \frac{1}{8\delta_1}(-\Omega a_{134}A_2 b_{12} + 3\Omega a_{333}A_3 b_{12} - \Omega a_{344}A_3 b_{12} + \cdots),\\
s_{24} &= \frac{1}{8\delta_2}(2\Omega a_{133}A_3 b_{32} - 2\Omega a_{114}A_2 b_{32} - \Omega a_{123}A_2 b_{32} + \cdots),\\
s_{34} &= \frac{1}{8\delta_1}(\Omega a_{144}A_2 b_{12} - 2\Omega a_{334}A_3 b_{12} - 3\Omega a_{333}A_4 b_{12} + \cdots),\\
s_{44} &= \frac{1}{8\delta_2}(\Omega a_{124}A_2 b_{32} - 2\Omega a_{113}A_2 b_{32} - \Omega a_{134}A_3 b_{32} + \cdots)
\end{aligned}$$

and the other coefficients have the same expressions as those given in Eq. (12).

The so-called frequency-response equations are given by

$$\begin{aligned}
&r_1^2(-\mu_1 + s_{11}r_1^2 + s_{12}r_2^2)^2 + r_1^2(\tfrac{1}{3}\sigma_2 - \rho_1 - s_{31}r_1^2 - s_{32}r_2^2)^2 - (s_{14}^2 + s_{34}^2)r_2^2 r_2^2 = 0,\\
&r_2^2(-\mu_2 + s_{21}r_1^2 + s_{22}r_2^2)^2 + r_2^2(\tfrac{1}{3}\sigma_2 - \rho_2 - s_{41}r_1^2 - s_{42}r_2^2)^2 - (s_{24}^2 + s_{44}^2)r_1^2 r_2^2 = 0.
\end{aligned} \tag{16}$$

Introducing the transformation $p_{21} = r_1\cos\gamma_{21}$, $q_{21} = r_1\sin\gamma_{21}$, $p_{22} = r_2\cos\gamma_{22}$, $q_{22} = r_2\sin\gamma_{22}$, into Eq. (15), and performing trigonometric manipulations leads to the following modulation equations in the Cartesian form

$$\begin{aligned}
\dot{p}_{21} = &-\mu_1 p_{21} + (\rho_1 - \tfrac{1}{3}\sigma_2)q_{21} + s_{11}p_{21}(p_{21}^2 + q_{21}^2) + s_{12}p_{21}(p_{22}^2 + q_{22}^2) + s_{31}q_{21}(p_{21}^2 + q_{21}^2)\\
&+ s_{32}q_{21}(p_{22}^2 + q_{22}^2) + s_{14}(p_{22}^2 - q_{22}^2) - 2s_{34}p_{22}q_{22},\\
\dot{q}_{21} = &-\mu_1 q_{21} - (\rho_1 - \tfrac{1}{3}\sigma_2)p_{21} + s_{11}q_{21}(p_{21}^2 + q_{21}^2) + s_{12}q_{21}(p_{22}^2 + q_{22}^2) - s_{31}p_{21}(p_{21}^2 + q_{21}^2)\\
&- s_{32}p_{21}(p_{22}^2 + q_{22}^2) - s_{34}(p_{22}^2 - q_{22}^2) - 2s_{14}p_{22}q_{22},\\
\dot{p}_{22} = &-\mu_2 p_{22} + (\rho_2 - \tfrac{1}{3}\sigma_2)q_{22} + s_{21}p_{22}(p_{21}^2 + q_{21}^2) + s_{22}p_{22}(p_{22}^2 + q_{22}^2) + s_{41}q_{22}(p_{21}^2 + q_{21}^2)\\
&+ s_{42}q_{22}(p_{22}^2 + q_{22}^2) - s_{24}(p_{21}q_{22} - q_{21}q_{22}) - s_{44}(p_{21}q_{22} + p_{22}q_{21}),\\
\dot{q}_{22} = &-\mu_2 q_{22} - (\rho_2 - \tfrac{1}{3}\sigma_2)p_{22} + s_{21}q_{22}(p_{21}^2 + q_{21}^2) + s_{22}q_{22}(p_{22}^2 + q_{22}^2) - s_{41}p_{22}(p_{21}^2 + q_{21}^2)\\
&- s_{42}p_{22}(p_{22}^2 + q_{22}^2) - s_{24}(p_{22}q_{21} + p_{21}q_{22}) - s_{44}(p_{21}p_{22} - q_{21}q_{22}).
\end{aligned} \tag{17}$$

