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First-passage failure of quasi-integrable Hamiltonian systems under time-delayed feedback control

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Abstract

A procedure for studying the first-passage failure of quasi-integrable Hamiltonian systems under time-delayed feedback control is proposed. The stochastic averaging method for quasi-integrable Hamiltonian systems under time-delayed feedback control is firstly introduced. A backward Kolmogorov equation governing the conditional reliability function and a set of generalized Pontryagin equations governing the conditional moments of first-passage time are then established. The conditional reliability function, the conditional probability density and moments of first-passage time are obtained by solving the backward Kolmogorov equation and generalized Pontryagin equations with suitable initial and boundary conditions. An example is given to illustrate the proposed procedure and the results from digital simulation are obtained to verify the effectiveness of the proposed procedure. The effects of time delay in feedback control forces on the conditional reliability function, conditional probability density and moments of first-passage time are analyzed.

1. Introduction

Time delay is usually unavoidable in feedback control systems due to time spent in measuring and estimating the system state, calculating and executing the control forces, etc. This time delay often leads to instability or poor performance of the controlled systems. Thus, the issue of handling time delay has drawn much attention of the control community.

Systems with time delay under deterministic excitation have been studied by many researchers [1–7]. The study on those systems under stochastic excitation is very limited. The linearly controlled system with deterministic and random time delays excited by Gaussian white noise has been treated by Grigoriu [8] and the stability of such a system has been investigated by means of Lyapunov exponent. The effects of time delay on the controlled linear systems under Gaussian random excitation has been studied by Di Paola and Pirrotta [9] using an approach based on the Taylor expansion of the control force and another approach to finding exact

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stationary solution. The stochastic averaging method for quasi-integrable Hamilton systems with time-delayed feedback control has been proposed by the present authors and the effects of time delay on system response and stability have been studied [10–12].

First-passage failure is a major failure model for mechanical and structural systems under random excitation and it is among the most difficult problems in the theory of random vibration or stochastic structural dynamics. At present, a mathematical exact solution is possible only if the random phenomenon in question can be treated as a diffusion process and the known solutions are limited to one-dimensional case. A feasible way to study the systems with two or higher dimensions is to use stochastic averaging method to reduce the system to averaged Itô equations. In the last three decades, many researchers applied the classical stochastic averaging method to study the first-passage failure problem [13–22]. Recently, Zhu and his coworkers applied the stochastic averaging method for quasi-Hamiltonian systems to study the first-passage problem and the optimal feedback control for maximizing the reliability and mean first-passage time of multi-degree-of-freedom (mdof) quasi-Hamiltonian systems [23–27].

In the present paper, the effects of time delay in control forces on the conditional reliability function, the conditional probability density and moments of first-passage time of controlled quasi-integrable Hamiltonian systems are studied. First, the stochastic averaging method for quasi-integrable Hamiltonian systems with time-delayed feedback control is introduced. The time-delayed feedback control forces are expressed in terms of the system states without time delay in the averaging sense. The motion equations of the system are reduced to a set of averaged Itô stochastic differential equations. Then, a backward Kolmogorov equation governing the conditional reliability function and a set of generalized Pontryagin equations governing the conditional moments of first-passage time are established. The conditional reliability function, the conditional probability density and moments of first-passage time are obtained from solving the backward Kolmogorov equation and generalized Pontryagin equations with suitable initial and boundary conditions. An example is given to illustrate the proposed procedure and the results from digital simulation are obtained to verify the effectiveness of the proposed procedure. The effects of time delay in control forces on the conditional reliability function, conditional probability density and moments of first-passage time are analyzed.

2. Quasi-integrable Hamiltonian systems with time-delayed feedback control

Consider an n-dof quasi-Hamiltonian system with time-delayed feedback control governed by the following Itô stochastic differential equations:

$$\begin{aligned}
\mathrm{d}Q_i &= \frac{\partial H'}{\partial P_i} \\
\mathrm{d}P_i &= -\left[\frac{\partial H'}{\partial Q_i} + \varepsilon c'_{ij}\frac{\partial H'}{\partial P_i} + \varepsilon F_i(\mathbf{Q}_\tau, \mathbf{P}_\tau)\right]\mathrm{d}t + \varepsilon^{1/2}\sigma_{ik}\,\mathrm{d}B_k(t) \\
& i,j = 1,2,\ldots,n, \quad k = 1,2,\ldots,m
\end{aligned} \tag{1}$$

where Q_i and P_i are generalized displacements and momenta, respectively, $\mathbf{Q} = [Q_1, Q_2, \ldots, Q_n]^{\mathrm{T}}$, $\mathbf{P} = [P_1, P_2, \ldots, P_n]^{\mathrm{T}}$; $H' = H'(\mathbf{Q},\mathbf{P})$ is twice differentiable Hamiltonian; ε is a small positive parameter; $\varepsilon c'_{ij} = \varepsilon c'_{ij}(\mathbf{Q},\mathbf{P})$ represent the coefficients of quasi-linear dampings; $B_k(t)$ are standard Wiener processes and $\varepsilon^{1/2}\sigma_{ik}$ represent the amplitudes of stochastic excitations; $\varepsilon F_i(\mathbf{Q}_\tau,\mathbf{P}_\tau)$ with $\mathbf{Q}_\tau = \mathbf{Q}(t-\tau)$ and $\mathbf{P}_\tau = \mathbf{P}(t-\tau)$ denote time-delayed feedback control forces, τ is the time delay, and $\varepsilon F_i(\mathbf{Q}_\tau,\mathbf{P}_\tau) = 0$ when $t \in [0, \tau]$.

Assume that the Hamiltonian H' associated with system (1) is separable and of the form

$$H' = \sum_{i=1}^{n} H'_i(q_i, p_i), \quad H'_i = \frac{1}{2}p_i^2 + G(q_i) \tag{2}$$

where $G(q_i) \geqslant 0$ is symmetric with respect to $q_i = 0$, and with minimum at $q_i = 0$, i.e., the Hamiltonian system with Hamiltonian H' is integrable and has a family of periodic solutions around the origin. Then the solution to Eq. (1) is of the form [27,28]

$$Q_i(t) = A_i \cos \Phi_i(t), \quad P_i(t) = -A_i \frac{\mathrm{d}\Theta_i}{\mathrm{d}t} \sin \Phi_i(t), \quad \Phi_i(t) = \Theta_i(t) + \Gamma_i(t) \tag{3}$$

where $\cos\Phi(t)$ and $\sin\Phi(t)$ are called generalized harmonic functions. For quasi-integrable Hamiltonian systems, $A_i(t)$ and $\Gamma_i(t)$ are slowly varying processes and the averaged value of the instantaneous frequency $\mathrm{d}\Theta_i/\mathrm{d}t$ is equal to $\omega_i(A_i)$. For small delay time τ, we have the following approximate expressions for time-delayed state variables:

$$\begin{aligned} Q_i(t-\tau) &= A_i(t-\tau) \cos \Phi_i(t-\tau) \\ &\approx A_i(t) \cos[\omega_i(t-\tau) + \Gamma_i(t)] \\ &= Q_i(t) \cos \omega_i\tau - \frac{P_i}{\omega_i} \sin \omega_i\tau \\ P_i(t-\tau) &= -A_i(t-\tau) \frac{\mathrm{d}\Theta_i(t-\tau)}{\mathrm{d}t} \sin \Phi_i(t-\tau) \\ &\approx -A_i(t)\omega_i \sin[\omega_i(t-\tau) + \Gamma_i(t)] \\ &= P_i \cos \omega_i\tau + Q_i(t)\omega_i \sin \omega_i\tau \end{aligned} \tag{4}$$

Thus, the time-delayed feedback control forces εF_i ($\mathbf{Q}_\tau$, $\mathbf{P}_\tau$) can be expressed approximately in terms of system state variables without time delay. Note that the numerical results in the present paper and in Refs. [10–12] show that Eq. (4) holds even for larger τ.

