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A correction method for the analysis of continuous linear one-dimensional systems under moving loads

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Abstract

A new correction procedure for dynamic analysis of linear, proportionally damped, continuous systems under traveling concentrated loads is proposed; both cases of non-parametric (moving forces) and parametric (moving mass) loads are considered. Improvement in the evaluation of the dynamic response is obtained by separating the contribution of the low-frequency (LF) modes from that of the high-frequency (HF) modes. The former is calculated, as usual, by classical modal analysis, while the latter is taken into account using a new series expansion of the corresponding particular solution. The advantage of the suggested method is immediately shown in the calculation of the stress distribution since it is able to capture the stress discontinuities due to the nature of the applied loads themselves. Numerical results are presented to discuss the convergence of the proposed series and to show the accuracy of the calculated response compared to the classical series expansion solution.

1. Introduction

Theoretical and experimental investigations on continuous systems under moving concentrated loads have been ongoing for more than a century [1]. This problem finds its widest field of application in bridge engineering where the dynamic nature of moving loads, combined with the presence of damage scenarios, either due to environmental loads (corrosion, material loss, supports deterioration) or to stress concentrations (cracks, joint failures), can dramatically reduce the life of the structure.

In this framework a correct evaluation of the stress distribution due to the moving loads is of fundamental importance in order to recognize when the structure is approaching an overstressed condition. However, the dynamic problem of a continuous system subjected to a moving load is quite cumbersome; first of all the difficulty in evaluating the response depends on the moving vehicle model. For the simplest moving force model, approximate analytical solutions are available primarily based on the series expansion of the unknown displacement function [2–4]. On the other hand for the moving mass problem (the inertia of the moving

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sub-system is taken into account) approximate solution methods have been presented on the basis of series expansions [5,6], iterative solution of integral equations [7], and finite element discretization [8,9]. The moving oscillator [10–12] as well as more complex vehicle models have also been investigated in the literature [13–16]. For beam type systems the Euler–Bernoulli model is generally employed, however when the travelling loads move very fast (close to the critical velocity) the Timoshenko model, that includes effects of shear deformation and rotatory inertia, appears to be more appropriate [17–20]. Recently, the presence of structural damage into the continuous system has been taken into account [21–23]. Moreover, a research field has been dealing with the identification of the moving loads from stress field measurements [24].

However, most of the above-mentioned research works use classical modal analysis approach and truncation procedures to calculate the system response: the whole procedure is known as modal displacement method (MDM) [25]. This approach leads to accurate estimations of displacements and their time derivatives, however the truncation of the HF-modes may lead to a significant underestimation of the stress distribution that is proportional to higher order spatial derivatives of displacements whose accuracy, in turn, is directly related to the truncation order. This is especially true for the case of moving concentrated loads, since the nature of the problem leads to stress discontinuities that could never be obtained by using a classical C^{∞} (class of ∞ time differentiable functions) series expansion of the unknown displacement function.

In order to take into account the contribution on the response due to HF-modes neglected in the classical MDM, several methods have been proposed for discrete (or discretized ones) structures: the mode-acceleration method (MAM) [26–33], where a pseudo-static response obtained by neglecting the inertial and damping forces in the original system is added to the MDM solution, the dynamic correction method (DCM) [34], where the particular solution of the equation of motion is evaluated [35,36]; the force derivative method (FDM) [37] whose correction term is built as a series expansion of derivatives of the forcing function.

The problem of transient response has only recently received attention [38]. An original approach based on the separation of the low-frequency (LF)- and high-frequency (HF)-response components has been presented in Di Paola and Failla [39] and the correction term is given in series form providing the convergence conditions of the proposed series as well.

To the authors' knowledge, no systematic research has been dedicated to correction procedures for continuous systems. The case of moving load excitations has been addressed in Refs. [40–42]. Particularly, in Pesterev et al. [40] an improved series representation for the dynamic solution of one-dimensional (1D) systems under a moving force and a moving oscillator is proposed. This solution is able to capture the jump in the shear force at the point of attachment of the oscillator in an approximate way. In Biondi and Muscolino [41] and Biondi et al. [42] two approaches are proposed: the first is an extension of the DCM to continuous systems, the second is based on the use of space–time-varying eigenfunctions; the latter appears to be more computationally convenient even though quite cumbersome in the implementation.

The authors [43] have recently proposed an extension to continuous systems of the correction procedure presented in Di Paola and Failla [39]. In this paper, the procedure exploited in Bilello et al. [43] is extended to the case of moving concentrated loads; both the cases of a moving force and a moving mass are considered. Although the Timoshenko beam and non-proportional damping model are certainly more effective to investigate the effects of the HF-modes on the system response in this paper the proportionally damped Euler–Bernoulli beam model is used both to highlight the correction procedure itself and to compare the results with those presented in the literature.

Advantages of the proposed method are discussed in detail and numerical results are presented and compared to the classical series expansion solution.

2. Problem statement

The equation of motion for a continuous 1D systems is given by

$$N[\ddot{w}(x,t)] + C[\dot{w}(x,t)] + L[w(x,t)] = f(x,t), \quad x \in D, \quad t \geqslant 0, \tag{1}$$

where $w(x,t)$ is the unknown displacement function, $D = \begin{bmatrix} 0 & l \end{bmatrix}$ is the space-domain, $f(x,t)$ is the forcing function, while $N[\cdot]$, $C[\cdot]$, and $L[\cdot]$ are linear homogeneous differential operators of mass, damping and stiffness. Hereinafter the given system is assumed to be proportionally damped, that is $C[\cdot] = aN[\cdot] + bL[\cdot]$

($a,b \in R^+$); the latter assumption is not strictly required to the aim of this paper but it allows to obtain a compact analytical form of the correction term presented later. The initial conditions associated to Eq. (1) are $w(x,0) = w_0(x)$ and $\dot{w}(x,0) = \dot{w}_0(x)$, while generic boundary conditions are assumed.

The solution of Eq. (1) can be found by expanding the unknown function in a series of the ortho-normal eigenfunctions $\varphi_j(x) \in C^\infty$, i.e.

$$w(x,t) = \sum_{j=1}^{\infty} \varphi_j(x) q_j(t), \tag{2}$$

where $q_j(t)$ are unknown modal coordinates to be calculated. By substituting Eq. (2) into Eq. (1) and using the well-known orthogonality relationships [2], one yields to the jth modal equation in the form

$$\ddot{q}_j(t) + 2\zeta_j \omega_j \dot{q}_j(t) + \omega_j^2 q_j(t) = g_j(t) \quad \text{for } j = 1, 2, \ldots \infty, \tag{3}$$

where ω_j is jth natural frequency ($\omega_{j-1} \leqslant \omega_j \leqslant \omega_{j+1}$), $\zeta_j = a + b\omega_j^2 / 2\omega_j$, and

$$g_j(t) = \int_D \varphi_j(x) f(x,t)\, \mathrm{d}x. \tag{4}$$

The relevant initial conditions are $q_j(0) = \int_D \varphi_j(x) N[w_0(x)]\, \mathrm{d}x$ and $\dot{q}_j(0) = \int_D \varphi_j(x) N[\dot{w}_0(x)]\, \mathrm{d}x$.

