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On energy transfer between linear and nonlinear oscillators

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Abstract

In this paper energy transfer in a dissipative mechanical system is analysed. Such system is composed of a linear and a nonlinear oscillator with a nonlinearizable cubic stiffness. Depending on initial conditions, we find energy transfer either from linear to nonlinear oscillator (energy pumping) or from nonlinear to linear. Such results are valid for two different potentials. However, under resonance and absence of external excitation, if the mass of the nonlinear oscillator is adequately small then the linear oscillator always loses energy. Our approach uses rigorous Regular Perturbation Theory. Besides, we have included the case of two linear oscillators under linear or cubic interactions. Comparisons with the earlier case are made.

1. Introduction

This work concerns energy transfer between oscillators. This is a natural way to study the vibration isolation in a mechanical system. Indeed, it would be interesting to know if there is an energy transfer from a linear oscillator to another one. The case of one of the oscillators having an essential cubic nonlinearity has been intensively investigated in several papers such as Refs. [1–6]. Indeed, in some of these papers the main idea is to study the dynamics of the undamped, *but perturbed system*, and from this study, we obtain results about the energy pumping of the damped system. In this line of research, the study of nonlinear normal modes (NNM), see Refs. [7,8], of the undamped system is a natural goal.

The approach presented here is very different from the ones used in the mentioned papers. In all cases treated in this work there are two oscillators for the unperturbed system. We are interested in the *oscillations of the energy of each oscillator calculated on the orbits of the perturbed system.* The energies of these oscillators are denoted by E_1 and E_2. A precise definition of this idea is given in the next section, see Eq. (3). *Such oscillations occur in a transient regime*.

It should be noticed that the main mechanical system investigated here is a spring–mass system weakly damped with two degrees of freedom. In a mathematically rigorous way, expansions of E_1 and E_2 are

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obtained in several important cases in Sections 3 and 4. All conclusions come from these expansions. The novelty of the results presented here is that they have been derived within a uniform theoretical framework based on Regular Perturbation Theory.

This paper is organized as follows: Section 2 gives a precise meaning of our interpretation of the energy transfer in the vibrating system given by Eq. (1). In Section 2.1 a particular spring–mass system is given. The study of this system, under particular interactions and couplings, constitutes the bulk of the paper. In Section 3, it is assumed that one of the anchor springs has a nonlinearizable cubic stiffness. This case has been implemented in a mechanical array in Ref. [4]. Here, the use of a special change of variables allows the use of Regular Perturbation Theory in a rigorous way. This approach permits one to see how the phenomenon of resonance appears here. Under the linear interaction and non-resonance condition, it is proved in Section 3.1 that there is no energy transfer between the oscillators. But in the presence of resonances and absence of external excitation, there is a more complex situation than the usual pumping considered in the current literature. Here, depending on the initial conditions, there is energy transfer from linear to nonlinear oscillator, and from nonlinear to linear oscillator. Moreover, there are initial conditions in which both oscillators lose energy. In all cases, precise regions of the phase space where such initial conditions occur are given by inequalities. We would like to emphasize that in an adequate two-dimensional projection of the phase space, there is a partition in regions where each situation of energy transfer from an oscillator to another one is shown. And, by gathering this information, the plot given in Fig. 2 is obtained. A useful consequence of the analysis in Section 3.1.2 is an estimate of the mass m_1, in order to get always a decreasing energy of the linear oscillator. These results show that the use of expansions given by Eq. (34) can give much information. Besides, Section 3.1.3 shows a relationship between the resonance condition and the nonlinear modes of the unperturbed system, with initial conditions given by Eq. (43). In Section 3.2 some comments are made on the case of cubic interactions, where analogous results can be obtained. Here, unlike in Section 3.1.2, detailed computations were not made. In Section 4, some additional computations are made about the linear case, i.e., when the anchor springs are linear. Particularly, there is energy transfer if resonance occurs. Similar to the results in Section 3.1.2, Figs. 3 and 4 are obtained. Under non-resonance condition, there is no energy transfer. Finally, there is an Appendix about elliptic functions.

Anyway, the results presented in this paper take into account a more general dynamic phenomenon than energy pumping. Here, energy pumping is understood as energy transfer from a linear oscillator to a nonlinear one.

2. A definition of energy transfer

Consider the following system:

$$\begin{cases} \dot{q}_1 = \dfrac{\partial H_1}{\partial p_1}(q_1,p_1) + \varepsilon R_1(q_1,p_1,q_2,p_2,t,\varepsilon), \\ \dot{p}_1 = -\dfrac{\partial H_1}{\partial q_1}(q_1,p_1) + \varepsilon R_2(q_1,p_1,q_2,p_2,t,\varepsilon), \\ \dot{q}_2 = \dfrac{\partial H_2}{\partial p_2}(q_2,p_2) + \varepsilon R_3(q_1,p_1,q_2,p_2,t,\varepsilon), \\ \dot{p}_2 = -\dfrac{\partial H_2}{\partial q_2}(q_2,p_2) + \varepsilon R_4(q_1,p_1,q_2,p_2,t,\varepsilon), \end{cases} \tag{1}$$

where $H_i, i = 1, 2$ and $R_j, j = 1, \ldots, 5$ are functions adequately smooth defined on open sets of $\mathbb{R}^2$ and $\mathbb{R}^6$, respectively. It is assumed that each open set contains the origin and each R_j is T-periodic in the variable t. Note that Eq. (1) is not necessarily a Hamiltonian system, because it eventually contains a dissipative term in the perturbation. Define

$$\begin{cases} E_1(t,\varepsilon,a,b,c,d) = H_1(q_1(t,\varepsilon),p_1(t,\varepsilon)), \\ E_2(t,\varepsilon,a,b,c,d) = H_2(q_2(t,\varepsilon),p_2(t,\varepsilon)), \end{cases} \tag{2}$$

where $(q_1(t,\varepsilon), p_1(t,\varepsilon), q_2(t,\varepsilon), p_2(t,\varepsilon))$ is the solution of Eq. (1) such that $(q_1(0,\varepsilon), p_1(0,\varepsilon), q_2(0,\varepsilon), p_2(0,\varepsilon)) = (a,b,c,d)$.