In the case of time-delayed feedback bang–bang control, i.e.,

$$\varepsilon F_i(\mathbf{Q}_\tau, \mathbf{P}_\tau) = \varepsilon u_i(P_{i\tau}) = -\varepsilon\eta_i \operatorname{sgn}(p_i(t-\tau)), \quad i = 1, 2, \ldots, n \tag{5}$$

where 'sgn' denotes sign function. εu_i ($P_{i\tau}$) have constant magnitude $\varepsilon\eta_i$ in the opposite direction of $P_{i\tau}$ and changes its direction at $P_{i\tau} = 0$. The time-delayed control forces εu_i ($P_{i\tau}$) can be equivalently replaced by $\varepsilon K_i u_i$ (P_i) in the sense of averaging, i.e.,

$$\int_0^{2\pi/\omega_i} \varepsilon u_i(P_i(t-\tau)) P_i(t)\, \mathrm{d}t = \int_0^{2\pi/\omega_i} \varepsilon K_i u_i(P_i(t)) P_i(t)\, \mathrm{d}t \tag{6}$$

Using the approximate expressions for P_i (t) in Eq. (3) and P_i $(t-\tau)$ in Eq. (4), and assuming that $\Gamma_i = 0$ and $\tau \in [0, 2\pi/\omega_i]$, we have

$$\int_0^{2\pi/\omega_i} \varepsilon u_i(P_i(t-\tau)) P_i(t)\, \mathrm{d}t = \int_0^{2\pi/\omega_i} -\varepsilon\eta_i \operatorname{sgn}(P_i(t-\tau)) P_i(t)\, \mathrm{d}t = -4\varepsilon\eta_i A_i \cos \omega_i\tau \tag{7}$$

and

$$\int_0^{2\pi/\omega_i} \varepsilon K_i u_i(P_i(t)) P_i(t)\, \mathrm{d}t = \int_0^{2\pi/\omega_i} \varepsilon K_i(-\eta_i) \operatorname{sgn}(P_i(t)) P_i(t)\, \mathrm{d}t = -4\varepsilon K_i \eta_i A_i \tag{8}$$

From Eqs. (7) and (8) we obtain $K_i = \cos\omega_i\tau$. Thus, the time-delayed feedback bang–bang control forces in Eq. (1) can be expressed in terms of state variables without time delay as follows:

$$\varepsilon u_i(P_{i\tau}) \doteq \varepsilon u(P_i(t)) \cos \omega_i\tau = -\varepsilon\eta_i \cos \omega_i\tau \operatorname{sgn}(P_i(t)) \tag{9}$$

The terms $\varepsilon F(\mathbf{Q}_\tau,\mathbf{P}_\tau)$ in Eq. (1) can be split into two parts: one has the effect of modifying the conservative forces and the other modifying the damping forces. The first part can be combined with $-\partial H'/\partial Q_i$ to form overall effective conservative forces $-\partial H/\partial Q_i$ with a new Hamiltonian $H = H(\mathbf{Q},\mathbf{P};\tau)$ and with $\partial H/\partial P_i = \partial H'/\partial P_i$. The second part may be combined with $-\varepsilon c'_{ij}\partial H'/\partial P_j$ to constitute effective damping forces $-\varepsilon m_{ij}\partial H/\partial P_i$ with $m_{ij}=m_{ij}(\mathbf{Q},\mathbf{P};\tau)$. With these accomplished, Eq. (1) can be rewritten as

$$\begin{aligned}\mathrm{d}Q_i &= \frac{\partial H}{\partial P_i}\,\mathrm{d}t\\ \mathrm{d}P_i &= -\left(\frac{\partial H}{\partial Q_i}+\varepsilon m_{ij}\frac{\partial H}{\partial P_j}\right)\mathrm{d}t+\varepsilon^{1/2}\sigma_{ik}\,\mathrm{d}B_k(t)\\ &i,j=1,2,\ldots,n,\quad k=1,2,\ldots,m\end{aligned} \tag{10}$$

where $H=H(\mathbf{Q},\mathbf{P};\tau)$, $m_{ij}=m_{ij}(\mathbf{Q},\mathbf{P};\tau)$. Eq. (10) is the Itô equations for quasi-integrable Hamiltonian systems without time-delayed feedback control.

3. Stochastic averaging of quasi-integrable Hamiltonian systems

Assume that the Hamiltonian system with Hamiltonian H is still integrable and nonresonant. That is, the Hamiltonian system has n independent first integrals H_1, $H_2,\ldots,H_n$, which are in involution. The word "in involution" implies that the Poisson bracket of any two of H_1, $H_2,\ldots,H_n$ vanishes. In principle, n pairs of action-angle variables I_i, θ_i can be introduced for an integrable Hamiltonian system of n-dof. Nonresonance means that the n frequencies $\omega_i=\mathrm{d}\theta_i/\mathrm{d}t$ do not satisfy the following resonant relation:

$$k_i^u\omega_i=0(\varepsilon) \tag{11}$$

where $k^u{}_i$ are integers.

Introduce transformations

$$H_r^\varepsilon=H_r(\mathbf{Q},\mathbf{P},\varepsilon),\quad r=1,2,\ldots n \tag{12}$$

The Itô stochastic differential equations for H_r^ε are obtained from Eq. (10) by using Itô differential rule as follows:

$$\begin{aligned}\mathrm{d}H_r^\varepsilon &= \varepsilon\left(-m_{ij}\frac{\partial H}{\partial P_j}\frac{\partial H_r^\varepsilon}{\partial P_i}+\frac{1}{2}\sigma_{ik}\sigma_{jk}\frac{\partial^2 H_r^\varepsilon}{\partial P_i\partial P_j}\right)\mathrm{d}t\\ &+\varepsilon^{1/2}\sigma_{ik}\frac{\partial H_r^\varepsilon}{\partial P_i}\,\mathrm{d}B_k(t),\quad r,i,j=1,2,\ldots n,\quad k=1,2,\ldots,m\end{aligned} \tag{13}$$

where P_i are replaced by H_r^ε in terms of Eq. (12). It is seen from Eqs. (10) and (12) that Q_i are rapidly varying processes while H_r^ε are slowly varying processes. According to the Khasminskii theorem [29], $\mathbf{H}^\varepsilon=[H_1^\varepsilon,H_2^\varepsilon,\ldots,H_n^\varepsilon]^\mathrm{T}$ converges weakly to an n-dimensional vector diffusion process $\mathbf{H}=[H_1,H_2,\ldots,H_n]^\mathrm{T}$ in a time interval $O(\varepsilon^{-1})$ as $\varepsilon\to 0$. For each bounded and continuous real-valued function $f(\mathbf{H})$, the word "H_r^ε converges weakly to H_r" means $\int f(\mathbf{H})\,\mathrm{d}P^\varepsilon(\mathbf{H})\to\int f(\mathbf{H})\,\mathrm{d}P(\mathbf{H})$ as $\varepsilon\to 0$, where $P^\varepsilon(\mathbf{H})$ and $P(\mathbf{H})$ are, respectively, the joint probability distributions of $\mathbf{H}^\varepsilon$ and $\mathbf{H}$. The error between the solutions of the original and averaged systems is of order ε.

The Itô stochastic differential equations for this n-dimensional vector diffusion process can be obtained by applying time averaging to Eq. (13). The result is

$$\mathrm{d}H_r=a_r(\mathbf{H})\,\mathrm{d}t+\overline{\sigma}_{rk}(\mathbf{H})\,\mathrm{d}\overline{B}_k(t),\quad r=1,2,\ldots,n,\quad k=1,2,\ldots,m \tag{14}$$

where $\overline{B}_k(t)$ are independent unit Wiener processes and

$$\begin{aligned}a_r(\mathbf{H}) &= \varepsilon\left\langle -m_{ij}\frac{\partial H}{\partial P_j}\frac{\partial H_r}{\partial P_i}+\frac{1}{2}\sigma_{ik}\sigma_{jk}\frac{\partial^2 H_r}{\partial P_i\partial P_j}\right\rangle_t\\ b_{rs}(\mathbf{H}) &= \overline{\sigma}_{rk}(\mathbf{H})\overline{\sigma}_{sk}(\mathbf{H})=\varepsilon\left\langle \sigma_{ik}\sigma_{jk}\frac{\partial H_r}{\partial P_i}\frac{\partial H_s}{\partial P_j}\right\rangle_t\\ \langle[\bullet]\rangle_t &= \lim_{T\to\infty}\frac{1}{T}\int_T^{t_0+T}[\bullet]\,\mathrm{d}t\end{aligned} \tag{15}$$

Note that H_r are kept constant in performing the time averaging.