The eigenvalues of the corresponding Jacobian matrix of Eq. (17) determine the stability of the steady-state solutions. The stability of the trivial solution is determined by the eigenvalues of the corresponding Jacobian matrix which is obtained by letting $p_{21} = q_{21} = p_{22} = q_{22} = 0$. The four eigenvalues for the trivial solutions are given by $-\mu_1 \pm ib_2$ and $-\mu_2 \pm ib_2$, where i is the imaginary unit, $b_2 = |\rho_2 - \frac{1}{3}\sigma_2|$. It is easy to note that the trivial solution is asymptotically stable if both $\mu_1 > 0$ and $\mu_2 > 0$.

Case III: $\Omega = 2\delta_1 + \delta_2 + \varepsilon\sigma_3$. In this case, the terms that produce secular or nearly secular terms are the trigonometric functions with the arguments of $(\delta_1 t + \phi_1)$, $(\delta_2 t + \phi_2)$, $(\Omega t - \delta_1 t - \delta_2 t - \phi_1 - \phi_2)$, and $(\Omega t - 2\delta_1 t - 2\phi_1)$ in the right-hand sides of the equation, which is obtained by substituting solution (7) in Eq. (8). Eliminating these terms gives rise to the averaged equations that determine the amplitudes and phases of the free-oscillation terms for additive resonance Case III:

$$\begin{aligned}
\dot{r}_1 &= -\mu_1 r_1 + s_{11} r_1^3 + s_{12} r_1 r_2^2 + s_{15} r_1 r_2 \cos(2\gamma_{31} + \gamma_{32}) - s_{35} r_1 r_2 \sin(2\gamma_{31} + \gamma_{32}), \\
r_1 \dot{\gamma}_{31} &= (\tfrac{1}{3}\sigma_3 - \rho_1) r_1 - s_{31} r_1^3 - s_{32} r_1 r_2^2 - s_{15} r_1 r_2 \sin(2\gamma_{31} + \gamma_{32}) - s_{35} r_1 r_2 \cos(2\gamma_{31} + \gamma_{32}), \\
\dot{r}_2 &= -\mu_2 r_2 + s_{21} r_1^2 r_2 + s_{22} r_2^3 + s_{25} r_1^2 \cos(2\gamma_{31} + \gamma_{32}) - s_{45} r_1^2 \sin(2\gamma_{31} + \gamma_{32}), \\
r_2 \dot{\gamma}_{32} &= (\tfrac{1}{3}\sigma_2 - \rho_2) r_2 - s_{41} r_1^2 r_2 - s_{42} r_2^3 - s_{25} r_1^2 \sin(2\gamma_{31} + \gamma_{22}) - s_{45} r_1^2 \cos(2\gamma_{31} + \gamma_{32}),
\end{aligned} \tag{18}$$

where $\gamma_{31} = \frac{1}{3}\sigma_3 T_0 - \phi_{31}$, $\gamma_{32} = \frac{1}{3}\sigma_3 T_0 - \phi_{32}$,

$$s_{15} = \frac{1}{8\delta_1}(2\Omega a_{133} A_3 b_{12} - 2\Omega a_{114} A_2 b_{12} - \Omega a_{123} A_2 b_{12} + \cdots),$$

$$s_{25} = \frac{1}{8\delta_2}(\Omega a_{113} A_3 b_{32} - 2\Omega a_{112} A_2 b_{32} - \Omega a_{223} A_3 b_{32} + \cdots),$$

$$s_{35} = \frac{1}{8\delta_1}(\Omega a_{124} A_2 b_{12} - 2\Omega a_{113} A_2 b_{12} - \Omega a_{134} A_3 b_{12} + \cdots),$$

$$s_{45} = \frac{1}{8\delta_2}(\Omega a_{122} A_2 b_{32} - \Omega a_{123} A_3 b_{32} - \Omega a_{113} A_4 b_{32} + \cdots)$$

and the other coefficients have the same expressions as those given in Eq. (12).