It is said that there is a transfer of energy, from the oscillator 1 to oscillator 2, at the point (a,b,c,d) of the phase space of the system given by Eq. (1), if the following condition is satisfied.

There is $T_0 = T_0(a,b,c,d) \geqslant 0$ such that for all finite time interval $[T_1, T]$, with $T_1 \geqslant T_0$, there exists $\varepsilon_0 = \varepsilon_0(a,b,c,d,T) > 0$ such that

$$\begin{aligned} &E_1(t,\varepsilon,a,b,c,d) < E_1(0,\varepsilon,a,b,c,d),\\ &E_2(t,\varepsilon,a,b,c,d) > E_2(0,\varepsilon,a,b,c,d),\\ &\text{for all } t \in [T_1, T] \text{ and } \varepsilon \in (0,\varepsilon_0). \end{aligned} \tag{3}$$

Moreover, it follows from Eq. (2) that

$$\begin{aligned} E_1(0,\varepsilon,a,b,c,d) &= H_1(a,b),\\ E_2(0,\varepsilon,a,b,c,d) &= H_2(c,d). \end{aligned}$$

Note that $E_1(t,\varepsilon,a,b,c,d)$ and $E_2(t,\varepsilon,a,b,c,d)$ are the energies of the unperturbed oscillators, obtained from Eq. (1) making $\varepsilon = 0$, computed on the perturbed trajectories. Additionally, in order to use correctly the Regular Perturbation Theory, it is necessary to work in a finite time interval. So, this restriction was included in the above definition.

2.1. A particular case: the spring–mass system with two degree of freedom

Consider the classical vibrating system in Fig. 1.

Assuming that an external excitation has been just applied on the body 1, the governing equations of this system are given by

$$\begin{cases} x'' + f(x) + \varepsilon\left(c_0 x' + \dfrac{\partial V}{\partial x}(x,y) + A\sin(\omega t)\right) = 0,\\ m_1 y'' + g(y) + \varepsilon\left(c_0 y' + \dfrac{\partial V}{\partial y}(x,y)\right) = 0, \end{cases} \tag{4}$$

where c_0 is the coefficient of the viscous damping . It is assumed that the bodies 1 and 2 have masses equal to 1 and m_1, respectively. Here x is the displacement of the body 1 from its equilibrium position and y is the displacement of the body 2. Besides, V is the potential energy associated with the coupling spring, A and ω, are, respectively, the amplitude and frequency of the external excitation. It is assumed that $V(0,0) = 0$. Making the following change of variables:

$$q_1 = x, \quad p_1 = x', \quad q_2 = y, \quad p_2 = y', \tag{5}$$

Eq. (4) can be written as a time-dependent perturbation of a Hamiltonian system whose energy is given by

$$H(q_1,p_1,q_2,p_2) = \frac{p_1^2}{2} + F(q_1) + \frac{p_2^2}{2} + G(q_2) + \varepsilon V(q_1,q_2). \tag{6}$$

where F, G are such that $F'(x) = f(x)$, $F(0) = 0, G'(y) = g(y)$, $G(0) = 0$. Clearly, in this case $H_1(q_1,p_1) = p_1^2/2 + F(q_1)$ and $H_2(q_2,p_2) = p_2^2/2 + G(q_2)$. For the cases $F(q_1) = q_1^2/2$, $G(q_2) = q_2^4/4$ and V is given by Eqs. (7) and (8) and $A = 0$, such models have been intensively investigated in Refs. [1,9] under the following

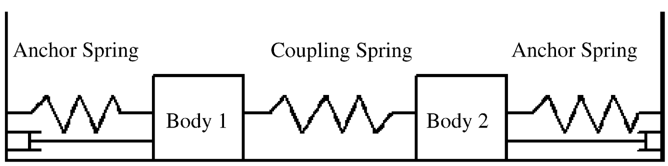

Fig. 1. A vibrating system.

potentials:

$$V(q_1, q_2) = \frac{(q_1 - q_2)^2}{2} \tag{7}$$

and

$$V(q_1, q_2) = \tfrac{1}{4}(q_1^4 - 4q_1^3 q_2 + 6q_1^2 q_2^2 - 4q_1 q_2^3). \tag{8}$$

Natural questions are the following ones: What happens with the energies of the system, defined by Eq. (2), in a finite time interval? Which energy increases or decreases? It would be interesting to know if there exists a transfer (pumping) of energy between the oscillators 1 and 2 in the sense of Eq. (3).