The moving force problem is simply obtained by substituting for $f(x,t)$ in Eqs. (1) and (4) the expression

$$f(x,t) = F(t)\delta[x - \xi(t)], \tag{5}$$

where $F(t)$ is the time-varying moving force, $\delta(\cdot)$ is the Dirac delta function, and $\xi(t)$ denotes the location of the moving force at time t.

Thus, using the properties of the Dirac delta function, the jth modal equation can be written as

$$\ddot{q}_j(t) + 2\zeta_j \omega_j \dot{q}_j(t) + \omega_j^2 q_j(t) = F(t)\varphi_j[\xi(t)] \quad \text{for } j = 1, 2, \ldots \infty. \tag{6}$$

This approach, commonly referred to as modal analysis, always requires an unavoidable truncation of the series (2), i.e. only a limited number of modes, say m, are retained in the analysis, and Eq. (2) is rewritten as

$$w(x,t) = \sum_{j=1}^{m} \varphi_j(x) q_j(t). \tag{7}$$

In most engineering problems this is justified by the fact that spatial distribution and frequency content of loading are such that the contribution to the system response of the HF modes is assumed to be negligible. Therefore accurate estimations of displacements are obtained, however the truncation may lead to a significant underestimation of the stress distribution that is proportional to higher order spatial derivatives of $w(x,t)$ whose accuracy, in turn, is directly related to the truncation order. This is especially true for the case of moving concentrated loads since the nature of the problem leads to stress discontinuities that could never be evaluated correctly by using a C^∞ series representation as that of Eq. (7).

In the next section the correction method proposed in Ref. [43], that accounts for the HF-modes neglected in the truncated series expansion (7), will be briefly summarized and extended to the case of moving load excitations.

3. Correction method

The idea behind the method is that of splitting the system response into its LF and HF contribution. In Ref. [43] the governing equations of motion for the LF — as well as for the HF-response are explicited and it is shown that the LF-response, denoted by $\hat{w}(x,t)$, is calculated by classical modal analysis using a limited number of modes m, while the steady-state HF-response, denoted by $\tilde{w}(x,t)$, is evaluated in a series form that, by a proper choice of the number m, can be proven to be uniformly and absolutely convergent. Specifically, the converge criterion can be stated as [43]

$$\omega_m \geqslant \Omega_{\max}, \tag{8}$$

where ω_m and $\Omega_{\max}$ are the mth natural frequency of the system and the maximum frequency content of $f(x,t)$, respectively.

Thus, by introducing the eigenfunctions vector $\hat{\mathbf{\Phi}}^{\mathrm{T}}(x) = [\,\varphi_1(x) \quad \varphi_2(x) \quad \ldots \quad \varphi_m(x)\,]$, the LF-response is calculated as

$$\hat{w}(x,t) = \hat{\mathbf{\Phi}}^{\mathrm{T}}(x)\hat{\mathbf{q}}(t), \tag{9}$$

where $\hat{\mathbf{q}}^{\mathrm{T}}(t) = [\,q_1(t) \quad q_2(t) \quad \ldots \quad q_m(t)\,]$ collects the first m modal components (evaluated by classical modal analysis), while the HF-response is asymptotically obtained in series form as [43]

$$\tilde{w}(x,t) = \sum_{p=1}^{\infty} \int_D \tilde{K}_p(x,y) \frac{\partial^{p-1} f(y,t)}{\partial t^{p-1}}\, \mathrm{d}y. \tag{10}$$

In Eq. (10) $\tilde{K}_p(x,y)$ are the *HF-iterative kernels* defined as

$$\tilde{K}_p(x,y) = K_p(x,y) - \hat{\mathbf{\Phi}}^{\mathrm{T}}(x)\mathbf{\Omega}_m^{-2} F_p \hat{\mathbf{\Phi}}(y), \tag{11}$$

where

$$K_p(x,y) = -\int_D K_1(x,z)\{C[K_{p-1}(z,y)] + N[K_{p-2}(z,y)]\}\, \mathrm{d}z, \tag{12}$$

$K_1(x,y)$ is the static Green function of the system and $K_0(x,y) = 0$. Moreover $\mathbf{\Omega}_m = \mathrm{diag}(\,\omega_1 \quad \omega_2 \quad \ldots \quad \omega_m\,)$, and $\mathbf{F}_p$ is a $m \times m$ matrix, calculated in a recursive form as

$$\mathbf{F}_p = -\mathbf{\Omega}_m^{-2}(\mathbf{\Lambda}_m \mathbf{F}_{p-1} + \mathbf{F}_{p-2}), \tag{13}$$

where $\mathbf{\Lambda}_m = \mathrm{diag}(\,2\zeta_1\omega_1 \quad 2\zeta_2\omega_2 \quad \ldots \quad 2\zeta_m\omega_m\,)$, $\mathbf{F}_1 = \mathbf{I}_m$, and $\mathbf{F}_0 = \mathbf{0}_m$ being $\mathbf{I}_m$ the $m \times m$ identity matrix and $\mathbf{0}_m$ the $m \times m$ null matrix; it is worth to notice that if the system is not proportionally damped the second term on the r.h.s. of Eq. (11), that removes the LF-modes contribution from the iterative kernels, could not be written in compact explicit form using Eq. (13). By summing up Eqs. (9) and (10) the total response is obtained in the form

$$w(x,t) = \hat{\mathbf{\Phi}}^{\mathrm{T}}(x)\hat{\mathbf{q}}(t) + \sum_{p=1}^{\infty} \int_D \tilde{K}_p(x,y) \frac{\partial^{p-1} f(y,t)}{\partial t^{p-1}}\, \mathrm{d}y. \tag{14}$$

In order to evaluate the stress distributions in the continuous system higher order derivatives of the displacement function are required; they are simply given in the form

$$\frac{\partial^n w(x,t)}{\partial x^n} = \frac{\mathrm{d}^n \hat{\mathbf{\Phi}}^{\mathrm{T}}(x)}{\mathrm{d}x^n}\hat{\mathbf{q}}(t) + \sum_{p=1}^{\infty} \int_D \frac{\partial^n \tilde{K}_p(x,y)}{\partial x^n} \frac{\partial^{p-1} f(y,t)}{\partial t^{p-1}}\, \mathrm{d}y. \tag{15}$$

It can be shown that the second term on the r.h.s. of Eq. (15) greatly improves the stress evaluation.

