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Forced response of a viscoelastically damped rod using the superposition of modal contribution functions

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Abstract

In this communication the axial vibration problem of a uniform elastic rod with a viscoelastic end damper is studied. The analysis is carried out in the frequency domain, the properties of the damper being characterised by a complex stiffness, and the viscoelastic damping being represented by an exponential model. First, an analytical solution for frequency response functions is obtained using a direct method. Next the computation of the system response is proposed, by means of the modal contribution functions (MCF) superposition method. This method allows evaluating the individual participation of the eigenmodes in the total response (even if the system is not self-adjoint and thus the classical modal superposition cannot be applied), providing important information for practical engineering applications that is lost otherwise. Finally, a numerical example is presented, in which the response provided by both, direct and MCF methods, is compared aimed at validating the latter.

1. Introduction

In machinery, transmission elements such as ball screws transmit dynamic axial forces [1], and consequently, axial vibrations are induced. These vibrations can be mitigated using dampers, which dissipate mechanical energy into heat. If dampers are characterised by viscous damping, the dissipative forces are proportional to the actual velocity, and the mechanical response may be analysed by means of classical methods for damped linear system vibrations (see e.g. Refs. [2,3]). However, if viscoelastic damping is considered, the response of the system depends on the complete history of the load, the relationship between forces and velocity being nonlinear (see the exhaustive work of Adhikari [4] for details on viscoelastic damping). In this sense, Golla and Hughes [5] proposed a time domain formulation for the analysis of structural systems with viscoelastic damping, and Adhikari [6] extends classical modal analysis to treat lumped-parameter nonviscously damped linear dynamic systems.

For the analysis of structural systems in the frequency domain, it should be taken into account that viscoelasticity implies frequency-dependent damping properties. Indeed, Cortés and Elejabarrieta developed

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approximate numerical methods for the extraction of eigenvalues and eigenvectors in structural systems in which damping is modelled by structural [7] and viscous [8] matrices whose coefficients vary with frequency.

Besides, if viscoelasticity is modelled by structural damping varying with frequency, modal superposition cannot be applied to calculate frequency response functions. Thus, direct methods are usually employed, leading to loss of information about the contribution of eigenmodes to the total response. Consequently, this work is aimed at obtaining the frequency response function of a uniform elastic rod with a viscoelastic end damper by means of the superposition of a certain kind of functions indicating modal participation.

Firstly, the problem is defined and solved by a direct method. The modal contribution superposition method is described next, and finally, a numerical example is presented comparing the results provided by both methods.

2. Problem definition and direct method solution

The axial vibration problem of a uniform elastic rod of length L with a viscoelastic end damper (see Fig. 1) is given next. The displacement of any plane cross section Ω_x is represented by $u(x,t)$, where x and t indicate spatial and time variables. The harmonic force excitation $F^*(t)$ is applied in the section $x = \ell$, and it is given by

$$F^*(t) = F_0 e^{i\omega t}, \tag{1}$$

where $(\cdot)^*$ represents a complex magnitude, F_0 is the force amplitude, ω represents the circular frequency and $i = \sqrt{-1}$ denotes the imaginary operator.

The left-hand side of the rod is clamped, and a viscoelastic damper actuates on the right-hand side, represented by a string with complex stiffness k^* [9] varying with frequency ω, defined as follows:

$$k^*(\omega) = k(\omega)[1 + i\eta(\omega)], \tag{2}$$

where $k(\omega)$ and $\eta(\omega)$ represent the stiffness and loss factor, respectively. The former is related to the elastic response of the damper and the latter to the dissipative behaviour.

The field equation (see e.g. Refs. [2,3])

$$ES\frac{\partial^2 u(x,t)}{\partial x^2} - \rho S\frac{\partial^2 u(x,t)}{\partial t^2} = 0, \tag{3}$$

must be solved for the displacement field $u(x,t)$ by a direct method, where E and ρ are the Young modulus and the density of the rod material, respectively, and S is the cross-sectional area. Due to the discontinuity induced by the force, the response $u(x,t)$ is given by two harmonic displacement functions, $u_1(x,t)$ for $0 \leqslant x \leqslant \ell$ and $u_2(x,t)$ for $\ell \leqslant x \leqslant L$. The following four boundary conditions must be considered:

1. Displacement of the left-hand side,

$$u_1(0,t) = 0. \tag{4}$$

2. Axial force on the right-hand side,

$$ES\left.\frac{\partial u_2(x,t)}{\partial x}\right|_{x=L} = -k^* u_2(L,t). \tag{5}$$

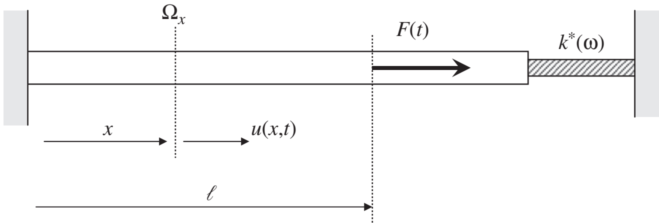

Fig. 1. Viscoelastically damped rod.

3. Continuity of displacement field for $x = \ell$,

$$u_1(\ell^-, t) = u_2(\ell^+, t). \tag{6}$$

4. Discontinuity of axial force for $x = \ell$,

$$ES\frac{\partial u_1(x,t)}{\partial x}\bigg|_{x=\ell^-} - ES\frac{\partial u_2(x,t)}{\partial x}\bigg|_{x=\ell^+} = F(t). \tag{7}$$

Thus, the displacements $u_1(x,t)$ and $u_2(x,t)$ as a function of circular frequency ω yield

$$u_1(x,t) = \frac{c\,\tan(\omega\ell/c)\sin(\omega x/c)}{ES\omega\left(1 - \dfrac{ES\omega\,\tan(\omega(L-\ell)/c) - ck^*(\omega)}{ES\omega + ck^*(\omega)\tan(\omega(L-\ell)/c)}\tan(\omega\ell/c)\right)\sin(\omega\ell/c)}F^*(t), \tag{8}$$

and

$$u_2(x,t) = \frac{\left(\dfrac{ES\omega\,\tan(\omega(L-\ell)/c) - ck^*(\omega)}{ES\omega + ck^*(\omega)\tan(\omega(L-\ell)/c)}\sin(\omega(x-\ell)/\omega) + \cos(\omega(x-\ell)/c)\right)}{ES\omega\left(1 - \dfrac{ES\omega\tan(\omega(L-\ell)/c) - ck^*(\omega)}{ES\omega + ck^*(\omega)\tan(\omega(L-\ell)/c)}\tan(\omega\ell/c)\right)\sin(\omega\ell/c)} \times c\,\tan(\omega\ell/c)F^*(t), \tag{9}$$

respectively, where $c = \sqrt{E/\rho}$ is the longitudinal wave velocity. From these two equations any frequency response function can be deduced. For example, that of the force application point direct one $H^*_{\ell,\ell}$ satisfies

$$H^*_{\ell,\ell}(\omega) = \frac{(ES\omega + ck^*(\omega)\tan(\omega(L-\ell)/c))\tan(\omega\ell/c)}{ES\omega(1 - \tan(\omega\ell/c)\tan(\omega(L-\ell)/c))(ES\omega + ck^*(\omega)\tan(\omega\ell/c))}. \tag{10}$$

3. Modal contribution functions superposition method

If the damper properties were constant with frequency, the system would be self-adjoint and the response would be calculated by modal superposition. Any two mode shapes $U_r^*(x)$ and $U_s^*(x)$ are complex and orthogonal to the mass and stiffness operators, satisfying

$$\int_L U_r^*(x)\rho S U_s^*(x)\,\mathrm{d}x = m_r^*\delta_{rs}, \tag{11}$$

and

$$\int_L \frac{\mathrm{d}U_r^*(x)}{\mathrm{d}x} ES \frac{\mathrm{d}U_s^*(x)}{\mathrm{d}x}\mathrm{d}x = k_r^*\delta_{rs}, \tag{12}$$

respectively, where m_r^* and k_r^* are the complex modal mass and stiffness of rth mode, respectively, and δ_{rs} is Kronecker's delta, defined as follows:

$$\delta_{rs} = \begin{cases} 0 & \text{if} \quad r \neq s, \\ 1 & \text{if} \quad r = s. \end{cases} \tag{13}$$