3. The energy transfer between a linear and a nonlinear oscillator with essential stiffness nonlinearity

In the remainder of this section, it is assumed that $f(x) = \omega_1^2 x$, $g(y) = y^3$ in Eq. (4). It follows from Eqs. (5) and (4) that

$$\begin{cases} \dot{q}_1 = p_1, \\ \dot{p}_1 = -\omega_1^2 q_1 - \varepsilon\left(c_0 p_1 + \dfrac{\partial V}{\partial q_1} + A\sin(\omega t)\right), \\ \dot{q}_2 = p_2, \\ \dot{p}_2 = -\dfrac{1}{m_1} q_2^3 - \varepsilon \dfrac{1}{m_1}\left(c_0 p_2 + \dfrac{\partial V}{\partial q_2}\right). \end{cases} \tag{9}$$

Hence, the unperturbed energies are given by

$$\begin{aligned} H_1(q_1, p_1) &= \frac{p_1^2 + \omega_1^2 q_1^2}{2}, \\ H_2(q_2, p_2) &= \frac{p_2^2}{2} + \frac{q_2^4}{4m_1}. \end{aligned} \tag{10}$$

Consider the elliptic Jacobian functions defined in the Appendix. For more details see Ref. [10]. Write $cn(t) = cn[t, \frac{1}{\sqrt{2}}], sn(t) = sn[t, \frac{1}{\sqrt{2}}], dn(t) = dn[t, \frac{1}{\sqrt{2}}]$. The function $cn(t)$ is periodic and satisfies the following differential equation $cn''(t) + (cn(t))^3 = 0$ (see Appendix, Eq. (63)). Its period is equal to $4K = 4K(1/\sqrt{2})$ (see Appendix, Eq. (62)). Moreover, its Fourier expansion is given by

$$\begin{aligned} cn\left(\frac{2K}{\pi}t\right) &= \sum_{m=0}^{\infty} \frac{2\sqrt{2}\pi}{K} \frac{q^{m+\frac{1}{2}}}{1+q^{2m-1}} \cos((2m+1)t) \\ &= \sum_{m=0}^{\infty} a_m \cos((2m+1)t), \end{aligned} \tag{11}$$

where q is the only one solution in the interval $(0, 1)$ of the equation

$$\left(\frac{2\sum_{l=0}^{\infty} q^{(l+\frac{1}{2})^2}}{1 + 2\sum_{l=0}^{\infty} q^{l^2}}\right)^4 = \frac{1}{2}$$

(see Appendix, Eqs. (64) and (65)). Furthermore, it follows from the definition of a_m, given in Eq. (11), that

$$0 < a_m \leqslant \text{const.} q^{m+1} \quad \text{for all } m \geqslant 0. \tag{12}$$

Now, take into account the following change of variables:

$$\begin{aligned} q_1 &= \sqrt{\frac{2I}{\omega_1}} \sin\theta, \quad p_1 = \sqrt{2\omega_1 I}\cos\theta, \\ q_2 &= \sqrt{m_1} J cn\varphi, \quad p_2 = -\sqrt{m_1} J^2 sn\varphi \, dn\varphi = \sqrt{m_1} J^2 cn'\varphi. \end{aligned} \tag{13}$$

Remark 1. A canonical transformation can be used, instead of foregoing change of variables. But Eq. $(13)_2$ is easier, from an algebraic point of view.

It therefore follows, from Eqs. (10) and (13), that

$$E_1 = I, \quad E_2 = m_1 \frac{J^4}{4}. \tag{14}$$

For the remainder of this paper, $b_i(t, e_1, \alpha, e_2, \beta)$ will denote a bounded function, where $0 \leqslant t < \infty$ and $(e_1, \alpha, e_2, \beta)$ belongs to a bounded region of $\mathbb{R}^4$.

Substituting Eq. (13) into Eq. (9), one obtains

$$\begin{pmatrix} \dot{I} \\ \dot{\theta} \\ \dot{J} \\ \dot{\varphi} \end{pmatrix} = \begin{pmatrix} 0 \\ \omega_1 \\ 0 \\ J \end{pmatrix} + \varepsilon \begin{pmatrix} -2c_0 I \cos^2\theta - \cos\theta \sqrt{\frac{2I}{\omega_1}} \frac{\partial V}{\partial q_1}\left(\sqrt{\frac{2I}{\omega_1}} \sin\theta, \sqrt{m_1} J cn\varphi\right) \\ c_0 (\sin\theta)(\cos\theta) + \frac{(\sin\theta)}{\sqrt{2\omega_1 I}} \frac{\partial V}{\partial q_1}\left(\sqrt{\frac{2I}{\omega_1}} \sin\theta, \sqrt{m_1} J cn\varphi\right) \\ -\frac{c_0 J cn'(\varphi)^2}{m_1} - \frac{1}{\sqrt{m_1^3}} \frac{cn'(\varphi)}{J} \frac{\partial V}{\partial q_2}\left(\sqrt{\frac{2I}{\omega_1}} \sin\theta, \sqrt{m_1} J cn\varphi\right) \\ \frac{c_0 cn(\varphi) cn'(\varphi)}{m_1} + \frac{1}{\sqrt{m_1^3}} \frac{cn(\varphi)}{J^2} \frac{\partial V}{\partial q_2}\left(\sqrt{\frac{2I}{\omega_1}} \sin\theta, \sqrt{m_1} J cn\varphi\right) \end{pmatrix}$$
$$+ \varepsilon \begin{pmatrix} -A\sqrt{\frac{2I}{\omega_1}}(\cos\theta)(\sin\omega t) \\ A\frac{(\sin\theta)(\sin\omega t)}{\sqrt{2\omega_1 I}} \\ 0 \\ 0 \end{pmatrix}. \tag{15}$$

Now, this system is in an adequate form in order to apply the Regular Perturbation Theory with success. Note that the use of Regular Perturbation Theory directly into Eq. (9) leads to a solution involving circular and elliptic functions in the zeroth approximation. But in the computation of the first-order approximation one has to deal with great analytical difficulties. A way to skirt them is to use the change of variables given by Eq. (13).