The time averaging in Eq. (15) may be replaced by space averaging. For example, suppose that the Hamiltonian is separable and equal to sum of n independent first integers, i.e.,

$$H(\mathbf{q}, \mathbf{p}) = \sum_{r=1}^{n} H_r(q_r, p_r) \tag{16}$$

and for each H_r there is a periodic orbit with period T_r. Then the averaged drift and diffusion coefficients in Eq. (15) become

$$\begin{aligned} a_r(H) &= \frac{\varepsilon}{T} \oint \left(-m_{ij} \frac{\partial H}{\partial P_j} \frac{\partial H_r}{\partial P_i} + \frac{1}{2} \sigma_{ik} \sigma_{jk} \frac{\partial^2 H_r}{\partial P_i \partial P_j} \right) \prod_{u=1}^{n} \left(1 \Big/ \frac{\partial H_u}{\partial P_u} \right) \mathrm{d}q_u \\ b_{rs}(H) &= \frac{\varepsilon}{T} \oint \left(\sigma_{ik} \sigma_{jk} \frac{\partial H_r}{\partial P_i} \frac{\partial H_r}{\partial P_j} \right) \prod_{u=1}^{n} \left(1 \Big/ \frac{\partial H_u}{\partial P_u} \right) \mathrm{d}q_u \end{aligned} \tag{17}$$

where $\oint [\bullet] \Pi_{u=1}^{n} (\cdot\cdot) \, \mathrm{d}q_u$ represents an n-fold loop integral and

$$T = T(\mathbf{H}) = \prod_{u=1}^{n} T_u = \oint \prod_{u=1}^{n} \left(1 \Big/ \frac{\partial H_u}{\partial P_u} \right) \mathrm{d}q_u \tag{18}$$

Note that averaged Eq. (14) is much simpler than original Eq. (10). The dimension of the former equation is only a half of that of the later equation. Averaged Eq. (14) contains only slowly varying process while Eq. (10) contains both rapidly and slowly varying processes. Furthermore, averaged equation can be used to study the long-term behavior of the system, such as stability, stationary response and first-passage failure, since the convergence of H_r^{ε} to diffusion process holds even for $t \to \infty$ [30,31].

4. Backward Kolmogorov equation and generalized Pontryagin equations

For most mechanical and structural systems, Hamiltonian H represents the total energy of the system, and H_r represents the energy of the rth dof of the system. H_r may vary between H_{r0} and ∞, where H_{r0} is a constant. The state of the averaged system of a quasi-integrable Hamiltonian system varies randomly in the n-dimensional domain defined by the direct product of the H_r intervals and the safety domain Ω is a bonded region with boundary Γ within the n-dimensional H_r domain. Suppose that the lower boundary of a safety domain for each H_r is at zero (it is always possible to make so by using coordinate transformation). Then the boundary Γ consists of Γ_0 (at least one of H_r vanishes) and critical boundary Γ_c. The first-passage failure occurs when $\mathbf{H}(t)$ crosses Γ_c for the first time, and it is characterized by the conditional reliability function, the conditional probability density or conditional moments of first-passage time, where the word "conditional" means the system is initially located somewhere in the safety domain.

The conditional reliability function, denoted by $R(t|\mathbf{H}_0)$, is defined as the probability of $\mathbf{H}(t)$ being in safety domain Ω within time interval $(0,t]$ given initial state $\mathbf{H}_0 = \mathbf{H}(0)$ being in Ω, i.e.,

$$R(t|\mathbf{H}_0) = P\{\mathbf{H}(s) \in \Omega, s \in (0, t] | \mathbf{H}_0 \in \Omega\} \tag{19}$$

It is the integral of the conditional transition probability density in Ω, which is the transition probability density of the sample functions that remain in Ω in time interval $[0, t]$. For an averaged system, the conditional transition probability density satisfies the backward Kolmogorov equation with drift and diffusion coefficients defined by Eq. (15) or (17). Thus, the following backward Kolmogorov equation can be derived for the conditional reliability function:

$$\frac{\partial R}{\partial t} = a_r(H_0) \frac{\partial R}{\partial H_{r0}} + \frac{1}{2} b_{rs}(H_0) \frac{\partial^2 R}{\partial H_{r0} \partial H_{s0}}, \quad r, s = 1, 2, \ldots, n \tag{20}$$

where $a_r(\mathbf{H}_0)$ and $b_{rs}(\mathbf{H}_0)$ are defined by Eq. (15) or (17) with $\mathbf{H}$ replaced by $\mathbf{H}_0$. The initial condition is

$$R(0|\mathbf{H}_0) = 1, \quad \mathbf{H}_0 \in \Omega \tag{21}$$

which implies that the system is initially in the safety domain. The boundary conditions are

$$R(t|\Gamma_0) = \text{finite} \tag{22a}$$

$$R(t|\Gamma_c) = 0 \tag{22b}$$

Eq. (22) implies that Γ_0 is a reflecting boundary while Γ_c is the absorbing boundary.

The first-passage time T is defined as the time when the system reaches critical boundary Γ_c for the first time given $\mathbf{H}_0$ being in Ω. Noting that the conditional probability of the first-passage failure is $F(t|\mathbf{H}_0) = 1-R(t|\mathbf{H}_0)$. The conditional probability density of the first-passage time can be obtained from the conditional reliability function as follows:

$$p(T|\mathbf{H}_0) = \left.\frac{-\partial R(t|\mathbf{H}_0)}{\partial t}\right|_{t=T} \tag{23}$$

The conditional moments of first-passage time are defined as

$$\mu_l(H_0) = \int_0^{\infty} T^l p(T|\mathbf{H}_0)\,\mathrm{d}t, \quad l = 1, 2, \ldots. \tag{24}$$

The equations governing the conditional moments of first-passage time can be obtained from Eq. (20) in terms of relations (23) and (24) as follows:

$$\begin{aligned}&\frac{1}{2} b_{rs}(\mathbf{H}_0)\frac{\partial^2 \mu_{l+1}}{\partial H_{r0}\partial H_{s0}} + a_r(\mathbf{H}_0)\frac{\partial \mu_{l+1}}{\partial H_{r0}} = -(l+1)\mu_l\\ &r, s = 1, 2, \ldots, n, \quad l = 0, 1, 2, \ldots\end{aligned} \tag{25}$$

It is easily seen from Eq. (24) that $\mu_0 = 1$. The boundary conditions associated with Eq. (25) are obtained from Eq. (22) in terms of Eqs. (23) and (24). They are

$$\mu_l(\Gamma_0) = \text{finite} \tag{26a}$$

$$\mu_l(\Gamma_c) = 0. \tag{26b}$$

The conditional reliability function is obtained from solving backward Kolmogorov Eq. (20) together with initial condition (21) and boundary conditions (22). The conditional probability density of first-passage time is obtained from the conditional reliability function by using Eq. (23). The conditional moments of first-passage time are obtained either from the conditional probability density of first-passage time by using definition (24) or directly from solving generalized Pontryagin Eq. (25) together with boundary conditions (26). Note that boundary conditions (22a) and (26a) are only qualitative, and can be made to be quantitative by using Eqs. (20) and (25), respectively, and the values of a_r,b_{rs} at Γ_0 [23–28].

5. Example

Consider linearly and nonlinearly coupled two linear oscillators with time-delayed feedback control subject to external and parametric excitations of Gaussian white noises. The equations of motion of the system are of the form

$$\begin{aligned}&\ddot{X}_1 + \alpha'_{11}\dot{X}_1 + \alpha'_{12}\dot{X}_2 + \beta_1(X_1^2 + X_2^2)\dot{X}_1 + \omega'^2_1 X_1 = F_{1\tau} + W_1(t) + X_1 W_2(t)\\ &\ddot{X}_2 + \alpha'_{21}\dot{X}_1 + \alpha'_{22}\dot{X}_2 + \beta_2(X_1^2 + X_2^2)\dot{X}_2 + \omega'^2_2 X_2 = F_{2\tau} + W_3(t) + X_2 W_4(t)\end{aligned} \tag{27}$$

where X_i are generalized coordinates; α'_{ij} and β_i are damping coefficients; ω'_i are the natural frequencies of the two linear oscillators; $W_k(t)(k = 1$–$4)$ are independent Gaussian white noises with intensities $2D_k$; $F_{i\tau}$ represent the time-delayed feedback control forces. Here we study the effects of time delay in feedback control forces on the statistics of the first-passage failure of system (27). Two different cases of $F_{i\tau}$ are considered.