The so-called frequency-response equations are then given by

$$\begin{aligned}
&r_1^2(-\mu_1 + s_{11} r_1^2 + s_{12} r_2^2)^2 + r_1^2(\tfrac{1}{3}\sigma_3 - \rho_1 - s_{31} r_1^2 - s_{32} r_2^2)^2 - (s_{15}^2 + s_{35}^2) r_1^2 r_2^2 = 0, \\
&r_2^2(-\mu_2 + s_{21} r_1^2 + s_{22} r_2^2)^2 + r_2^2(\tfrac{1}{3}\sigma_3 - \rho_2 - s_{41} r_1^2 - s_{42} r_2^2)^2 - (s_{25}^2 + s_{45}^2) r_1^2 r_1^2 = 0.
\end{aligned} \tag{19}$$

Introducing the transformation $p_{31} = r_1 \cos\gamma_{31}$, $q_{31} = r_1 \sin\gamma_{31}$, $p_{32} = r_2 \cos\gamma_{32}$, $q_{32} = r_2 \sin\gamma_{32}$, into Eq. (18), and performing algebraic manipulations leads to the following modulation equations in the Cartesian form

$$\begin{aligned}
\dot{p}_{31} =& -\mu_1 p_{31} + (\rho_1 - \tfrac{1}{3}\sigma_3) q_{31} + s_{11} p_{31}(p_{31}^2 + q_{31}^2) + s_{12} p_{31}(p_{32}^2 + q_{32}^2) + s_{31} q_{31}(p_{31}^2 + q_{31}^2) \\
&+ s_{32} q_{31}(p_{32}^2 + q_{32}^2) + s_{15}(p_{31} p_{32} - q_{31} q_{32}) - s_{35}(p_{31} q_{32} + p_{32} q_{31}), \\
\dot{q}_{31} =& -\mu_1 q_{31} - (\rho_1 - \tfrac{1}{3}\sigma_3) p_{31} + s_{11} q_{31}(p_{31}^2 + q_{31}^2) + s_{12} q_{31}(p_{32}^2 + q_{32}^2) - s_{31} p_{31}(p_{31}^2 + q_{31}^2) \\
&- s_{32} p_{31}(p_{32}^2 + q_{32}^2) - s_{15}(p_{31} q_{32} - p_{32} q_{31}) - s_{35}(p_{31} p_{32} - q_{31} q_{32}), \\
\dot{p}_{32} =& -\mu_2 p_{32} + (\rho_2 - \tfrac{1}{3}\sigma_3) q_{32} + s_{21} p_{32}(p_{31}^2 + q_{31}^2) + s_{22} p_{32}(p_{32}^2 + q_{32}^2) + s_{41} q_{32}(p_{31}^2 + q_{31}^2) \\
&+ s_{42} q_{32}(p_{32}^2 + q_{32}^2) + s_{25}(p_{31}^2 - q_{31}^2) - 2 s_{45} p_{31} q_{31}, \\
\dot{q}_{32} =& -\mu_2 q_{32} - (\rho_2 - \tfrac{1}{3}\sigma_3) p_{32} + s_{21} q_{32}(p_{31}^2 + q_{31}^2) + s_{22} q_{32}(p_{32}^2 + q_{32}^2) - s_{41} p_{32}(p_{31}^2 + q_{31}^2) \\
&- s_{42} p_{32}(p_{32}^2 + q_{32}^2) - s_{45}(p_{31}^2 - q_{31}^2) - 2 s_{25} p_{31} q_{31}.
\end{aligned} \tag{20}$$

It is easy to notice that the stability of the trivial solution is determined by the following eigenvalues: $-\mu_1 \pm ib_3$ and $-\mu_2 \pm ib_3$, where i is the imaginary unit, $b_3 = |\rho_2 - \frac{1}{3}\sigma_3|$. Therefore, the trivial solution is asymptotically stable if both $\mu_1 > 0$ and $\mu_2 > 0$.

The averaged equations for three cases of additive resonances have been obtained analytically. In the next section, the steady-state solutions and their stability of the averaged equations will be studied by illustrative examples for a given set of the system parameters.