As stated in Ref. [43] Eq. (14) strictly holds at the steady state, moreover the calculation of the iterative kernels $\tilde{K}_p(x,y)$ does not require the knowledge of the HF-eigenvalues and eigenfunctions since one can take full advantage of the fact that the $K_1(x,y)$ is known in an analytical form for different stiffness operators and boundary cases, thus the iterative kernels $K_p(x,y)$ can be calculated in closed form using Eq. (12). The improved solution provided by Eq. (14) is derived in the most general form but it can be made explicit for any given load function $f(x,t)$, including the moving load-mass case as shown in the next sections.

Hereinafter some considerations on the proposed solution (14) and the correction method in Ref. [41] for the moving oscillator case are reported. The generalized form of the DCM solution proposed by Biondi and Muscolino [41] (using the notation introduced in the present work) is given as

$$w_{\mathrm{DCM}}(x,t) = \hat{\mathbf{\Phi}}^{\mathrm{T}}(x)\hat{\mathbf{q}}(t) + \left[w_{\mathrm{par}}(x,t) - \hat{\mathbf{\Phi}}^{\mathrm{T}}(x)\hat{\mathbf{q}}_{\mathrm{par}}^{\mathrm{UND}}(t)\right], \tag{16}$$

where

$$w_{\text{par}}(x,t) = \sum_{p=1}^{\infty} \int_D K_p^{\text{UND}}(x,y) \frac{\partial^{p-1} f(y,t)}{\partial t^{p-1}}\, \mathrm{d}y. \tag{17}$$

$K_p^{\text{UND}}(x,y)$ are the undamped-iterative kernels obtained from Eq. (12) by neglecting the damping operator and $\hat{\mathbf{q}}_{\text{par}}^{\text{UND}}(t)$ is the vector collecting the undamped particular solutions of the first m modes. However, the series on the r.h.s. of Eq. (17) is not convergent if $\Omega_{\max} > \omega_1$; to prove this let us assume $f(x,t) = p(x)\,\mathrm{e}^{\mathrm{i}\Omega t}$, i.e. the forcing function is a mono-component signal of frequency $\omega_1 < \Omega < \omega_s$, then $K_p^{\text{UND}}(x,y)$ in Eq. (17) can be expanded in the form [43]:

$$w_{\text{par}}(x,t) = \sum_{p=1}^{\infty} d_p \frac{\varphi_p(x) \int_D \varphi_p(y) p(y)\, \mathrm{d}y}{\omega_p^2}, \tag{18}$$

where $d_p = \sum_{r=0}^{\infty} (-1)^r (\Omega/\omega_p)^r$. Since $\omega_1 < \Omega < \omega_s$ all the coefficients d_p of the series (18) up to the sth would diverge. Conversely, if $\Omega < \omega_1$ the ratios $\Omega/\omega_p < 1$ for $p = 1, 2, \ldots \infty$, thus using the binomial expansion yields

$$d_p = \frac{\omega_p^2}{\omega_p^2 - \Omega^2} \tag{19}$$

and Eq. (18) is recast in the form

$$w_{\text{par}}(x,t) = \sum_{p=1}^{\infty} \frac{\varphi_p(x) \int_D \varphi_p(y) p(y)\, \mathrm{d}y}{\omega_p^2 - \Omega^2} \tag{20}$$

which is the exact expression for the particular solution of the undamped continuous system. It follows that the $w_{\text{par}}(x,t)$ converges to the exact solution only if $\Omega/\omega_p < 1$. On the contrary the series expansion in Eq. (14) is always convergent as demonstrated in Ref. [43].

3.1. The moving force case

Let us assume the forcing function $f(x,t)$ be given by Eq. (5) and $\xi(t) = vt$, where v is the constant speed of the moving force; the latter hypothesis is not strictly required but it will be used herein to simplify the calculations. Inserting Eq. (5) into Eq. (10) leads to

$$\tilde{w}(x,t) = \sum_{p=1}^{\infty} \int_D \tilde{K}_p(x,y) \frac{\partial^{p-1}}{\partial t^{p-1}} [F(t)\delta(y - vt)]\, \mathrm{d}y. \tag{21}$$

Eq. (21) may be rewritten in an explicit form by using the product derivative rule, i.e.

$$\frac{\mathrm{d}^r}{\mathrm{d}t^r}[g_1(t) g_2(t)] = \sum_{s=1}^{r+1} c_{s-1}^{r+1-s} \frac{\mathrm{d}^{r+1-s} g_1(t)}{\mathrm{d}t^{r+1-s}} \frac{\mathrm{d}^{s-1} g_2(t)}{\mathrm{d}t^{s-1}}, \tag{22}$$

where $c_n^m = \begin{pmatrix} m+n \\ m \end{pmatrix}$ are the binomial coefficients (Pascal's triangle coefficients). By inserting Eq. (22) into Eq. (21) $\tilde{w}(x,t)$ is given as

$$\tilde{w}(x,t) = \sum_{p=1}^{\infty} \sum_{s=1}^{p} c_{s-1}^{p-s} \frac{\mathrm{d}^{p-s} F(t)}{\mathrm{d}t^{p-s}} \int_D \tilde{K}_p(x,y) \frac{\mathrm{d}^{s-1} \delta(y - vt)}{\mathrm{d}t^{s-1}}\, \mathrm{d}y. \tag{23}$$

It can be easily shown that the time-derivative applied to the Dirac delta function on the r.h.s. of Eq. (23) may be changed in a y-derivative using the relationship

$$\frac{\partial^r}{\partial x_2^r} \delta(x_1 - a x_2) = (-1)^r a^r \frac{\partial^r}{\partial x_1^r} \delta(x_1 - a x_2) \tag{24}$$

thus Eq. (23) is rewritten in the form

$$\tilde{w}(x,t)=\sum_{p=1}^{\infty}\sum_{s=1}^{p}(-1)^{s-1}v^{s-1}c_{s-1}^{p-s}\frac{\mathrm{d}^{p-s}F(t)}{\mathrm{d}t^{p-s}}\int_D \tilde{K}_p(x,y)\frac{\mathrm{d}^{s-1}\delta(y-vt)}{\mathrm{d}y^{s-1}}\,\mathrm{d}y. \tag{25}$$