5.1. Case 1

$F_{i\tau}$ are time-delayed linear feedback control forces, i.e., $F_{i\tau} = -\eta_i \dot{X}_{i\tau}$. Following Eq. (4), the time-delayed feedback control forces $F_{i\tau}$ can be expressed in terms of system state variables without time delay as follows:

$$F_{i\tau} = -\eta_i \dot{X}_{i\tau} = -\eta_i \dot{X}_i \cos \omega_i' \tau - \eta_i \omega_i' X_i \sin \omega_i' \tau, i = 1, 2 \tag{28}$$

On the right-hand side of Eq. (28), the first terms are dissipative while the second terms are conservative. They can be combined, respectively, with the damping terms and conservative terms of Eq. (27) to constitute the effective damping terms and effective conservative terms. Let $X_1 = Q_1$, $X_2 = Q_2$, $\dot{X}_1 = P_1, \dot{X}_2 = P_2$, Eq. (27) can be converted into Itô stochastic differential Eq. (10) with

$$\begin{aligned}
&H = H_1 + H_2, \quad H_i = (P_i^2 + \omega_i^2 Q_i^2)/2, \\
&\varepsilon m_{11} = \alpha_{11} + \beta_1 (Q_1^2 + Q_2^2), \quad \varepsilon m_{12} = \alpha_{12}', \\
&\varepsilon m_{21} = \alpha_{21}', \quad \varepsilon m_{22} = \alpha_{22} + \beta_2 (Q_1^2 + Q_2^2), \\
&\varepsilon^{1/2} \sigma_{11} = 1, \quad \varepsilon^{1/2} \sigma_{12} = Q_1, \quad \varepsilon^{1/2} \sigma_{23} = 1, \quad \varepsilon^{1/2} \sigma_{24} = Q_2, \\
&\omega_i^2 = \omega_i'^2 + \eta_i \omega_i' \sin \omega_i' \tau \\
&\alpha_{ii} = \alpha_{ii}' + \eta_i \cos \omega_i' \tau \\
&i,j = 1, 2
\end{aligned} \tag{29}$$

The Hamiltonian system with Hamiltonian H is integrable. Thus, system (27) is a quasi-integrable Hamiltonian system. By using the stochastic averaging method for quasi-integrable Hamiltonian systems, the following averaged Itô equations can be obtained in the nonresonant case:

$$\begin{aligned}
&\mathrm{d}H_r = a_r(H_1, H_2)\,\mathrm{d}t + \overline{\sigma}_{rk}(H_1, H_2)\,\mathrm{d}B_k(t) \\
&r = 1, 2, \quad k = 1, 2, 3, 4
\end{aligned} \tag{30}$$

where

$$\begin{aligned}
a_1 &= -\alpha_{11} H_1 - \frac{\beta_1}{2\omega_1^2} H_1^2 - \frac{\beta_1}{\omega_2^2} H_1 H_2 + D_1 + \frac{D_2}{\omega_1^2} H_1 \\
a_2 &= -\alpha_{22} H_2 - \frac{\beta_2}{2\omega_2^2} H_2^2 - \frac{\beta_2}{\omega_1^2} H_1 H_2 + D_3 + \frac{D_4}{\omega_2^2} H_2 \\
b_{11} &= \overline{\sigma}_{1k} \overline{\sigma}_{1k} = 2D_1 H_1 + D_2 \frac{H_1^2}{\omega_1^2} \\
b_{22} &= \overline{\sigma}_{2k} \overline{\sigma}_{2k} = 2D_3 H_2 + D_4 \frac{H_2^2}{\omega_2^2}
\end{aligned} \tag{31}$$

It is seen that H_i vary from 0 to ∞. So, the state of averaged system (30) varies randomly in the first quadrant of plane (H_1,H_2). Suppose that the limit state of the system is $H = H_1 + \mathrm{H}_2 = H_\mathrm{c})H = H_1 + H_2 = H_c$, i.e.,

$$\Gamma_c : H_1 + H_2 = H_c, H_1, H_2 \geqslant 0 \tag{32}$$

The safety domain of the system is the inside of a right triangle with boundaries Γ_c defined by Eq. (32) and Γ_0 consisting of the following Γ_{01} and Γ_{02}:

$$\begin{aligned}
&\Gamma_{01} : H_1 = 0, \quad 0 \leqslant H_2 < H_c \\
&\Gamma_{02} : H_2 = 0, \quad 0 \leqslant H_1 < H_c
\end{aligned} \tag{33}$$

as shown in Fig. 1.

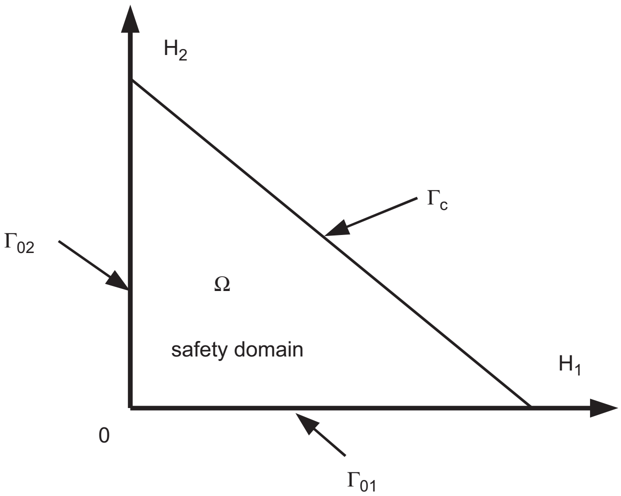

Fig. 1. Safety domain Ω on plane (H_1, H_2) and its boundary for system (27).

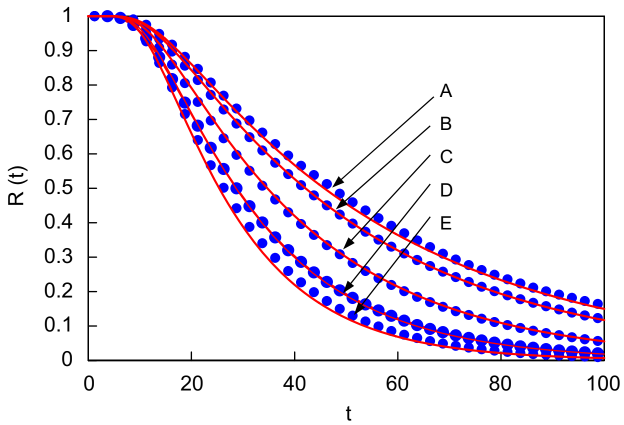

Fig. 2. Reliability function of system (27) in case 1 for different values of time delay τ in feedback control: $\alpha'_{11} = 0.02$, $\alpha'_{12} = 0.01$, $\beta_1 = 0.02$, $\omega'_1 = 1.0$, $2D_1 = 0.02$, $2D_2 = 0.02$, $\eta_1 = 0.03$, $\alpha'_{21} = 0.01$, $\alpha'_{22} = 0.02$, $\beta_2 = 0.02$, $\omega'_2 = 1.414$, $2D_3 = 0.04$, $2D_4 = 0.04$, $\eta_2 = 0.03$, $H_{10} = H_{20} = 0$. A: $\tau = 0$; B: $\tau = 0.5$; C: $\tau = 1.0$; D: $\tau = 1.5$; E: $\tau = 2.0$; (——) analytical result obtained by using the proposed method; and (●) from digital simulation.

Following Eq. (20), the conditional reliability function $R(t|H_{10},H_{20})$ of system (27) is governed by the following backward Kolmogorov equation:

$$\frac{\partial R}{\partial t} = a_1 \frac{\partial R}{\partial H_{10}} + a_2 \frac{\partial R}{\partial H_{20}} + \frac{1}{2} b_{11} \frac{\partial^2 R}{\partial H_{10}^2} + \frac{1}{2} b_{22} \frac{\partial^2 R}{\partial H_{20}^2} \tag{34}$$

where a_1,a_2,b_{11} and b_{22} are defined by Eq. (31) with H_1,H_2 replaced by H_{10} and H_{20}, respectively. The initial condition is Eq. (21) with $\mathbf{H} = [H_{10},H_{20}]^{\mathrm{T}}$. One boundary condition is Eq. (22b) with Γ_c defined by Eq. (32). The other qualitative boundary condition is Eq. (22a) with Γ_0 defined by Eq. (33), which is a reflecting boundary and can be converted into a quantitative boundary condition by inserting the coefficients in Eq. (31) at Γ_0 into Eq. (34).