3. Numerical examples

This section gives numerical results of the dynamic behaviour of additive resonance response of the system. As an illustrative example, consider a specific system with the system parameters in Eq. (3) given by $\mu = 0.1$, $\omega = 1.0$, $p = -0.4$, $q = -0.40219$, $\alpha = 0.4$, $\beta = 0.5$, $k_1 = 0.2$, $k_4 = 0.5$, and $k_2 = k_3 = 0.0$. It was found that an interaction of non-resonant Hopf bifurcations occurs when $\tau = 5.46397$, where the frequencies of the two bifurcations are $\delta_{01} = 1.28038$ and $\delta_{02} = 0.71582$. It is easy to find from Eqs. (9) to (11) that for this specific system, the combinational resonances of the additive type may appear in a certain interval of a small neighbourhood of the forcing frequencies at $\Omega_0 = 0.998096$ for Case I; $\Omega_0 = 2.71201$ for Case II; and $\Omega_0 = 3.27657$ for Case III.

When $\alpha_1 = 0.001$, $\alpha_2 = -0.0053$, $e_0 = 0.12$, $\Omega_0 = 0.998$, the averaged equations for additive resonance Case I, which are obtained by substituting the numerical values of the corresponding coefficients in Eq. (12), are found to be

$$\begin{aligned}
\dot{r}_1 &= 0.02102 r_1 - 0.2335 r_1^3 - 0.0478 r_1 r_2^2 - 0.016089 r_2 \cos(\gamma_{11} + \gamma_{12}) \\
&\quad + 0.02052 r_2 \sin(\gamma_{11} + \gamma_{12}), \\
r_1 \dot{\gamma}_{11} &= (\tfrac{1}{2}\sigma_1 - 0.099652) r_1 + 0.5545 r_1^3 + 1.0091 r_1 r_2^2 + 0.016089 r_2 \sin(\gamma_{11} + \gamma_{12}) \\
&\quad + 0.02052 r_2 \cos(\gamma_{11} + \gamma_{12}), \\
\dot{r}_2 &= -0.0473715 r_2 - 1.5127 r_1^2 r_2 - 0.6932 r_2^3 + 0.03968 r_1 \cos(\gamma_{11} + \gamma_{12}) \\
&\quad + 0.022477 r_1 \sin(\gamma_{11} + \gamma_{12}), \\
r_2 \dot{\gamma}_{12} &= (\tfrac{1}{2}\sigma_1 - 0.059653) r_2 + 1.7226 r_1^2 r_2 + 0.5886 r_2^3 - 0.03968 r_1 \sin(\gamma_{11} + \gamma_{12}) \\
&\quad + 0.022477 r_1 \cos(\gamma_{11} + \gamma_{12}).
\end{aligned} \tag{21}$$

The steady-state solutions can be obtained by setting $\dot{r}_1$, $r_1\dot{\gamma}_{11}$, $\dot{r}_2$, and $r_2\dot{\gamma}_{12}$ in Eq. (21) equal to zero, and numerically solving the resulting algebraic equations. The stability of the solutions can be examined by finding the corresponding eigenvalues of the Jacobian matrix. Because two frequencies δ_1 and δ_2 are incommensurable, the steady-state solutions of Eq. (21) correspond to quasi-periodic motions of the nonlinear system (3) under additive resonance Case I.

Figs. 1(a) and (b) show typical frequency-response curves for additive resonance Case I in the interval $\sigma_1 \in [0.0483, 0.06101]$. Stable solutions are represented by solid lines and unstable solutions by dashed lines. Solid and dashed lines will also be used to denote stable solutions and unstable solutions shown in frequency-response curves for Cases II and III. The frequency-response curve is an isolated closed curve in Figs. 1(a) and (b). The non-trivial solutions do not exist outside this closed curve. The closed curve consists of two branches. The upper branch of r_1 is stable with the four eigenvalues having negative real part. The lower branch of r_1 is unstable with the eigenvalues having positive real part. On the contrary, the upper branch of r_2 is unstable and the lower branch of r_2 is stable. The isolated two non-trivial solutions coexist with the trivial solution that is unstable in the interval σ_1.