Take the following initial conditions for Eq. (15):

$$I(0) = e_1, \quad \theta(0) = \alpha, \quad J(0) = e_2, \quad \varphi(0) = \beta. \tag{16}$$

It follows from Eq. (16) that the corresponding initial conditions of the system given in Eq. (9) are given by

$$\begin{aligned} q_1(0) &= \sqrt{\frac{2e_1}{\omega_1}} \sin\alpha, \quad p_1(0) = \sqrt{\frac{2e_1}{\omega_1}} \cos\alpha, \\ q_2(0) &= \sqrt{m_1} e_2\, cn\beta, \quad p_2(0) = \sqrt{m_1} e_2^2 cn'\beta. \end{aligned} \tag{17}$$

From the basic theorem on the Differentiability of the Flow [11], given $T > 0$, there is $\varepsilon_0 > 0$ such that

$$\begin{pmatrix} I \\ \theta \\ J \\ \varphi \end{pmatrix} = \begin{pmatrix} I_0 \\ \theta_0 \\ J_0 \\ \varphi_0 \end{pmatrix} + \varepsilon \begin{pmatrix} I_1 \\ \theta_1 \\ J_1 \\ \varphi_1 \end{pmatrix} + O(\varepsilon^2) \tag{18}$$

holds for all $t \in [0, T]$ and $0 < \varepsilon < \varepsilon_0$. Of course, all $I_i, \theta_i, J_i, \varphi_i$ depend on e_1, α, e_2, β.

The substitution of Eqs. (18), (16) in Eq. (15) yields

$$\begin{pmatrix} \dot{I}_0 \\ \dot{\theta}_0 \\ \dot{J}_0 \\ \dot{\varphi}_0 \end{pmatrix} = \begin{pmatrix} 0 \\ \omega_1 \\ 0 \\ J_0 \end{pmatrix}, \quad \begin{pmatrix} I_0(0) \\ \theta_0(0) \\ J_0(0) \\ \varphi_0(0) \end{pmatrix} = \begin{pmatrix} e_1 \\ \alpha \\ e_2 \\ \beta \end{pmatrix} \tag{19}$$

and

$$\begin{pmatrix} \dot{I}_1 \\ \dot{\theta}_1 \\ \dot{J}_1 \\ \dot{\varphi}_1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} I_1 \\ \theta_1 \\ J_1 \\ \varphi_1 \end{pmatrix}$$
$$+ \begin{pmatrix} -2c_0 I_0 \cos^2\theta_0 - \cos\theta_0 \sqrt{\dfrac{2I_0}{\omega_1}} \dfrac{\partial V}{\partial q_1}\left(\sqrt{\dfrac{2I_0}{\omega_1}} \sin\theta_0, \sqrt{m_1} J_0 cn\varphi_0 \right) \\ c_0(\sin\theta_0)(\cos\theta_0) + \dfrac{(\sin\theta_0)}{\sqrt{2\omega_1 I_0}} \dfrac{\partial V}{\partial q_1}\left(\sqrt{\dfrac{2I_0}{\omega_1}} \sin\theta_0, \sqrt{m_1} J_0 cn\varphi_0 \right) \\ -\dfrac{c_0 J_0 cn'(\varphi_0)^2}{m_1} - \dfrac{cn'(\varphi_0)}{\sqrt{m_1^3} J_0} \dfrac{\partial V}{\partial q_2}\left(\sqrt{\dfrac{2I_0}{\omega_1}} \sin\theta_0, \sqrt{m_1} J_0 cn\varphi_0 \right) \\ \dfrac{c_0 cn(\varphi_0) cn'(\varphi_0)}{m_1} + \dfrac{cn(\varphi_0)}{\sqrt{m_1^3} J_0^2} \dfrac{\partial V}{\partial q_2}\left(\sqrt{\dfrac{2I_0}{\omega_1}} \sin\theta_0, \sqrt{m_1} J_0 cn\varphi_0 \right) \end{pmatrix}$$
$$+ \begin{pmatrix} -A\sqrt{\dfrac{2I_0}{\omega_1}}(\cos\theta_0)(\sin\omega t) \\ A\dfrac{(\sin\theta_0)(\sin\omega t)}{\sqrt{2\omega_1 I_0}} \\ 0 \\ 0 \end{pmatrix},$$

$$\begin{pmatrix} I_1(0) \\ \theta_1(0) \\ J_1(0) \\ \varphi_1(0) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}. \tag{20}$$

Due to Eq. (14), it is only interesting to compute

$$\begin{cases} I(t, e_1, \alpha, e_2, \beta, \varepsilon) = I_0(t, e_1, \alpha, e_2, \beta) + \varepsilon I_1(t, e_1, \alpha, e_2, \beta) + O(\varepsilon^2), \\ J(t, e_1, \alpha, e_2, \beta, \varepsilon) = J_0(t, e_1, \alpha, e_2, \beta) + \varepsilon J_1(t, e_1, \alpha, e_2, \beta) + O(\varepsilon^2). \end{cases}$$

Hence, it is necessary to calculate I_0, J_0, I_1, J_1.