Then by applying the properties of the Dirac delta function derivatives, i.e.,

$$\int_D g(x_1)\frac{\partial^r}{\partial x_1^r}\delta(x_1-ax_2)\,\mathrm{d}x_1=(-1)^r\frac{\partial^r g(x_1)}{\partial x_1^r}\bigg|_{x_1=ax_2},\quad x_1\in D \tag{26}$$

yields the equation

$$\tilde{w}(x,t)=\sum_{p=1}^{\infty}\sum_{s=1}^{p}c_{s-1}^{p-s}v^{s-1}\frac{\mathrm{d}^{p-s}F(t)}{\mathrm{d}t^{p-s}}\frac{\partial^{s-1}\tilde{K}_p(x,y)}{\partial y^{s-1}}\bigg|_{y=vt}. \tag{27}$$

Thus the moving force solution that accounts for the HF contributions is written as

$$w(x,t)=\hat{\mathbf{\Phi}}^{\mathrm{T}}(x)\hat{\mathbf{q}}(t)+\sum_{p=1}^{\infty}\sum_{s=1}^{p}c_{s-1}^{p-s}v^{s-1}\frac{\mathrm{d}^{p-s}F(t)}{\mathrm{d}t^{p-s}}\frac{\partial^{s-1}\tilde{K}_p(x,y)}{\partial y^{s-1}}\bigg|_{y=vt}, \tag{28}$$

where the vector $\hat{\mathbf{q}}(t)$ is obtained by solving the first m equation of the system (6). The spatial derivatives of the response function can be evaluated as

$$\frac{\partial^n w(x,t)}{\partial x^n}=\frac{\mathrm{d}^n\hat{\mathbf{\Phi}}^{\mathrm{T}}(x)}{\mathrm{d}x^n}\hat{\mathbf{q}}(t)+\sum_{p=1}^{\infty}\sum_{s=1}^{p}c_{s-1}^{p-s}v^{s-1}\frac{\mathrm{d}^{p-s}F(t)}{\mathrm{d}t^{p-s}}\frac{\partial^{n+s-1}\tilde{K}_p(x,y)}{\partial x^n\partial y^{s-1}}\bigg|_{y=vt}. \tag{29}$$

Finally, for the special case in which $F(t)=F=\text{cost}$, Eq. (28) writes as follows:

$$w(x,t)=\hat{\mathbf{\Phi}}^{\mathrm{T}}(x)\hat{q}(t)+F\sum_{p=1}^{\infty}v^{p-1}\frac{\partial^{p-1}\tilde{K}_p(x,y)}{\partial y^{p-1}}\bigg|_{y=vt}. \tag{30}$$

It should be pointed out that the first term of the series in Eq. (30) coincides with that proposed in Ref. [40] that is substantially a MAM extension to continuous systems. A discussion on the number of dynamic modes that have to be included in the analysis will be presented in Section 4.

3.2. The moving mass case

In this section, the extension to the case of a moving mass is presented. For this case the forcing function $f(x,t)$ is given by Eq. (5) where

$$F(t)=M_v\left(g-\frac{\mathrm{d}^2w(x,t)}{\mathrm{d}t^2}\bigg|_{x=vt}\right), \tag{31}$$

M_v is the moving mass and g is the acceleration of gravity. The double time-derivative of $w(x,t)$ on the r.h.s. of Eq. (31) can be evaluated by using the compound functions derivative rules, i.e.,

$$A(t)=\frac{\mathrm{d}^2w(x,t)}{\mathrm{d}t^2}\bigg|_{x=vt}=\left[\frac{\partial^2w(x,t)}{\partial t^2}+2v\frac{\partial^2w(x,t)}{\partial x\partial t}+v^2\frac{\partial^2w(x,t)}{\partial x^2}\right]\bigg|_{x=vt}. \tag{32}$$

Eqs. (31) and (32) clearly show that the forcing function depends on the system response itself; the latter circumstance makes the solution process more cumbersome than the moving force problem. As in fact, substituting Eqs. (32) and (31) into Eq. (28) yields

$$w(x,t)=\hat{\mathbf{\Phi}}^{\mathrm{T}}(x)\hat{\mathbf{q}}(t)+M_vg\sum_{p=1}^{\infty}v^{p-1}\frac{\partial^{p-1}\tilde{K}_p(x,y)}{\partial y^{p-1}}\bigg|_{y=vt}-M_v\sum_{p=1}^{\infty}\sum_{s=1}^{p}c_{s-1}^{p-s}v^{s-1}\frac{\mathrm{d}^{p-s}A(t)}{\mathrm{d}t^{p-s}}\frac{\partial^{s-1}\tilde{K}_p(x,y)}{\partial y^{s-1}}\bigg|_{y=vt}, \tag{33}$$

where the third term on the r.h.s. is due to the inertia of the moving mass. The vector $\hat{\mathbf{q}}(t)$ is obtained by solving a set of m second-order coupled differential equations with time-varying coefficients, that is written in compact matrix form as

$$\mathbf{M}(t)\ddot{\hat{\mathbf{q}}}(t) + \mathbf{C}(t)\dot{\hat{\mathbf{q}}}(t) + \mathbf{K}(t)\hat{\mathbf{q}}(t) = Mg\hat{\mathbf{\Phi}}(vt), \tag{34}$$

where $\mathbf{M}(t)$, $\mathbf{C}(t)$, and $\mathbf{K}(t)$ are m-dimensional time-dependent square matrices

$$\begin{aligned}
\mathbf{M}(t) &= \mathbf{I}_m - M_v\hat{\mathbf{\Phi}}^{\mathrm{T}}(vt)\hat{\mathbf{\Phi}}(vt),\\
\mathbf{C}(t) &= \mathbf{\Lambda}_m - 2M_v v\hat{\mathbf{\Phi}}^{\mathrm{T}}(vt)\left.\frac{\mathrm{d}\hat{\mathbf{\Phi}}(x)}{\mathrm{d}x}\right|_{x=vt},\\
\mathbf{K}(t) &= \mathbf{\Omega}_m^2 - M_v v^2\hat{\mathbf{\Phi}}^{\mathrm{T}}(vt)\left.\frac{\mathrm{d}^2\hat{\mathbf{\Phi}}(x)}{\mathrm{d}x^2}\right|_{x=vt}.
\end{aligned} \tag{35}$$

Eq. (33) cannot be solved directly since the function $A(t)$ is dependent on the unknown function $w(x,t)$ (see Eq. (32)). Two approaches may be proposed to overcome this difficulty. The first approach, using a procedure similar to that proposed in Pesterev and Bergman [40] for the moving oscillator problem, consists in considering that the displacement function $w(x,t)$ and its time derivatives can be approximated accurately by the dynamic contribution due to the first few modes, then in Eq. (32) it may be assumed $w(x,t) \cong \hat{w}(x,t)$, and

$$A(t) \cong \left.\frac{\mathrm{d}^2\hat{w}(x,t)}{\mathrm{d}t^2}\right|_{x=vt} = \left.\hat{\mathbf{\Phi}}^{\mathrm{T}}(x)\right|_{x=vt}\ddot{\hat{\mathbf{q}}}(t) + 2v\left.\frac{\mathrm{d}\hat{\mathbf{\Phi}}^{\mathrm{T}}(x)}{\mathrm{d}x}\right|_{x=vt}\dot{\hat{\mathbf{q}}}(t) + v^2\left.\frac{\mathrm{d}^2\hat{\mathbf{\Phi}}^{\mathrm{T}}(x)}{\mathrm{d}x^2}\right|_{x=vt}\hat{\mathbf{q}}(t). \tag{36}$$

Inserting Eq. (36) into Eq. (33) provides accurate results as shown in the numerical application.