To examine the orthogonality property, the results for these modal functions are taken from Ref. [11]:

$$U_r^*(x) = A_r^*\,\sin(\beta_r^* x), \tag{14}$$

where A_r^* is the normalised complex modal constant and β_r^* the complex wavenumber. This wavenumber was obtained from the eigenvalue equation

$$ES\beta_r^* + k^*\,\tan(\beta_r^* L) = 0, \tag{15}$$

and for the sth mode,

$$ES\beta_s^* + k^* \tan(\beta_s^* L) = 0. \tag{16}$$

By multiplying Eq. (15) by β_s^* and Eq. (16) by β_r^*, the subtraction gives

$$k^*\beta_s^* \sin(\beta_r^* L)\cos(\beta_s^* L) - k^*\beta_r^* \sin(\beta_s^* L)\cos(\beta_r^* L) = 0. \tag{17}$$

Besides, for $\beta_r^* \neq \beta_s^*$, the integral (11) gives

$$\int_L U_r^*(x)\rho S U_s^*(x)\mathrm{d}x = \rho S \frac{A_r^* A_s^*}{\beta_r^{*2} - \beta_s^{*2}}[\beta_s^* \sin(\beta_r^* L)\cos(\beta_s^* L) - \beta_r^* \sin(\beta_r^* L)\cos(\beta_s^* L)], \tag{18}$$

which is zero by virtue of Eq. (17), demonstrating the orthogonality property with respect to the mass operator. A similar procedure may be carried out to verify Eq. (12).

Thus, according to the expansion theorem, the time response may be developed in a series using the mode shapes and natural frequencies, which yields

$$u(x,t) = \mathrm{Re}\left(\sum_{r=1}^{\infty} \frac{2a_r^* \sin(\omega_r^* \ell/c)\sin(\omega_r^* x/c)}{m_b(\omega_r^{*2} - \omega^2)} F^*(t)\right), \tag{19}$$

where $m_b = \rho SL$ is the mass of the rod, and ω_r^* and a_r^* are the rth complex natural frequency and modal constant normalised with respect to the unit mass. In this way, the frequency response function (FRF) $H_{x,\ell}^*(\omega)$ yields in

$$H_{x,\ell}^*(\omega) = \sum_{r=1}^{\infty} {}_rH_{x,\ell}^*(\omega), \tag{20}$$

where ${}_rH_{x,\ell}^*(\omega)$ indicates the individual participation for the rth mode, given by

$${}_rH_{x,\ell}^*(\omega) = \frac{2}{m_b}\frac{a_r^{*2} \sin(\omega_r^* x/c)\sin(\omega_r^* \ell/c)}{\omega_r^{*2} - \omega^2}. \tag{21}$$

However, by keeping in mind the frequency dependence of the complex stiffness k^*, the self-adjoint character of the system is lost; thus, the modal superposition method cannot be applied. In effect, in this case Eq. (17) becomes

$$k^*(\beta_r^*)\beta_s^* \sin(\beta_r^* L)\cos(\beta_s^* L) - k^*(\beta_s^*)\beta_r^* \sin(\beta_s^* L)\cos(\beta_r^* L) = 0. \tag{22}$$

Thus, the orthogonality property of Eq. (18) is not satisfied because, in a general case,

$$k^*(\beta_r^*) \neq k^*(\beta_s^*), \tag{23}$$

due to the frequency dependence of the complex stiffness of the spring.

Therefore, next the superposition of modal contribution functions (MCF) method is proposed, which was employed by the authors in a lumped matrix system [10]. This method considers that the frequency response function may be obtained by the superposition of individual functions representing the response of respective eigenmodes,

$$H_{x,\ell}^*(\omega) = \sum_{r=1}^{\infty} {}_r\mathrm{MCF}_{x,\ell}^*(\omega), \tag{24}$$

where the ${}_r\mathrm{MCF}_{x,\ell}^*(\omega)$ function takes into account the contribution of frequency variable modes (not having the strict sense of eigenmodes), given by

$${}_r\mathrm{MCF}_{x,\ell}^*(\omega) = \frac{2}{m_b}\frac{a_{r,\omega}^{*2} \sin(\omega_{r,\omega}^* x/c)\sin(\omega_{r,\omega}^* \ell/c)}{\omega_{r,\omega}^{*2} - \omega^2}, \tag{25}$$

where $\omega_{r,\omega}^*$ and $a_{r,\omega}^*$ represent the complex natural frequency and normalised modal constant for the rth mode, respectively, both dependent on frequency ω, which may be evaluated as was presented by the authors in Ref. [11].