Eqs. (19), (20) lead to

$$\begin{cases} I_0(t, e_1, \alpha, e_2, \beta) = e_1, \\ \theta_0(t, e_1, \alpha, e_2, \beta) = \omega_1 t + \alpha, \\ J_0(t, e_1, \alpha, e_2, \beta) = e_2, \\ \varphi_0(t, e_1, \alpha, e_2, \beta) = e_2 t + \beta \end{cases}$$

and

$$\begin{cases} I_1(t, e_1, \alpha, e_2, \beta) \\ \quad = -\frac{c_0 e_1}{2} t \\ \quad - \int_0^t \begin{pmatrix} \sqrt{\frac{2e_1}{\omega_1}}(\cos(\omega_1 s + \alpha)) \\ \quad \times \frac{\partial V}{\partial q_1}\left(\sqrt{\frac{2e_1}{\omega_1}} \sin(\omega_1 s + \alpha), \sqrt{m_1} e_2 cn(e_2 s + \beta)\right) \\ \quad + A\sqrt{\frac{2e_1}{\omega_1}}(\cos(\omega_1 s + \alpha))(\sin \omega s) \end{pmatrix} \mathrm{d}s \\ \quad + b_1(t, e_1, \alpha, e_2, \beta), \\ J_1(t, e_1, \alpha, e_2, \beta) \\ \quad = -\frac{c_0 e_2}{m_1} \int_0^t (cn'(e_2 s + \beta))^2 \, \mathrm{d}s \\ \quad - \frac{1}{\sqrt{m_1^3}} \int_0^t \begin{pmatrix} \frac{cn'((e_2 s + \beta))}{e_2} \\ \quad \times \frac{\partial V}{\partial q_2}\left(\sqrt{\frac{2e_1}{\omega_1}} \sin(\omega_1 s + \alpha), \sqrt{m_1} e_2 cn(e_2 s + \beta)\right) \end{pmatrix} \mathrm{d}s. \end{cases} \tag{21}$$

So

$$\begin{aligned} I(t, e_1, \alpha, e_2, \beta, \varepsilon) &= e_1 + \varepsilon I_1(t, e_1, \alpha, e_2, \beta) + O(\varepsilon^2), \\ J(t, e_1, \alpha, e_2, \beta, \varepsilon) &= e_2 + \varepsilon J_1(t, e_1, \alpha, e_2, \beta) + O(\varepsilon^2), \end{aligned} \tag{22}$$

where I_1, J_1 are given by Eq. (21). By using Eqs. (22), (14), (16) and (13), the following ones are obtained:

$$\begin{aligned} & E_1\left(t, \varepsilon, \sqrt{\frac{2e_1}{\omega_1}} \sin \alpha, \sqrt{\frac{2e_1}{\omega_1}} \cos \alpha, \sqrt{m_1} e_2 cn\beta, \sqrt{m_1} e_2^2 cn'\beta\right) \\ & \quad = e_1 + I_1(t, e_1, \alpha, e_2, \beta)\varepsilon + O(\varepsilon^2), \\ & E_2\left(t, \varepsilon, \sqrt{\frac{2e_1}{\omega_1}} \sin \alpha, \sqrt{\frac{2e_1}{\omega_1}} \cos \alpha, \sqrt{m_1} e_2 cn\beta, \sqrt{m_1} e_2^2 cn'\beta\right) \\ & \quad = m_1 \frac{e_2^4}{4} + m_1 e_2^3 J_1(t, e_1, \alpha, e_2, \beta)\varepsilon + O(\varepsilon^2). \end{aligned} \tag{23}$$

Clearly

$$\begin{aligned} E_1\left(0, \varepsilon, \sqrt{\frac{2e_1}{\omega_1}} \sin \alpha, \sqrt{\frac{2e_1}{\omega_1}} \cos \alpha, \sqrt{m_1} e_2 cn\beta, \sqrt{m_1} e_2^2 cn'\beta\right) &= e_1, \\ E_2\left(0, \varepsilon, \sqrt{\frac{2e_1}{\omega_1}} \sin \alpha, \sqrt{\frac{2e_1}{\omega_1}} \cos \alpha, \sqrt{m_1} e_2 cn\beta, \sqrt{m_1} e_2^2 cn'\beta\right) &= m_1 \frac{e_2^4}{4}. \end{aligned}$$

Now, Eq. (21) will be rewritten in a more adequate form for the computations that will be made in the next sections.

It follows from Eq. (11)

$$-c_0 e_2 \int_0^t (cn'(e_2 s+\beta))^2\,\mathrm{d}s = -\frac{1}{2}c_0 e_2 k t + b_2(t,e_1,\alpha,e_2,\beta), \tag{24}$$

where

$$k=\sum_{m=0}^{\infty}(2m+1)^2 a_m^2.$$

Using the change of variables $(2K/\pi)u = e_2 s+\beta$ and Eq. (24) into Eq. (21), it follows that