In the second approach Eq. (33) can be solved iteratively such as, at the kth step, one may write

$$\begin{aligned}
w^{(k)}(x,t) &= \hat{\mathbf{\Phi}}^{\mathrm{T}}(x)\hat{\mathbf{q}}(t) + M_v g\sum_{p=1}^{\infty} v^{p-1}\left.\frac{\partial^{p-1}\tilde{K}_p(x,y)}{\partial y^{p-1}}\right|_{y=vt}\\
&\quad - M_v\sum_{p=1}^{\infty}\sum_{s=1}^{p} c_{s-1}^{p-s}v^{s-1}\left.\frac{\mathrm{d}^{p-s}A^{(k)}(t)}{\mathrm{d}t^{p-s}}\frac{\partial^{s-1}\tilde{K}_p(x,y)}{\partial y^{s-1}}\right|_{y=vt},\\
A^{(k)}(t) &= \left.\left[\frac{\partial^2 w^{(k-1)}(x,t)}{\partial t^2} + 2v\frac{\partial^2 w^{(k-1)}(x,t)}{\partial x\partial t} + v^2\frac{\partial^2 w^{(k-1)}(x,t)}{\partial x^2}\right]\right|_{x=vt}.
\end{aligned} \tag{37}$$

The iterations end when a convergence criterion is satisfied. As an example, it may be computational convenient to define it as

$$\varepsilon^{(k)} = \int_0^t |A^{(k)}(t) - A^{(k-1)}(t)|\,\mathrm{d}t \leqslant \text{tolerance}. \tag{38}$$

The two approaches have been compared through different numerical applications and their results are identical although the first appeared to be computationally more efficient. Results of such analysis are not reported here for brevity's sake.

4. Terms to be included in the low-frequency response

Questions may arise concerning the number of terms to be included into the LF-response $\hat{w}(x,t)$ for the case of traveling loads. As mentioned earlier, the convergence criterion is stated by Eq. (8), however, in the case under investigation, it is not straightforward to define the maximum frequency $\Omega_{\max}$. For clarity sake, let us assume the continuous system to be an Euler–Bernoulli beam of stiffness EI and constant linear mass density ρ, traversed by a force moving at constant speed v, thus the mth modal equation would be given in the form

$$\ddot{q}_m(t) + 2\zeta_m\omega_m\dot{q}_m(t) + \omega_m^2 q_m(t) = F\varphi_m(\Omega_m t), \tag{39}$$

where $\omega_m = \lambda_m^2\sqrt{EI/\rho l^4}$, $\Omega_m = \lambda_m v/l$, λ_m is the mth dimensionless eigenvalue of the beam (for example in a simply supported beam $\lambda_m = m\pi$) and $\varphi_m(\Omega_m t)$, for m sufficiently large, can be considered as a

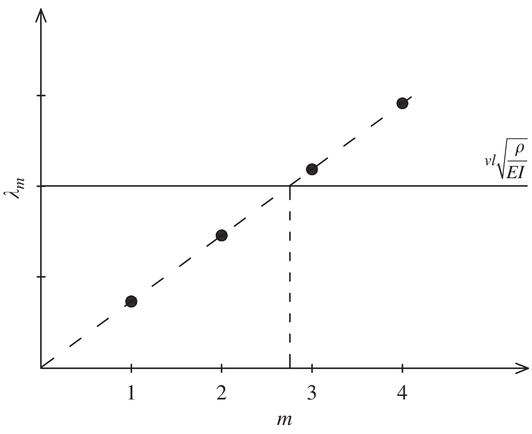

Fig. 1. Selection of the number of modes to be included into the LF-response of Euler–Bernoulli beam subject to a moving force.

mono-component signal of frequency Ω_m. Thus, the condition (8) yields

$$\lambda_m \geqslant vl\sqrt{\frac{\rho}{EI}}. \tag{40}$$

Since the continuous system has an infinite number of eigenvalues, there will always be an integer m such as Eq. (40) is satisfied (Fig. 1). If the moving force is also time-dependent the condition (40) should be rewritten as

$$\lambda_m \geqslant \max\left(vl\sqrt{\frac{\rho}{EI}}, l\sqrt[4]{\frac{\rho\Omega_F^2}{EI}}\right), \tag{41}$$

where Ω_F denotes the maximum frequency content of $F(t)$.

It could be easily shown that, for different types of continuous 1D systems, it is always possible to introduce conditions like Eq. (40) (or Eq. (41)) useful to define the number of terms to be included in the LF-response $\hat{w}(x,t)$.

5. Numerical application

Let us consider a uniform simply supported Euler–Bernoulli beam traversed by a mass moving at constant speed. The beam parameters are the same as those used in Refs. [7,10–12]: $L = 6\,\text{m}$, $EI = 275.4408\,\text{N}\,\text{m}^2$, $\rho = 1\,\text{N}\,\text{s}^2/\text{m}^2$, $M_v/\rho L = 0.2$, $F = M_v g = 11.78\,\text{N}$, $v = 6\,\text{m/s}$. The damping operator is given as $C[\cdot] = a$ and $a/\rho = 2.0\,\text{s}^{-1}$.

First the inertia of the moving mass is neglected leading to the moving force problem. For the given beam parameters and moving load speed the convergence criterion (40) provides $m = 1$, i.e. only 1 term is strictly required for the evaluation of the LF-response. In Fig. 2 the shear force distribution along the beam length at time $\bar{t} = l/5v = 0.2\,\text{s}$ is reported. Only the first mode is included in the LF-response and the series on the r.h.s. of Eq. (30) appears to be convergent by using 3 terms. As shown in Fig. 3 including few more modes ($m = 4$) into the LF response makes the convergence much faster, in fact only the first term of the correction series is needed in order to obtain the convergence of the proposed solution. This is obviously due to the nature of the system response which is strictly non-stationary, i.e. the homogeneous contribution of the first few modes (say $m+1, m+2, \ldots$), which would be lost if included in the HF-response, cannot be neglected.