As σ_1 increases from a small value, the amplitude r_1 of the free-oscillation term with natural frequency δ_1 grows, whereas the amplitude r_2 of the free-oscillation term having natural frequency δ_2 decreases. The amplitude r_1 is much higher than amplitude r_2, which means the motion corresponding to δ_1 dominates while the motion relating to δ_2 is smaller in amplitude but not negligible. Stable non-trivial solutions indicate that nonlinear response of the system under additive resonances comprises both the free-oscillation terms and forced terms as given in Eq. (7). Because the natural frequencies δ_1 and δ_2 are incommensurable, the system response under additive resonance Case I will exhibit quasi-periodic motion. Figs. 1(c) and (d) show the time history and phase portrait of quasi-periodic motion of the system for $\Omega_0 = 1.025$. A frequency spectral analysis has confirmed that the motion consists of three harmonic components, which are the natural

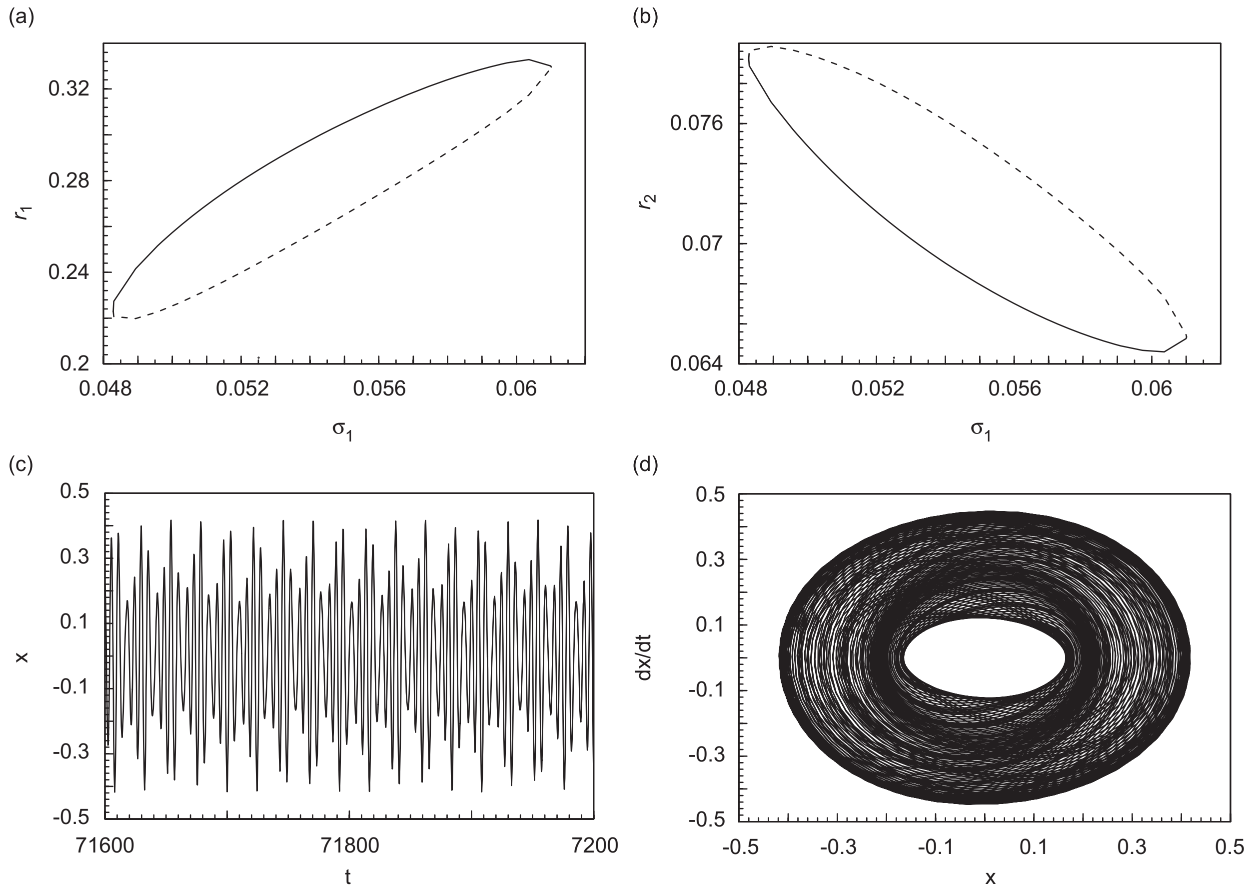

Fig. 1. *Case* I: (a) frequency-response curve of r_1 for $\Omega_0 = 0.998$; (b) frequency-response curve of r_2 for $\Omega_0 = 0.998$; (c) time history of the quasi-periodic motion at $\Omega_0 = 1.025$; (d) phase portrait of the quasi-periodic motion at $\Omega_0 = 1.025$. Solid lines in (a) and (b) denote stable steady-state solutions and dashed lines denote unstable solutions.

frequencies δ_1 and δ_2, and the frequency of the excitation Ω satisfying relationship (9). The numerical result on the existence of quasi-periodic motion is in good agreement with the analytical prediction.