$$\begin{aligned}
&I_1(t,e_1,\alpha,e_2,\beta)\\
&\quad= -\frac{c_0 e_1}{2}t - \frac{2K}{\pi e_2}\sqrt{\frac{2e_1}{\omega_1}} \times \int_{\frac{\pi\beta}{2K}}^{\frac{\pi}{2K}(e_2 t+\beta)}
\left(\left(\begin{array}{l}\cos\left(\dfrac{2K\omega_1}{\pi e_2}u-\dfrac{\omega_1\beta}{e_2}+\alpha\right)\\
\times\dfrac{\partial V}{\partial q_1}\left(\begin{array}{l}\sqrt{\dfrac{2e_1}{\omega_1}}\sin\left(\dfrac{2K\omega_1}{\pi e_2}u-\dfrac{\omega_1\beta}{e_2}+\alpha\right),\\ \sqrt{m_1}e_2 cn\left(\dfrac{2K}{\pi}u\right)\end{array}\right)\end{array}\right)\right.\\
&\qquad\left. +A\left(\cos\left(\frac{2K\omega_1}{\pi e_2}u-\frac{\omega_1\beta}{e_2}+\alpha\right)\right)\left(\sin\left(\frac{2K\omega}{\pi e_2}u-\frac{\beta\omega}{e_2}\right)\right)\right)\mathrm{d}u\\
&\qquad + b_1(t,e_1,\alpha,e_2,\beta),\\
&J_1(t,e_1,\alpha,e_2,\beta)\\
&\quad= -\frac{c_0 e_2 k}{2m_1}t - \frac{2K}{\pi e_2^2}\frac{1}{\sqrt{m_1^3}}\\
&\qquad\times\int_{\frac{\pi\beta}{2K}}^{\frac{\pi}{2K}(e_2 t+\beta)}\left(\begin{array}{l} cn'\left(\dfrac{2K}{\pi}u\right)\\
\quad\times\dfrac{\partial V}{\partial q_2}\left(\sqrt{\dfrac{2e_1}{\omega_1}}\sin\left(\dfrac{2K\omega_1}{\pi e_2}u-\dfrac{\omega_1\beta}{e_2}+\alpha\right),\sqrt{m_1}e_2 cn\left(\dfrac{2K}{\pi}u\right)\right)\end{array}\right)\mathrm{d}u\\
&\qquad + b_2(t,e_1,\alpha,e_2,\beta).
\end{aligned} \tag{25}$$

It is emphasized that Eqs. (23) and (25) are the main formulae in this section. All results in the next subsections are going to be deduced from them.

Next, two cases will be considered: a linear interaction and a cubic one.

3.1. Linear interaction

If V, in Eq. (25), is replaced by Eq. (7), we obtain

$$\begin{aligned}
&I_1(t,e_1,\alpha,e_2,\beta)\\
&\quad= -\frac{c_0 e_1}{2}t + \frac{2K}{\pi}\sqrt{\frac{2m_1 e_1}{\omega_1}}
\end{aligned}$$

$$\times \int_{\frac{\pi\beta}{2K}}^{\frac{\pi}{2K}(e_2t+\beta)} \cos\left(\frac{2K\omega_1}{\pi e_2}u - \frac{\omega_1\beta}{e_2} + \alpha\right) cn\left(\frac{2K}{\pi}u\right) \mathrm{d}u - \frac{2AK}{\pi e_2}\sqrt{\frac{2e_1}{\omega_1}}$$
$$\times \int_{\frac{\pi\beta}{2K}}^{\frac{\pi}{2K}(e_2t+\beta)} \left(\cos\left(\frac{2K\omega_1}{\pi e_2}u - \frac{\omega_1\beta}{e_2} + \alpha\right)\right)\left(\sin\left(\frac{2K\omega}{\pi e_2}u - \frac{\beta\omega}{e_2}\right)\right) \mathrm{d}u + b_3(t, e_1, \alpha, e_2, \beta).$$

But it follows from Eq. (11)

$$\frac{2K}{\pi}\sqrt{\frac{2m_1e_1}{\omega_1}} \int_{\frac{\pi\beta}{2K}}^{\frac{\pi}{2K}(e_2t+\beta)} \cos\left(\frac{2K\omega_1}{\pi e_2}u - \frac{\omega_1\beta}{e_2} + \alpha\right) cn\left(\frac{2K}{\pi}u\right) \mathrm{d}u$$
$$= \frac{2K}{\pi}\sqrt{\frac{2m_1e_1}{\omega_1}} \int_{\frac{\pi\beta}{2K}}^{\frac{\pi e_2}{2K}(t+\frac{\beta}{e_2})} \begin{pmatrix} \cos\left(\frac{2K\omega_1}{\pi e_2}u - \frac{\omega_1\beta}{e_2} + \alpha\right) \\ \times\sum_{m=0}^{\infty} a_m \cos((2m+1)u) \end{pmatrix} \mathrm{d}u$$
$$= \begin{pmatrix} \sum\limits_{m=0}^{\infty} \frac{2K}{\pi}\sqrt{\frac{2m_1e_1}{\omega_1}} a_m \cos\left(\alpha - \frac{\omega_1\beta}{e_2}\right) \\ \times \int_{\frac{\pi\beta}{2K}}^{\frac{\pi e_2}{2K}(t+\frac{\beta}{e_2})} \begin{pmatrix} \frac{1}{2}\cos\left(\left(-2\frac{K\omega_1}{\pi e_2} + 2m+1\right)u\right) \\ +\frac{1}{2}\cos\left(\left(2\frac{K\omega_1}{\pi e_2} + 2m+1\right)u\right) \end{pmatrix} \mathrm{d}u \end{pmatrix}$$
$$- \begin{pmatrix} \sum\limits_{m=0}^{\infty} \frac{2K}{\pi}\sqrt{\frac{2m_1e_1}{\omega_1}} a_m \sin\left(\alpha - \frac{\omega_1\beta}{e_2}\right) \\ \times \int_{\frac{\pi\beta}{2K}}^{\frac{\pi e_2}{2K}(t+\frac{\beta}{e_2})} \begin{pmatrix} \frac{1}{2}\sin\left(\left(2\frac{K\omega_1}{\pi e_2} + 2m+1\right)u\right) \\ -\frac{1}{2}\sin\left(\left(-2\frac{K\omega_1}{\pi e_2} + 2m+1\right)u\right) \end{pmatrix} \mathrm{d}u \end{pmatrix}.$$