In Fig. 4 the MDM solution using 50 terms of the classical series expansion (2) is compared to the solution obtained by using the correction method with $m = 1$ and $p = 3$, $m = 4$ and $p = 1$, and $m = 5$ and $p = 1$. The MDM solution oscillates around the proposed solution: the conventional series expansion (obtained as sum of C^∞ function) is not able to capture the discontinuity in the shear force distribution even if a large number of modes is used and the Gibbs effect clearly appears. On the other hand the shear jump is captured accurately by

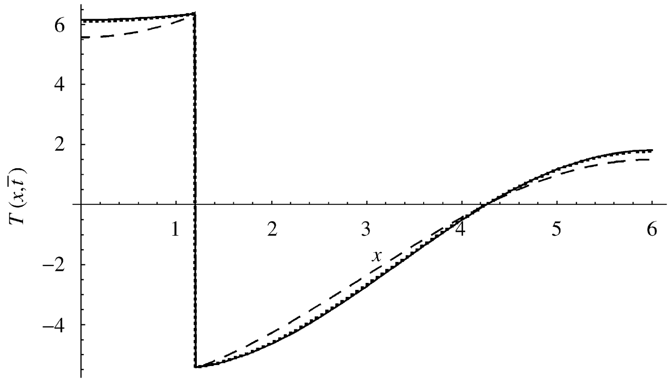

Fig. 2. Moving force case: shear force distribution at the time $\bar{t}$ by the proposed method using $m = 1$ and $p = 1$ (long dashed), $m = 1$ and $p = 3$ (dotted), $m = 1$ and $p = 5$ (solid).

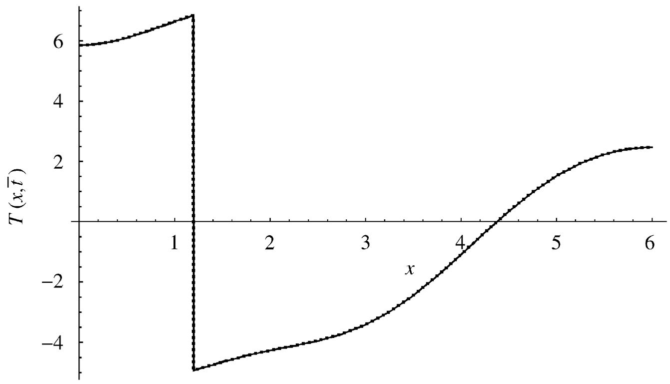

Fig. 3. Moving force case: shear force distribution at the time $\bar{t}$ by the proposed method using $m = 4$ and $p = 1$ (long dashed), $m = 4$ and $p = 3$ (dotted), and $m = 4$ and $p = 5$ (solid).

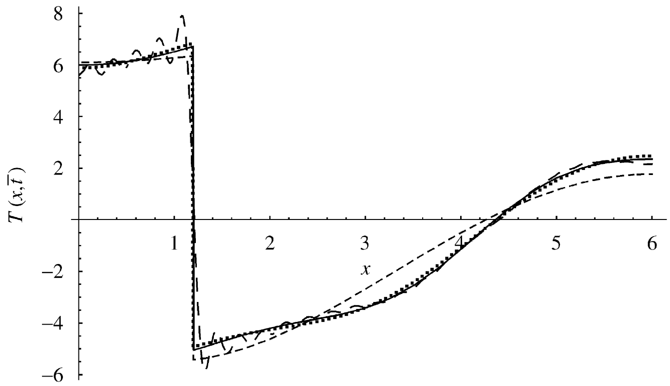

Fig. 4. Moving force case: shear force distribution at the time $\bar{t}$ by the proposed method using $m = 1$ and $p = 3$ (short dashed), $m = 4$ and $p = 1$ (dotted), $m = 5$ and $p = 1$ (solid), and MDM solution by using $m = 50$ (long dashed).

the proposed solution procedure (even for $m = 1$), but the convergence is obtained only by increasing the number of modes included into the LF-response from $m = 1$ (strictly required) to $m = 4$. It is worth to point out that these numerical results are consistent with those obtained in Ref. [40] even though the basic concept behind the proposed correction term [43] is quite different.

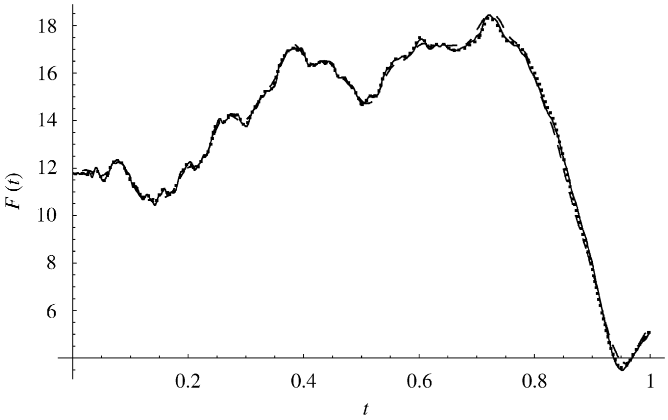

Fig. 5. Moving mass case: coupling force calculated by MDM using $m = 5$ (▬ ▬), $m = 8$ (▪▪▪▪▪▪▪▪), and $m = 12$ (solid).

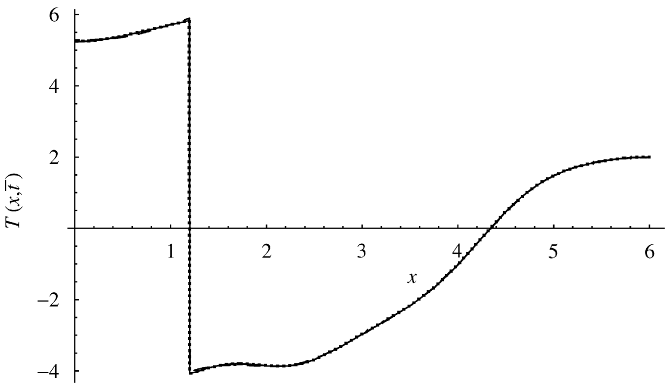

Fig. 6. Moving mass case: shear force distribution at the time $\bar{t}$ by the proposed method using $m = 8$ and $p = 1$ (▬ ▬), $m = 8$ and $p = 3$ (▪▪▪▪▪▪▪▪), and $m = 8$ and $p = 5$ (solid).