Eq. (34) with initial and boundary conditions can be solved numerically by using the finite difference method to yield the conditional reliability function of system (27). The conditional probability density of the first-passage time of system (27) is then obtained from the conditional reliability function by using Eq. (23).

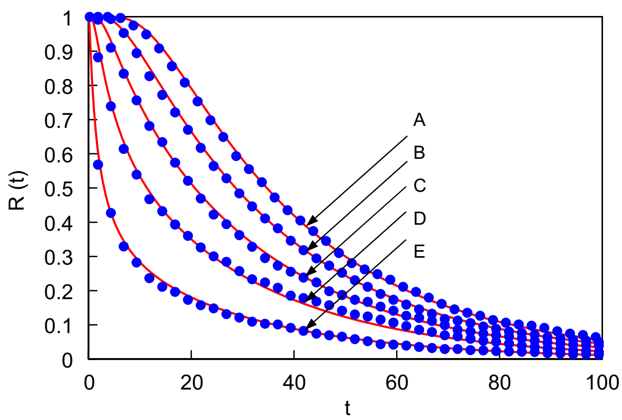

Fig. 3. Reliability function of system (27) in case 1 for different values of initial energy. The parameters are the same as those in Fig. 2 and $\tau = 1$: (A) $H_{10} = H_{20} = 0$; (B) $H_{10} = H_{20} = 0.1$; (C) $H_{10} = H_{20} = 0.2$; (D) $H_{10} = H_{20} = 0.3$; (E) $H_{10} = H_{20} = 0.4$; (——) analytical result obtained by using the proposed method; and (●) from digital simulation.

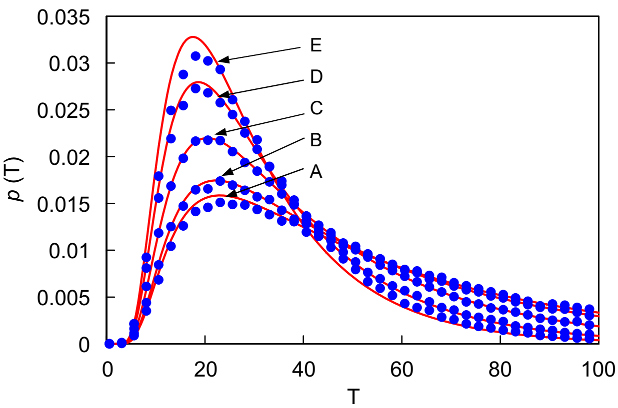

Fig. 4. Probability density of first-passage time of system (27) in case 1 for different values of delay time τ in feedback control. The parameters are the same as those in Fig. 2: (A) $\tau = 0$; (B) $\tau = 0.5$; (C) $\tau = 1.0$; (D) $\tau = 1.5$; (E) $\tau = 2.0$; (——) analytical result by using the proposed method; and (●) from digital simulation.

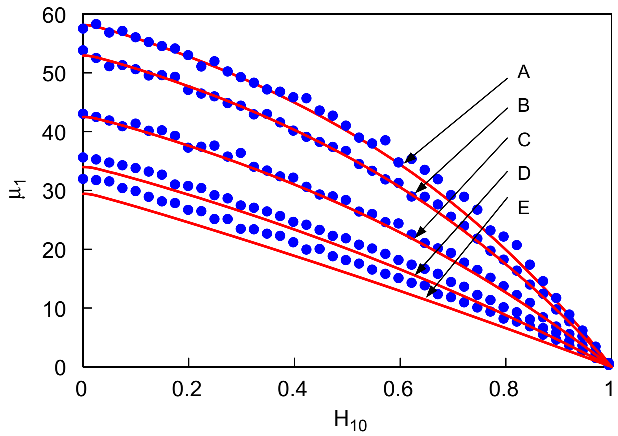

Fig. 5. Mean first-passage time of system (27) in case 1 as function of H_{10} for different values of time delay τ in feedback control. The parameters are the same as those in Fig. 2: (A) $\tau = 0$; (B) $\tau = 0.5$; (C) $\tau = 1.0$; (D) $\tau = 1.5$; (E) $\tau = 2.0$; (——) analytical result obtained by using the proposed method; and (●) from digital simulation.

Similarly, the generalized Pontryagin equations for the conditional moments of the first-passage time of system (27) can be derived from the averaged Itô Eq. (30) as follows:

$$\frac{1}{2}b_{11}\frac{\partial^2\mu_{l+1}}{\partial H_{10}^2}+\frac{1}{2}b_{22}\frac{\partial^2\mu_{l+1}}{\partial H_{20}^2}+a_1\frac{\partial\mu_{l+1}}{\partial H_{10}}+a_2\frac{\partial\mu_{l+1}}{\partial H_{20}}=-(l+1)\mu_l \tag{35}$$

where a_1,a_2, b_{11} and b_{22} are the same as those in Eq. (34). One boundary condition is Eq. (26b) with Γ_c defined by Eq. (32) and the other is qualitative boundary condition Eq. (26a) with Γ_0 defined by Eq. (33), which can be converted into quantitative boundary condition by inserting the coefficients in Eq. (31) at Γ_0 into Eq. (35). Eq. (35) with boundary conditions can be solved numerically by using the finite difference method to yield the conditional moments of first-passage time of system (27).