Moreover, the inequalities

$$\begin{aligned} &\left|\int_{\frac{\pi\beta}{2K}}^{\frac{\pi e_2}{2K}(t+\frac{\beta}{e_2})} \left(\frac{1}{2}\cos\left(\left(2\frac{K\omega_1}{\pi e_2} + 2m+1\right)u\right)\right) \mathrm{d}u\right| \leqslant 1, \\ &\left|\int_{\frac{\pi\beta}{2K}}^{\frac{\pi e_2}{2K}(t+\frac{\beta}{e_2})} \left(\frac{1}{2}\sin\left(\left(2\frac{K\omega_1}{\pi e_2} + 2m+1\right)u\right)\right) \mathrm{d}u\right| \leqslant 1, \\ &\left|\int_{\frac{\pi\beta}{2K}}^{\frac{\pi e_2}{2K}(t+\frac{\beta}{e_2})} \left(\frac{1}{2}\sin\left(\left(-2\frac{K\omega_1}{\pi e_2} + 2m+1\right)u\right)\right) \mathrm{d}u\right| \leqslant D_0, \end{aligned} \tag{26}$$

are valid for all m, t, α, β, where D_0 is a positive constant that depends only on e_2 and ω_1. From Eqs. (12) and (26) it is not difficult to verify that

$$\frac{2K}{\pi}\sqrt{\frac{2m_1e_1}{\omega_1}} \int_{\frac{\pi\beta}{2K}}^{\frac{\pi}{2K}(e_2t+\beta)} \cos\left(\frac{2K\omega_1}{\pi e_2}u - \frac{\omega_1\beta}{e_2} + \alpha\right) cn\left(\frac{2K}{\pi}u\right) \mathrm{d}u$$
$$= \frac{2K}{\pi}\sqrt{\frac{2m_1e_1}{\omega_1}} \sum_{m=0}^{\infty} \begin{pmatrix} a_m \cos\left(\alpha - \frac{\omega_1\beta}{e_2}\right) \\ \times \int_{\frac{\pi\beta}{2K}}^{\frac{\pi e_2}{2K}(t+\frac{\beta}{e_2})} \left(\frac{1}{2}\cos\left(\left(-2\frac{K\omega_1}{\pi e_2} + 2m+1\right)u\right)\right) \mathrm{d}u \end{pmatrix}$$
$$+ b_4(t, e_1, \alpha, e_2, \beta).$$

Hence

$$
\begin{aligned}
I_1(t,e_1,\alpha,e_2,\beta) = &-\frac{c_0 e_1}{2}t \\
&+\frac{2K}{\pi}\sqrt{\frac{2m_1e_1}{\omega_1}}\sum_{m=0}^{\infty}\left(\begin{array}{l} a_m\cos\left(\alpha-\dfrac{\omega_1\beta}{e_2}\right) \\ \qquad\times\int_{\frac{\pi\beta}{2K}}^{\frac{\pi e_2}{2K}(t+\frac{\beta}{e_2})}\dfrac{1}{2}\cos\left(\left(-2\dfrac{K\omega_1}{\pi e_2}+2m+1\right)u\right)\mathrm{d}u\end{array}\right) \\
&-\frac{2AK}{\pi e_2}\sqrt{\frac{2e_1}{\omega_1}}\int_{\frac{\pi\beta}{2K}}^{\frac{\pi}{2K}(e_2t+\beta)}\left(\begin{array}{l}\cos\left(\dfrac{2K\omega_1}{\pi e_2}u-\dfrac{\omega_1\beta}{e_2}+\alpha\right)\\ \qquad\times\left(\sin\left(\dfrac{2K\omega}{\pi e_2}u-\dfrac{\beta\omega}{e_2}\right)\right)\end{array}\right)\mathrm{d}u \\
&+b_5(t,e_1,\alpha,e_2,\beta).
\end{aligned}
\tag{27}
$$

Besides, Eqs. (7) and (25) yield

$$
\begin{aligned}
J_1(t,e_1,\alpha,e_2,\beta) = &-\frac{c_0e_2k}{2m_1}t+\frac{2K}{\pi e_2^2}\frac{1}{\sqrt{m_1^3}}\sqrt{\frac{2e_1}{\omega_1}} \\
&\times\int_{\frac{\pi\beta}{2K}}^{\frac{\pi}{2K}(e_2t+\beta)} cn'\left(\frac{2K}{\pi}u\right)\sin\left(\frac{2K\omega_1}{\pi e_2}u-\frac{\omega_1\beta}{e_2}+\alpha\right)\mathrm{d}u \\
&+b_6(t,e_1,\alpha,e_2,\beta).
\end{aligned}
\tag{28}
$$