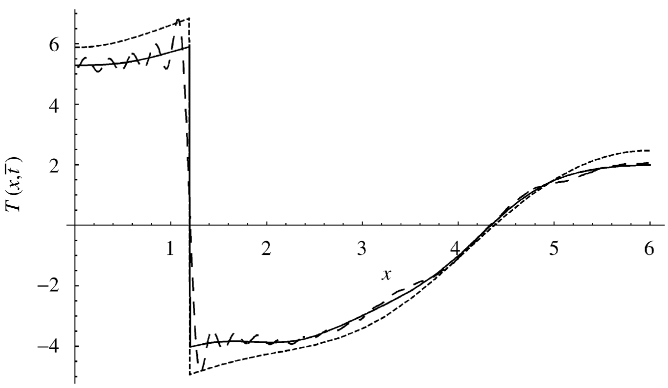

Fig. 7. Moving mass case: shear force distribution at the time $\bar{t}$ by the proposed method using $m = 8$ and $p = 1$ (solid), MDM solution by using $m = 50$ (▬ ▬), moving force solution by the proposed method using $m = 4$ and $p = 1$ (▬ ▬ ▬).

The moving concentrated load model, that is the most used in the literature, is a mathematical idealization, however from an engineering point of view it is commonly accepted when the ratio of the load area to the main system dimension is very small, for example this is the case of vehicles moving on bridges. In real structures the stress discontinuity that arises from this model becomes a rapid change of the stress sign in a small region, as it

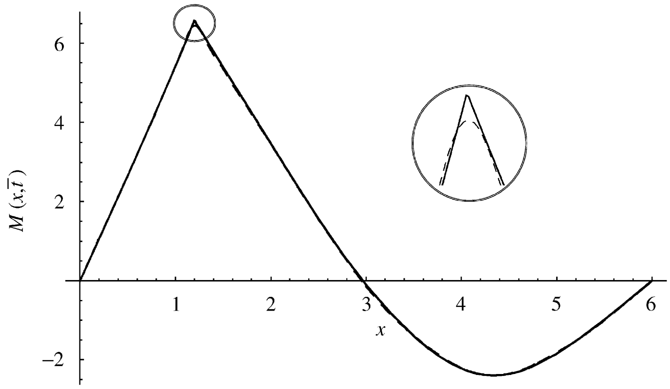

Fig. 8. Moving mass case: bending moment distribution at the time $\bar{t}$ by the proposed method using $m = 8$ and $p = 1$ (solid), MDM solution by using $m = 5$ (▬▬ ▬▬).

would be obtained using a moving load model distributed on a small area; however, the aim of this paper is to present a correction form of the classical series solution to account for the HF modes contribution and not to discuss whether the moving concentrated load is the best model to deal with this type of problems.

The moving mass case is investigated in Figs. 5–7 by using the first approach presented in Section 3.2. The force $F(t)$ acting on the beam from the moving mass is reported in Fig. 5 in the time interval $[0 \quad l/v]$ by using 5, 8 and 12 modes of the MDM solution. Compared to the moving oscillator case [40], where the coupling force depends only on the displacement function and its time derivative, the moving mass coupling force requires few more modes to converge, however, beginning with $m = 8$, the approximation can be considered quite accurate.

These force and the time-dependent coefficients $\hat{q}_i(t)$ (for $i = 1,2,\ldots,8$) of the MDM were substituted in Eq. (29) to calculate the shear force distribution at $\bar{t} = l/5v = 0.2\,\text{s}$ by the proposed method.

In Fig. 6 the shear distribution along the beam calculated at $\bar{t}$ by using $m = 8$ and $p = 1$, $m = 8$ and $p = 3$, $m = 8$ and $p = 4$ are reported. Due to the large number of modes included into the LF-response all curves overlap, i.e. convergence is obtained by using just 1 term ($p = 1$) of the series on the r.h.s. of Eq. (33). In Fig. 7 the moving mass as well as the moving force solution calculated by the proposed method are reported. The MDM solution for the moving mass case using 50 terms of the classical series expansion is also reported for comparison. The conventional series is still not able to capture the shear discontinuity. Moreover the effect of the inertia of the moving load on the system response is clearly noticed; it can be shown that such a difference becomes more relevant as the mass and the speed of the moving load increase.

Although not shown, the second approach proposed in Section 3.2 leads to the same results, as those presented here providing that the number of modes included into the LF-response is sufficient to obtain a good approximation of the coupling force.

Finally, in Fig. 8 the bending moment distribution is reported; also in this case the proposed solution is shown to be more accurate compared to the MDM solution calculated by using 50 terms.

6. Conclusions and remarks

A correction procedure to improve the evaluation of the dynamic response of linear, proportionally damped, continuous 1D systems traversed by moving loads has been presented. The method is based on the separation of the LF- and HF- response contribution: the first is evaluated by classical modal analysis while the second is obtained as a series expansion of the particular solution.

Compared to the DCM the proposed solution is more accurate and suitable since it provides clearly a convergence criterion and it takes into account the structural damping into the correction term, moreover it includes the MAM correction term into its formulation.

As shown by the numerical example the proposed corrected solution greatly improves the stress response resolution. Moreover the method is also computationally efficient, in fact few terms of the correction series are required to obtain accurate solutions. For the case under investigation, where the system response is strictly non-stationary, the number of terms is related to the number of modes included into the LF response: if the modes strictly required for the convergence are used, 3–4 terms of the correction series are needed while if few more modes are included into the LF response just 1 term of the correction series can be sufficient. It should also be remarked that the HF iterative kernels that appear into the series expansion are calculated in closed form only once for any given continuous system and then used for any type of forcing function (moving oscillator problem, multiple moving loads, combinations of moving and dynamic loads).

The proposed solution is general and can be used to analyze different load scenarios, however it requires the knowledge of the frequency content of the forcing function in order to guarantee the robustness of the method; for most of natural excitation sources this is known with good confidence, otherwise one may use a preliminary signal analysis in order to estimate the maximum frequency content of the forcing function.

The authors are currently working on the possibility of applying the proposed solution procedure to moving forces identification problems.

Acknowledgments

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References

[1] G.G. Stokes, Discussion of a differential equation relating to the breaking of railway bridges, *Transactions of the Cambridge Philosophical Society* 8 (1849) 707–735 (reprinted in 1883, Mathematical and Physical Papers, 2, pp. 178–220).

[2] L. Meirovitch, *Analytical Methods in Vibrations*, MacMillan, London, 1967.

[3] L. Fryba, *Vibrations of Solids and Structures Under Moving Loads*, Thomas Telford, London, 1999.

[4] A.V. Pesterev, B. Yang, L.A. Bergman, C.A. Tan, Revisiting the moving force problem, *Journal of Sound and Vibration* 261 (2003) 75–91.

[5] M.M. Stanisic, J.A. Euler, S.T. Montgomery, On a theory concerning the dynamical behavior of structures considering moving mass, *Ingenieur Archives* 43 (1974) 295–305.