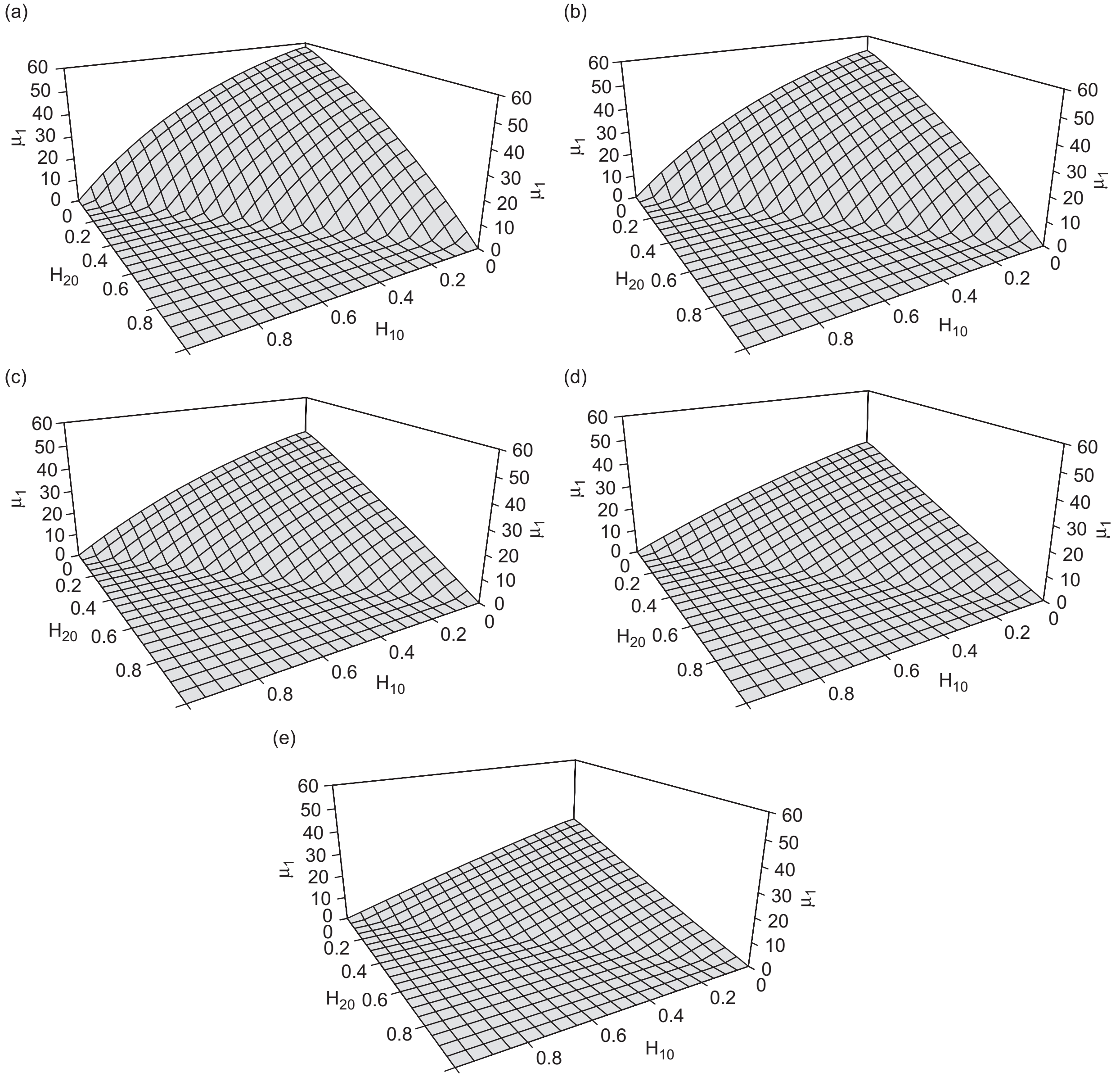

Fig. 6. Mean first-passage time of system (27) in case 1 for different values of delay time τ in feedback control. The parameters are the same as those in Fig. 2: (a) $\tau = 0$; (b) $\tau = 0.5$; (c) $\tau = 1.0$; (d) $\tau = 1.5$; and (e) $\tau = 2.0$.

Some numerical results for the conditional reliability function, the conditional probability density and the conditional mean of the first-passage time of system (27) with time-delayed feedback control obtained by using the above procedure are shown in Figs. 2–7. The results from digital simulations for system (27) are also shown in the figures for comparison. It is seen that the two results are in excellent agreement. It is seen from Figs. 2 and 3 that the conditional reliability function is a monotonously decreasing function of time. Figs. 2 and 3 show that the delay time τ increases, the conditional reliability function and the mean of first-passage

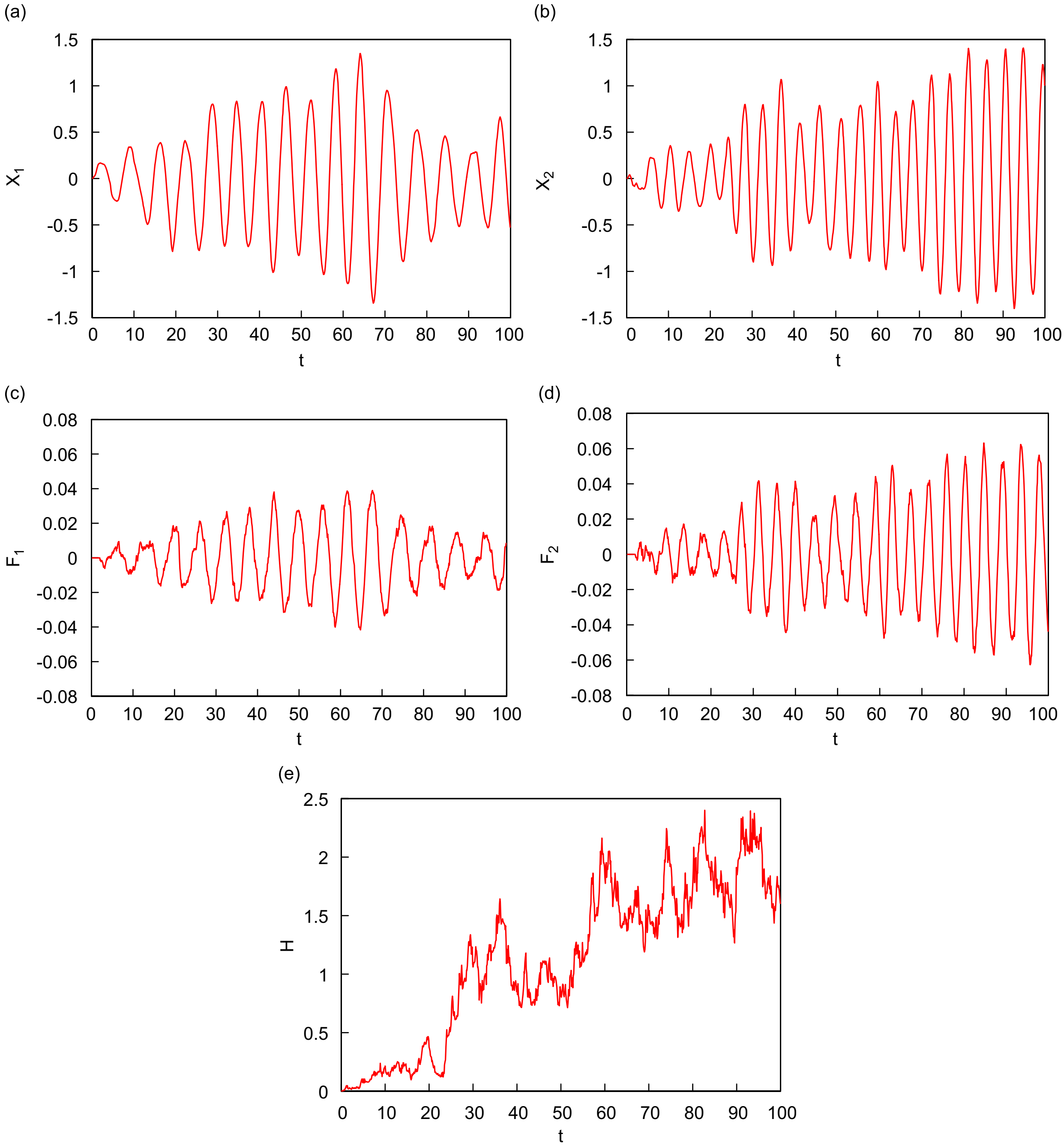

Fig. 7. Samples of displacements, control forces and total energy of system (27) in case 1 with delay time $\tau = 2.0$ in feedback control. The parameters are the same as those in Fig. 2: (a) displacement of the first oscillator; (b) displacement of the second oscillator; (c) control force of the first oscillator; (d) control force of the second oscillator; and (e) total energy H of system.

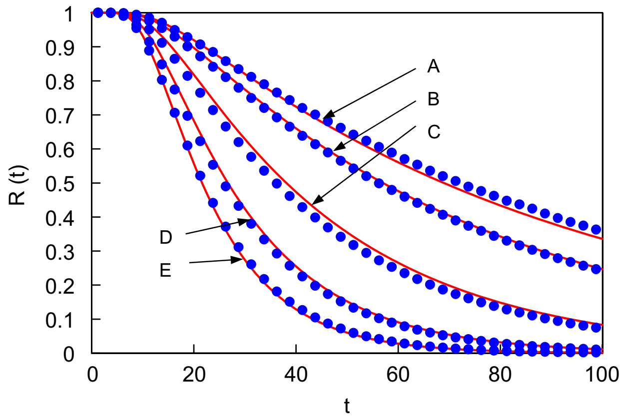

Fig. 8. Reliability function of system (27) for different values of delay time τ in feedback bang-bang control. The parameters are the same as those in Fig. 2: (A) $\tau = 0$; (B) $\tau = 0.5$; (C) $\tau = 1.0$; (D) $\tau = 1.5$; (E) $\tau = 2.0$; (——) analytical result obtained by using the proposed method; and (●) from digital simulation.

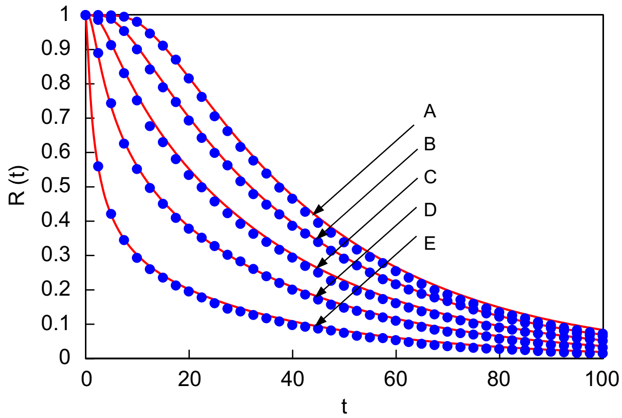

Fig. 9. Reliability function of system (27) for different values of initial energy in feedback bang–bang control. The parameters are the same as those in Fig. 2 and $\tau = 1$: (A) $H_{10} = H_{20} = 0$; (B) $H_{10} = H_{20} = 0.1$; (C) $H_{10} = H_{20} = 0.2$; (D) $H_{10} = H_{20} = 0.3$; (E) $H_{10} = H_{20} = 0.4$; (——) analytical result obtained by using the proposed method; and (●) from digital simulation.