Numerical simulations have also been performed for additive resonance Cases II and III. The values of system parameters and dummy parameters remained unchanged as those for Case I except the frequency of the excitation. Topologically equivalent response curves have been obtained for Case II. Figs. 2(a) and (b) show frequency-response curves for Case II in the interval $\sigma_2 \in [0.04924, 0.0857233]$. Figs. 2(c) and (d) show a quasi-periodic motion at $\Omega_0 = 2.712$. Figs. 2(a) and (b) indicate that the motion relating to δ_2 dominates while the motion corresponding to δ_1 is smaller in amplitude. The quasi-periodic motion shown in Fig. 2(c) is a quasi-periodic motion on a three-dimensional (3D) torus and contains three individual harmonic components: δ_1, δ_2, and Ω_0. These three frequencies satisfy the additive resonance condition that is given by Eq. (10).

For additive resonance Case III, numerical simulation results are shown in Fig. 3, where Figs. 3(a) and (b) show frequency-response curves in the interval $\sigma_3 \in [0.0758735, 0.095176]$. Figs. 3(c) and (d) show a quasi-periodic motion at $\Omega_0 = 3.278$. It is noted in Figs. 3(a) and (b) that the amplitudes of two motions relating to δ_1 and δ_2 are comparable. The meeting points of stable and unstable branches were found to have saddle-node bifurcations. As the detuning σ_3 increases, the amplitude r_2 increases until it reaches the maximum and then decreases, whereas the amplitude r_1 increases until it meets the saddle-node point. This indicates that vibrational energy is transferred between these two motions via additive resonances. The quasi-periodic motion shown in Fig. 3(c) involves three harmonic components: δ_1, δ_2, and Ω_0, which satisfy the resonance condition given by Eq. (11).