[6] A.V. Pesterev, L.A. Bergman, A contribution to the moving mass problem, *Journal of Vibration and Acoustic* 120 (1998) 824–826.

[7] S. Sadiku, H.H.E. Leipholz, On the dynamic effects of elastic systems with moving concentrated mass, *Ingenieur Archives* 57 (1987) 223–242.

[8] J.E. Akin, M. Mofid, Numerical solution for response of beam with moving mass, *Journal of Structural Engineering* 115 (1989) 120–131.

[9] M. Mofid, M. Shadam, On the response of beams with internal hinges, under moving mass, *Advances in Engineering Software* 31 (2000) 323–328.

[10] A.V. Pesterev, L.A. Bergman, Response of elastic continuum carrying moving linear oscillator, *Journal of Engineering Mechanics* 123 (1997) 878–884.

[11] A. Pesterev, L.A. Bergman, Vibration of elastic continuum carrying accelerating oscillator, *Journal of Engineering Mechanics* 123 (1997) 886–889.

[12] B. Yang, C.A. Tan, L.A. Bergman, Direct numerical procedure for solution of moving oscillator problems, *Journal of Engineering Mechanics* 126 (2000) 462–469.

[13] P. Swannel, C.W. Miller, Theoretical and experimental studies of a bridge–vehicle system, *Proceedings of the Institution of Civil Engineers* 83 (1987) 613–635.

[14] J.L. Humar, A.M. Kashif, Dynamic response of bridges under traveling loads, *Canadian Journal of Civil Engineers* 20 (1993) 287–298.

[15] P.K. Chatterjee, T.K. Datta, C.S. Surana, Vibration of continuous bridges under moving vehicles, *Journal of Sound and Vibration* 169 (1994) 619–632.

[16] Y.B. Yang, L.D. Yau, Vehicle–bridge interaction element for dynamic analysis, *Journal of Structural Engineering* 123 (1997) 1512–1518.

[17] S. Mackeretich, Moving load on a Timoshenko beam, *Journal of the Acoustic Society of America* 88 (1990) 1175–1178.

[18] H.-P. Lee, The dynamic response of a Timoshenko beam subjected to a moving mass, *Journal of Sound and Vibration* 198 (1996) 249–256.

[19] R.-T. Wang, Vibration of multi-span Timoshenko beams to moving force, *Journal of Sound and Vibration* 207 (1997) 731–742.

[20] T.-H. Chen, D.-S. Chen, Timoshenko beam with tuned mass dampers to moving loads, *Journal of Bridge Engineering* 9 (2004) 167–177.

[21] C. Bilello, L.A. Bergman, Vibration of damaged beams under a moving mass: theory and experimental validation, *Journal of Sound and Vibration* 274 (2004) 567–582.

[22] D.R. Parhi, A.K. Behera, Dynamic deflection of a cracked shaft subjected to moving mass, *Transactions of the CSME* 21 (1997) 295–316.
[23] M.A. Mahamoud, M.A. Abou Zaid, Dynamic response of a beam with a crack subjected to a moving mass, *Journal of Sound and Vibration* 256 (2002) 591–603.
[24] T.H.T. Chan, T.H. Yung, L. Yu, S.S. Law, Moving force identification I: theory, *Journal of Sound and Vibration* 247 (2001) 59–76.
[25] R.R. Craig, *Structural Dynamics*, Wiley, New York, 1981.
[26] N.R. Maddox, On the number of modes necessary for accurate response and resulting forces in dynamic analyses, *ASME Journal of Applied Mechanics* 42 (1975) 516–517.
[27] O.E. Hansteen, K. Bell, On the accuracy of mode superposition analysis in structural dynamics, *Earthquake Engineering and Structural Dynamics* 7 (1979) 405–411.
[28] R.E. Cornwell, R.R. Craig, C.P. Johnson, On the application of the mode-acceleration method to structural engineering problems, *Earthquake Engineering and Structural Dynamics* 11 (1983) 679–688.
[29] M.A. Akgun, A new family of mode-superposition methods for response calculations, *Journal of Sound and Vibration* 167 (1983) 289–302.
[30] A. Ibrahimbegovic, E.L. Wilson, Simple numerical algorithms for the mode superposition analysis of linear structural systems with non-proportional damping, *Computers & Structures* 33 (1989) 523–531.
[31] S.H. KulKarni, Inclusion of higher modes in the analysis non-classically damped systems, *Earthquake Engineering and Structural Dynamics* 21 (1992) 543–549.
[32] J.M. Dickens, J.M. Nakagawa, M.J. Wittbrodt, A critique of mode acceleration and modal augmentation methods for modal response analysis, *Computers & Structures* 62 (1997) 985–998.
[33] A. Dutta, C.V. Ramakrishnan, Accurate computation of design sensitivities for structures under transient dynamic loads with constraints on stresses, *Computers & Structures* 66 (1998) 463–472.
[34] G. Borino, G. Muscolino, Mode-superposition methods in dynamic analysis of classically and non-classically damped structures, *Earthquake Engineering and Structural Dynamics* 14 (1986) 705–717.
[35] S. Benfratello, G. Muscolino, Mode superposition correction method for deterministic and stochastic analysis of structural systems, *Computers & Structures* 79 (2001) 2471–2480.
[36] R.R. Arnold, R.L. Citerley, M. Chargin, D. Galant, Application of Ritz vectors for dynamic analysis of large structures, *Computers & Structures* 21 (1985) 901–907.
[37] C.J. Camarda, R.T. Haftka, M.F. Riley, An evaluation of higher-order modal methods for calculating transient structural response, *Computers & Structures* 27 (1987) 89–101.
[38] A. D'Aveni, G. Muscolino, Improved dynamic correction method in seismic analysis of both classically and non-classically damped structures, *Earthquake Engineering and Structural Dynamics* 30 (2001) 501–517.
[39] M. Di Paola, G. Failla, A correction method for dynamic analysis of linear system, *Computers & Structures* 82 (2004) 1217–1226.
[40] A.V. Pesterev, L.A. Bergman, C.A. Tan, A new method for calculating bending moment and shear force in moving load problems, *ASME Journal of Applied Mechanics* 68 (2001) 252–259.
[41] B. Biondi, G. Muscolino, New improved series expansion for solving the moving oscillator problem, *Journal of Sound and Vibration* 281 (2005) 99–117.
[42] B. Biondi, G. Muscolino, A. Sidoti, Methods for calculating bending moment and shear force in the moving mass problem, *Journal of Sound and Vibration* 126 (2004) 542–552.
[43] C. Bilello, M. Di Paola, S. Salamone, A correction method for dynamic analysis of linear continuous systems, *Computers & Structures* 83 (2005) 662–670.