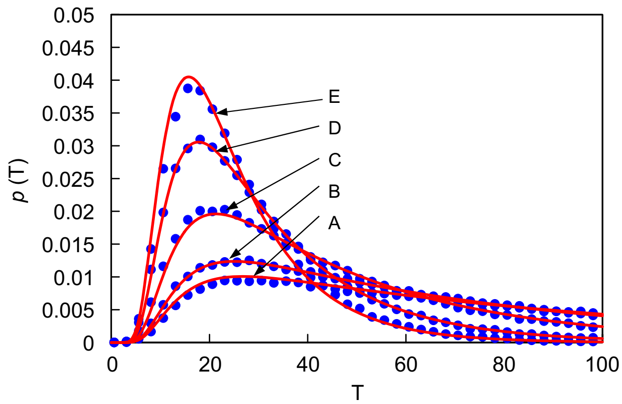

Fig. 10. Probability density of first-passage time of system (27) for different values of delay time τ in feedback bang–bang control. The parameters are the same as those in Fig. 2: (A) $\tau = 0$; (B) $\tau = 0.5$; (C) $\tau = 1.0$; (D) $\tau = 1.5$; (E) $\tau = 2.0$; (——) analytical result obtained by using the proposed method; and (●) from digital simulation.

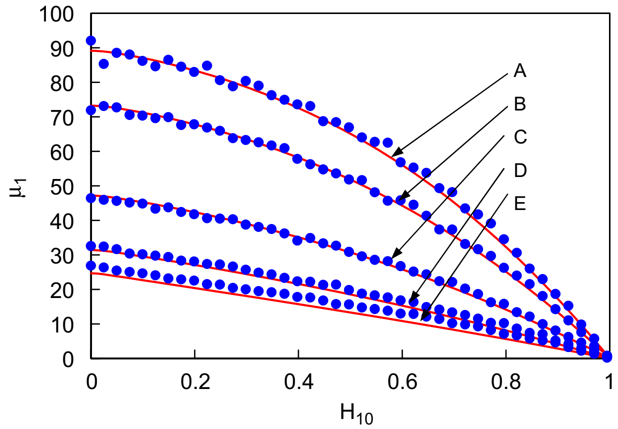

Fig. 11. Mean first-passage time of system (27) as function of H_{10} for different values of delay time τ in feedback bang–bang control. The parameters are the same as those in Fig. 2: (A) $\tau = 0$; (B) $\tau = 0.5$; (C) $\tau = 1.0$; (D) $\tau = 1.5$; (E) $\tau = 2.0$; (——) analytical result obtained by using the proposed method; and (●) from digital simulation.

time of system (27) decrease. It means that time delay in feedback control forces can immensely reduce the reliability of controlled system. It is also seen from Fig. 3 that as the initial energy increases, the conditional reliability function decreases more rapidly.

5.2. Case 2

$F_{i\tau}$ are time-delayed feedback bang–bang control forces, i.e. $F_{i\tau} = -\eta_i\,\mathrm{sgn}(\dot{X}_1)$. Following Eq. (9), the time-delayed feedback bang–bang control forces $F_{i\tau}$ can be expressed in terms of system state variables without time delay as follows:

$$F_{i\tau} = -\eta_i\,\mathrm{sgn}(\dot{X}_{i\tau}) = -\eta_i\,\cos\,\omega_i'\tau\,\mathrm{sgn}(\dot{X}_i) \tag{36}$$

Let $X_1 = Q_1, X_2 = Q_2, \dot{X}_1 = P_1, \dot{X}_2 = P_2$, Eq. (27) can be converted into Itô stochastic differential Eq. (10) with

$$\begin{aligned}
&H = H_1 + H_2, H_i = \left(P_i^2 + \omega_i^2 Q_i^2\right)/2,\\
&\varepsilon m_{11} = \alpha_{11} + \beta_1(Q_1^2 + Q_2^2) + \eta_1\,\cos\,\omega_1\tau/|P_1|, \quad \varepsilon m_{12} = \alpha_{12}',\\
&\varepsilon m_{21} = \alpha_{21}', \quad \varepsilon m_{22} = \alpha_{22} + \beta_2(Q_1^2 + Q_2^2) + \eta_2\,\cos\,\omega_2\tau/|P_2|,\\
&\varepsilon^{1/2}\sigma_{11} = 1, \quad \varepsilon^{1/2}\sigma_{12} = Q_1, \quad \varepsilon^{1/2}\sigma_{23} = 1, \quad \varepsilon^{1/2}\sigma_{24} = Q_2,\\
&\omega_i^2 = \omega_i'^2, \quad \alpha_{ii} = \alpha_{ii}',\\
&i,j = 1,2
\end{aligned} \tag{37}$$

The averaged Itô equations can be obtained in the form of Eq. (30) with the following drift and diffusion coefficients:

$$\begin{aligned}
a_1 &= -\alpha_{11}H_1 - \frac{\beta_1}{2\omega_1^2}H_1^2 - \frac{\beta_1}{\omega_2^2}H_1H_2 - \frac{2\eta_1}{\pi}\sqrt{2H_1} + D_1 + \frac{D_2}{\omega_1^2}H_1\\
a_2 &= -\alpha_{22}H_2 - \frac{\beta_2}{2\omega_2^2}H_2^2 - \frac{\beta_2}{\omega_1^2}H_1H_2 - \frac{2\eta_2}{\pi}\sqrt{2H_2} + D_3 + \frac{D_4}{\omega_2^2}H_2\\
b_{11} &= \bar{\sigma}_{1k}\bar{\sigma}_{1k} = 2D_1H_1 + D_2\frac{H_1^2}{\omega_1^2}\\
b_{22} &= \bar{\sigma}_{2k}\bar{\sigma}_{2k} = 2D_3H_2 + D_4\frac{H_2^2}{\omega_2^2}
\end{aligned} \tag{38}$$