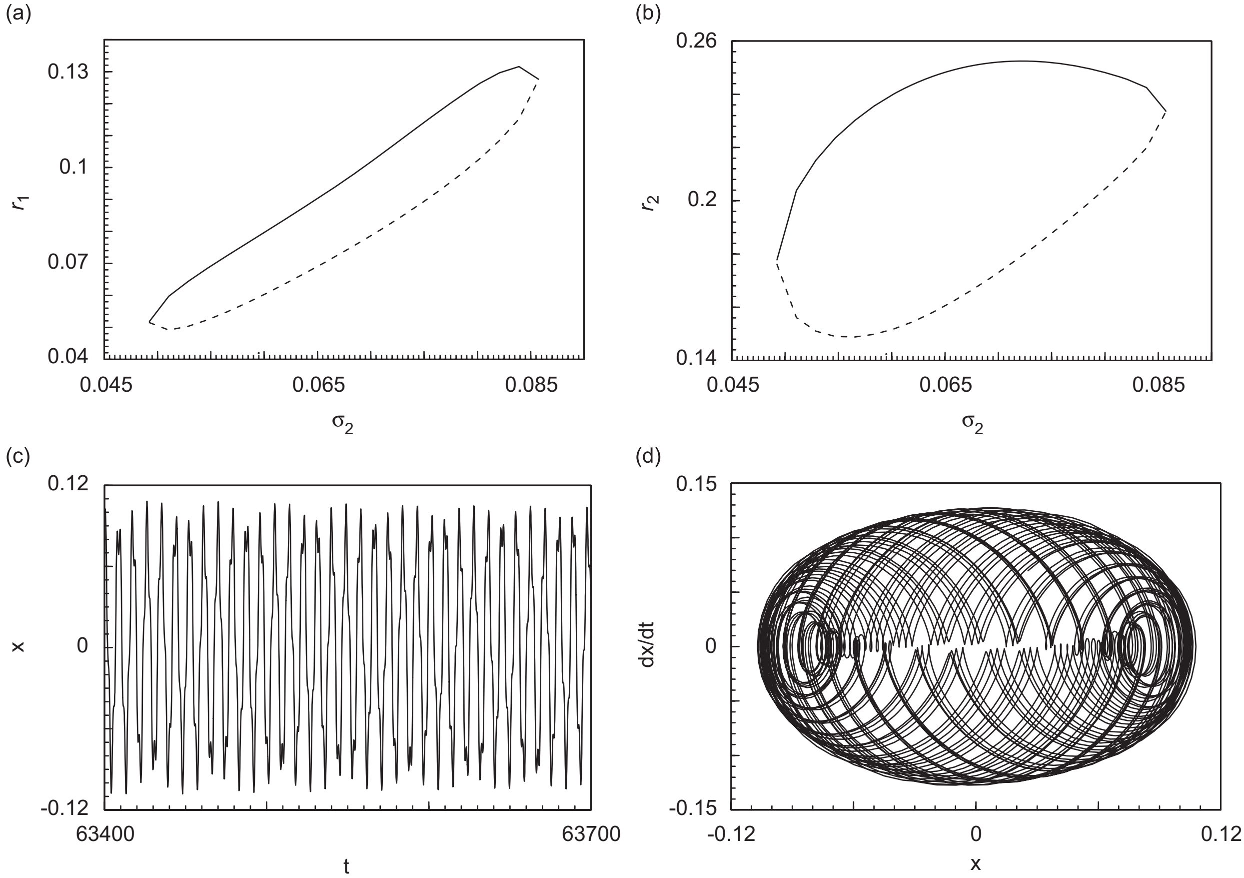

Fig. 2. *Case* II: (a) frequency-response curve of r_1 for $\Omega_0 = 2.71$; (b) frequency-response curve of r_2 for $\Omega_0 = 2.71$; (c) time history of the quasi-periodic motion at $\Omega_0 = 2.712$; (d) phase portrait of the quasi-periodic motion at $\Omega_0 = 2.712$. Solid lines in (a) and (b) denote stable steady-state solutions and dashed lines denote unstable solutions.

When the combinational resonances of the additive types are not excited, the nonlinear response of the system was numerically found to exhibit periodic motion, which consists of the forced terms only and has the forcing frequency.

4. Conclusion

In a controlled van der Pol–Duffing oscillator without external excitation, periodic or quasi-periodic solutions may appear after the trivial equilibrium loses its stability via a non-resonant interaction of Hopf–Hopf bifurcations. Presence of a periodic external excitation in the controlled van der Pol–Duffing oscillator at non-resonant bifurcations of co-dimension two can induce three types of additive resonances as a result of interactions of the bifurcating solutions and the periodic excitation. Such interactions between the bifurcating solutions and the excitation could lead to very interesting phenomena. Particularly, when their natural frequencies are incommensurable, interactions may produce quasi-periodic motions.

The method of multiple scales has been used to study the steady-state solutions of a system of coupled, weakly 4D nonlinear differential equations, which represents the local flow on the centre manifold in the neighbourhood of non-resonant bifurcations of co-dimension two occurring in the controlled oscillator Three cases are considered for additive resonances: $2\Omega = \delta_1 + \delta_2 + \varepsilon\sigma_1$; $\Omega = \delta_1 + 2\delta_2 + \varepsilon\sigma_2$; and $\Omega = \delta_1 + \delta_2 + \varepsilon\sigma_3$.

If the additive resonances are excited, the amplitudes of the free-oscillation terms admit three solutions; two non-trivial solutions and the trivial solution. Of two non-trivial solutions one is stable. The trivial solution is unstable. A stable non-trivial solution corresponds to a quasi-periodic motion of the original system.