Having reached this point, it is important to remark that the so-called nonviscous modes have not been considered. In effect, viscoelastic damping introduces a set of extra real eigenvalues into the system [12,13], whose nonviscous modes are overcritically damped modes, i.e. they do not show oscillatory behaviour. Their contribution to the total response disappears with time, having no special relevance in stationary analysis.

4. Numerical application

Next the analysis of the direct FRF for $\ell = L/2$ is carried out considering an exponential model [14] for the damper complex stiffness k^*, considering a constant elastic component k and a variable loss η factor given by

$$\eta(\beta) = \eta_{\max} \frac{2\omega_m \omega}{\omega_m^2 + \omega^2}, \tag{26}$$

where $k = ES/L$, $\omega_m = 4/c$ and $\eta_{\max} = 1$ have been chosen, as in Ref. [11].

Fig. 2 shows the first four modal contribution functions. On the abscissa axis the dimensionless frequency $\beta L = \omega L/c$ is represented, from 0 to 12, aimed at studying the influence of the first four modes, β being the wavenumber. The magnitude, represented in logarithmic scale on the ordinate axis, is normalised with respect to the static response of the system with a free right-hand side, $1/2k_b$, where $k_b = ES/L$ denotes the stiffness of the rod. The phase is represented in degrees, from -180° to 0°.

The curves represented in Fig. 2 provide information about the participation of the eigenmodes into the total response, information that is lost if a direct method is employed. The superposition of these four MCF is compared in Fig. 3(a) with the response provided by the direct solution given by Eq. (10).