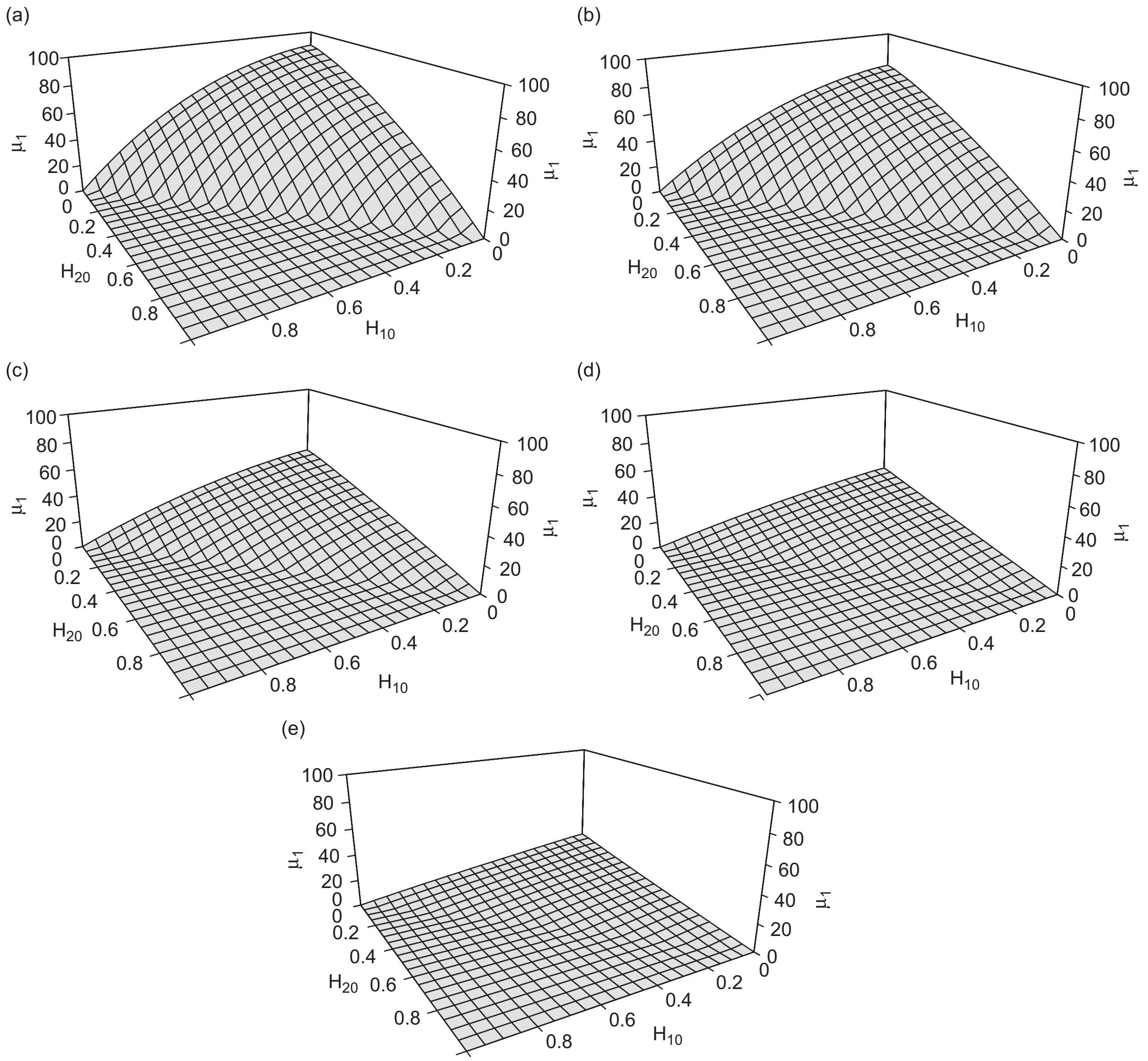

Fig. 12. Mean first-passage time of system (27) for different values of delay time τ in feedback bang–bang control. The parameters are the same as those in Fig. 2: (a) $\tau = 0$; (b) $\tau = 0.5$; (c) $\tau = 1.0$; (d) $\tau = 1.5$; and (e) $\tau = 2.0$.

The safety domain of the system is the same as in case 1. The conditional reliability function $R(t|H_{10},H_{20})$, the conditional probability density $p(T|H_{10},H_{20})$ and the conditional moments $\mu_{l+1}(H_{10},H_{20})$ of first-passage time can be obtained similarly as in case 1.

Some numerical results for the conditional reliability function, the conditional probability density and the conditional mean of the first-passage time of system (27) with time-delayed feedback bang–bang control obtained by using the proposed procedure are shown in Figs. 8–13. The results from digital simulation for system (27) are also obtained for comparison. It is seen that the two results are in excellent agreement. It is seen from Figs. 8 and 11 that, as the delay time τ increases, the conditional reliability function and the mean first-passage time of system (27) decrease. Comparing with case 1, the bang–bang control is more effective in increasing the reliability and mean first-passage than the linear feedback control for the case of without time delay. However, the performance of bang–bang control is easier affected by time delay than the linear feedback control.

6. Conclusions

In the present paper, a procedure for studying the effects of time delay in feedback control on the statistics of the first-passage failure such as the conditional reliability function, the conditional probability density and moments of the first-passage time of quasi-integrable Hamiltonian systems has been proposed based on the

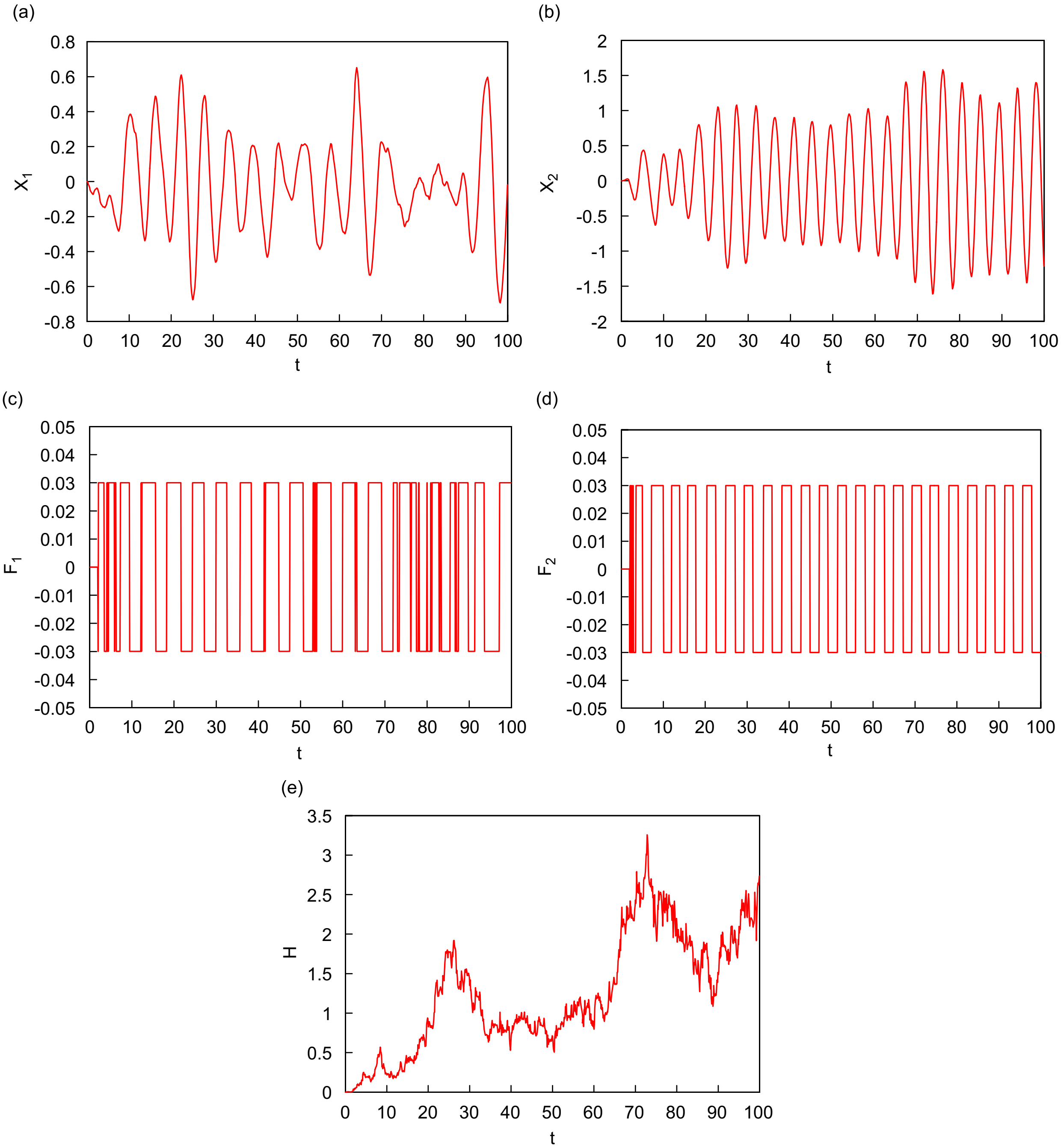

Fig. 13. Samples of displacements, control forces and total energy of system (27) with delay time $\tau = 2.0$ in feedback bang–bang control. The parameters are the same as those in Fig. 2: (a) displacement of the first oscillator; (b) displacement of the second oscillator; (c) control force of the first oscillator; (d) control force of the second oscillator; and (e) total energy H of system.

stochastic averaging method for quasi-integrable Hamiltonian systems. The time-delayed feedback control forces have been expressed approximately in terms of the system state variables without time delay. The backward Kolmogorov equation governing the conditional reliability function and the generalized Pontryagin equations governing the conditional moments of first-passage time have been derived from the averaged Itô equations of the systems. The backward Kolmogorov equation and generalized Pontryagin equations for an example have been solved by using the finite difference method. The effects of time delay in feedback control forces on the statistics of the first-passage time of the example system have been analyzed. The results show that time delay in the feedback control forces may remarkably reduce the conditional reliability and the mean first-passage time of the controlled systems.

Acknowledgments

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