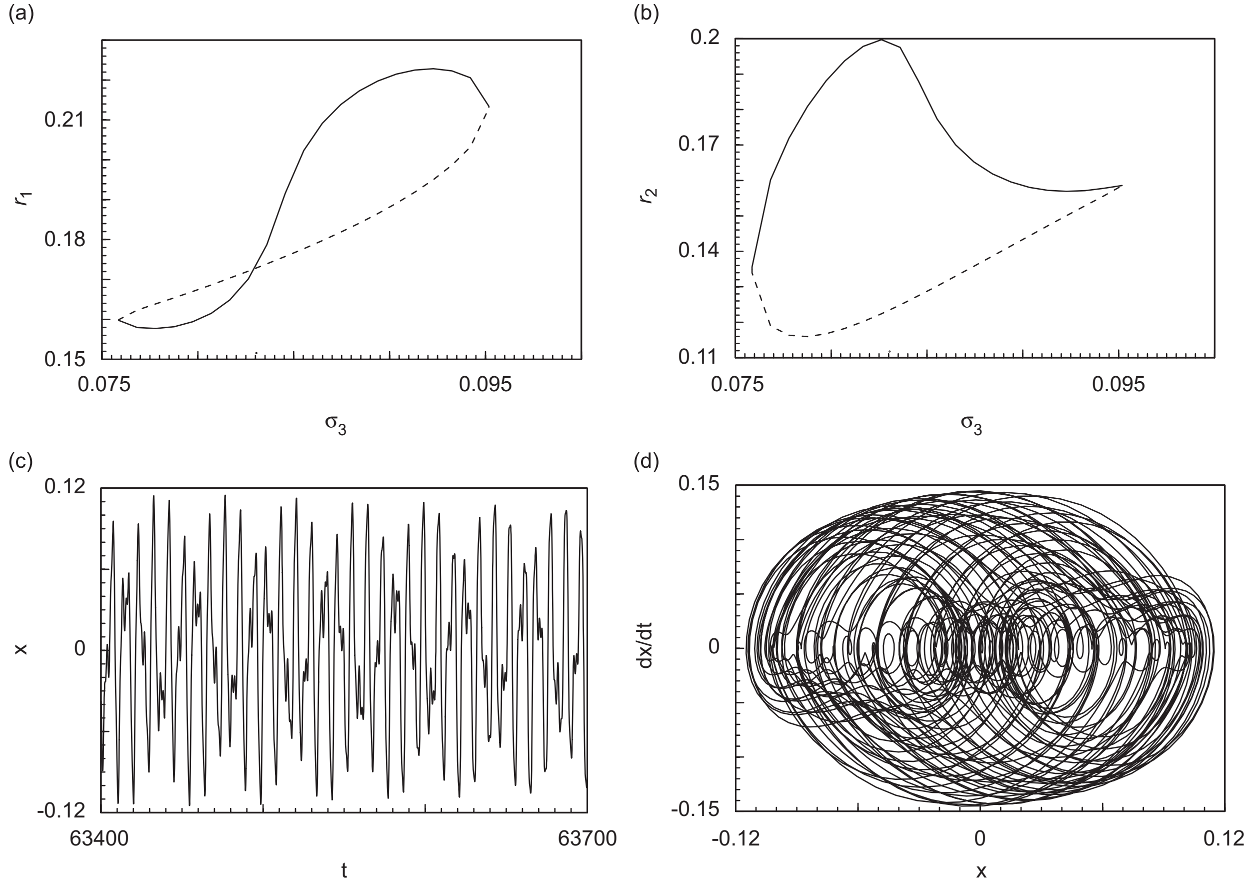

Fig. 3. *Case* III: (a) frequency-response curve of r_1 for $\Omega_0 = 3.27$; (b) frequency-response curve of r_2 for $\Omega_0 = 3.27$; (c) time history of the quasi-periodic motion at $\Omega_0 = 3.278$; (d) phase portrait of the quasi-periodic motion at $\Omega_0 = 3.278$. Solid lines in (a) and (b) denote stable steady-state solutions and dashed lines denote unstable solutions.

The quasi-periodic motion consists of three components having the frequency of the first Hopf bifurcation, the frequency of the second Hopf bifurcation, and the frequency of the excitation, respectively. The quasi-periodic motion is on a 3Dtorus, which can be viewed as a motion by adding a third periodic motion resulting from the periodic excitation to a quasi-period motion on a 2D torus having frequencies of two Hopf bifurcations. The three frequencies satisfy the additive resonance conditions.

The nonlinear system given by Eq. (1) is an infinite dimensional system in a mathematical sense. Under additive resonances, this system has been found to exhibit quasi-periodic motions involving three frequencies. Except the forcing frequency, the other two frequencies are difficult to identify from the original system (1), as they do not relate to the natural frequency ω, nor to its multiples. They came from two Hopf bifurcations occurring in the corresponding autonomous system. The observable behaviour of the non-autonomous system under additive resonances provides useful information for fault diagnosis. Presence of a complicated behaviour may indicate that the original system has undergone a certain bifurcation and induced certain types of additive resonances.

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