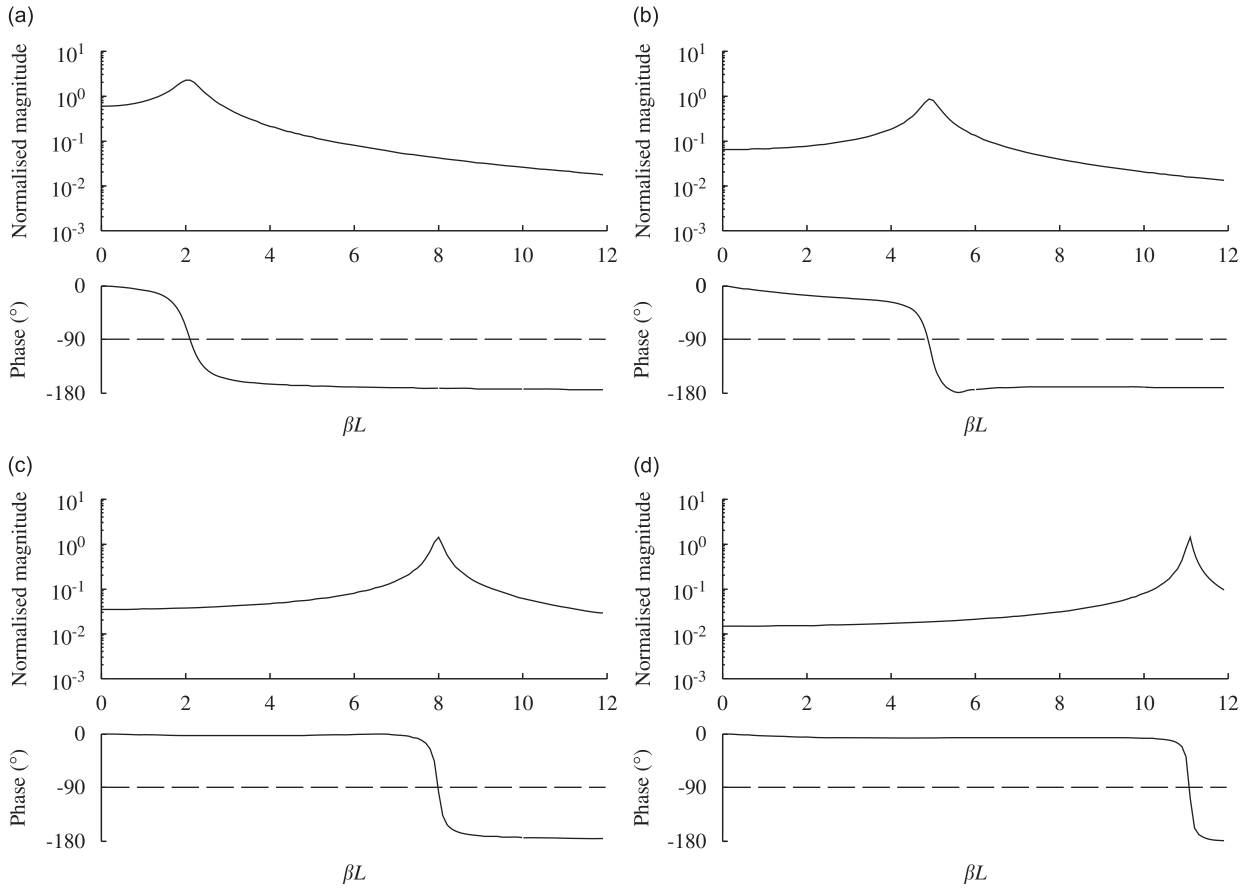

Fig. 2. Amplitude curve for the (a) first, (b) second, (c) third and (d) fourth modal contribution functions.

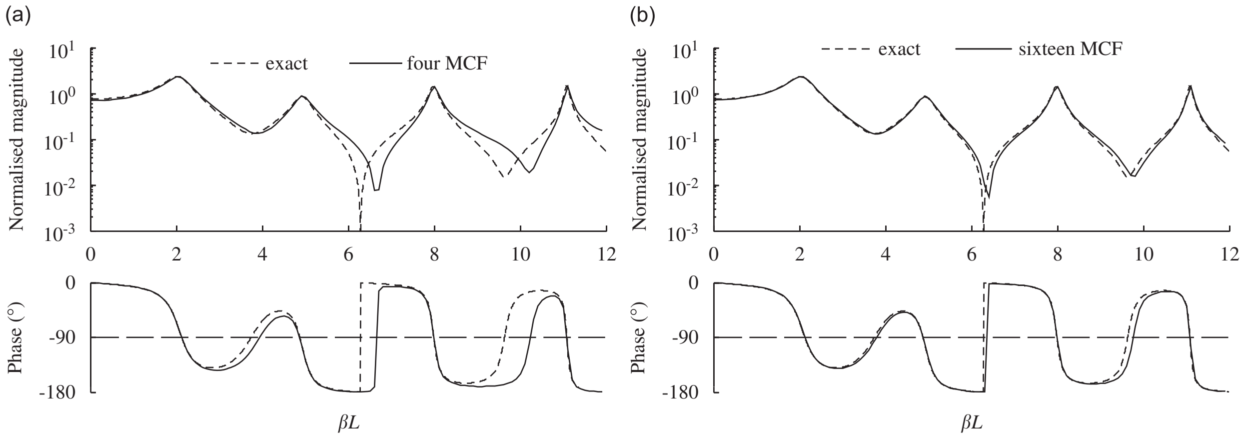

Fig. 3. Comparison between exact and MCF superposition using (a) four and (b) sixteen MCF.

In this figure it can be pointed out that the differences are due to the modal truncation. It should be remarked that these differences diminish as a higher number of modes are taken into account. This fact can be verified in Fig. 3(b), in which sixteen MCF have been superposed: close resonance peaks; there are no significant differences between both curves, but these differences become more remarkable in the vicinity of anti-resonance frequencies, differences that, as has been previously mentioned, would be reduced if a higher number of modes are taken into account.

5. Concluding remarks

In this paper the axial vibration problem of a uniform elastic rod with a viscoelastic end damper has been solved in the frequency domain by means of the superposition of modal contribution functions (MCF). In effect, the properties of the damper have being characterised by a complex stiffness with properties dependent on frequency. Hence, modal superposition cannot be applied, and direct methods are normally employed, leading to loss of information on modal participation. By applying the MCF superposition method, this modal participation has been retrieved, which may be important for some engineering applications.

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