Moreover, using an argument similar to that utilized in the proof of Eq. (27), we obtain

$$
\begin{aligned}
&\frac{2K}{\pi e_2^2}\frac{1}{\sqrt{m_1^3}}\sqrt{\frac{2e_1}{\omega_1}}\int_{\frac{\pi\beta}{2K}}^{\frac{\pi}{2K}(e_2t+\beta)} cn'\left(\frac{2K}{\pi}u\right)\sin\left(\frac{2K\omega_1}{\pi e_2}u-\frac{\omega_1\beta}{e_2}+\alpha\right)\mathrm{d}u \\
&=\frac{2K}{\pi e_2^2}\frac{1}{\sqrt{m_1^3}}\sqrt{\frac{2e_1}{\omega_1}}\left(\begin{array}{l}\cos\left(-\dfrac{\omega_1\beta}{e_2}+\alpha\right)\\ \qquad\times\int_{\frac{\pi\beta}{2K}}^{\frac{\pi}{2K}(e_2t+\beta)} cn'\left(\dfrac{2K}{\pi}u\right)\left(\sin\left(\dfrac{2K\omega_1}{\pi e_2}u\right)\right)\mathrm{d}u\\ \quad+\sin\left(-\dfrac{\omega_1\beta}{e_2}+\alpha\right)\\ \qquad\times\int_{\frac{\pi\beta}{2K}}^{\frac{\pi}{2K}(e_2t+\beta)} cn'\left(\dfrac{2K}{\pi}u\right)\left(\cos\left(\dfrac{2K\omega_1}{\pi e_2}u\right)\right)\mathrm{d}u\end{array}\right) \\
&=-\frac{1}{e_2^2}\frac{1}{\sqrt{m_1^3}}\sqrt{\frac{2e_1}{\omega_1}}\cos\left(-\frac{\omega_1\beta}{e_2}+\alpha\right) \\
&\quad\times\sum_{m=0}^{\infty}\left(\begin{array}{l}(2m+1)a_m\\ \qquad\times\int_{\frac{\pi\beta}{2K}}^{\frac{\pi}{2K}(e_2t+\beta)}\left(\dfrac{1}{2}\cos\left(\left(2m+1-\dfrac{2K\omega_1}{\pi e_2}\right)u\right)\right)\mathrm{d}u\end{array}\right) \\
&\quad+b_7(t,e_1,\alpha,e_2,\beta).
\end{aligned}
$$

So

$$\begin{aligned}
&\frac{2K}{\pi e_2^2}\frac{1}{\sqrt{m_1^3}}\sqrt{\frac{2e_1}{\omega_1}}\int_{\frac{\pi\beta}{2K}}^{\frac{\pi}{2K}(e_2 t+\beta)} cn'\left(\frac{2K}{\pi}u\right)\sin\left(\frac{2K\omega_1}{\pi e_2}u-\frac{\omega_1\beta}{e_2}+\alpha\right)\mathrm{d}u \\
&= -\frac{1}{e_2^2}\frac{1}{\sqrt{m_1^3}}\sqrt{\frac{2e_1}{\omega_1}}\cos\left(-\frac{\omega_1\beta}{e_2}+\alpha\right) \\
&\times\sum_{m=0}^{\infty}\left(\begin{array}{l}(2m+1)a_m \\ \times\int_{\frac{\pi\beta}{2K}}^{\frac{\pi}{2K}(e_2 t+\beta)}\left(\frac{1}{2}\cos\left(\left(2m+1-\frac{2K\omega_1}{\pi e_2}\right)u\right)\right)\mathrm{d}u\end{array}\right) \\
&+ b_7(t,e_1,\alpha,e_2,\beta). \qquad (29)
\end{aligned}$$

Upon substituting Eq. (29) in Eq. (28), one obtains

$$\begin{aligned}
J_1(t,e_1,\alpha,e_2,\beta) = &-\frac{c_0 e_2 k}{2m_1}t \\
&-\frac{1}{e_2^2}\frac{1}{\sqrt{m_1^3}}\sqrt{\frac{2e_1}{\omega_1}}\cos\left(-\frac{\omega_1\beta}{e_2}+\alpha\right) \\
&\times\sum_{m=0}^{\infty}\left(\begin{array}{l}(2m+1)a_m \\ \times\int_{\frac{\pi\beta}{2K}}^{\frac{\pi}{2K}(e_2 t+\beta)}\left(\frac{1}{2}\cos\left(\left(2m+1-\frac{2K\omega_1}{\pi e_2}\right)u\right)\right)\mathrm{d}u\end{array}\right) \\
&+ b_8(t,e_1,\alpha,e_2,\beta). \qquad (30)
\end{aligned}$$

In Eqs. (27) and (30) if the term $-2(K\omega_1/\pi e_2)+2m+1$ is zero for some m, after integration, an unbounded term is obtained and this has consequences for the energy transfer. This case is called internal resonance. If $\omega_1=\omega$ in Eq. (27), there is another unbounded term, and we have the external resonance. In the next subsections, these cases, non-resonant and resonant, will be considered.

3.1.1. The non-resonant case

Here we assume

$$\begin{aligned}
&\text{(a) } e_2 \neq \frac{2K\omega_1}{(2m+1)\pi} \quad \text{for all } m\in\mathbb{N}, \\
&\text{(b) } \omega_1\neq\omega. \qquad (31)
\end{aligned}$$

From Eqs. (23), (27), (30) and (31) one obtains that

$$\begin{aligned}
&E_1\left(t,\varepsilon,\sqrt{\frac{2e_1}{\omega_1}}\sin\alpha,\sqrt{\frac{2e_1}{\omega_1}}\cos\alpha,\sqrt{m_1}e_2 cn\beta,\sqrt{m_1}e_2^2 cn'\beta\right) \\
&= e_1+\left(-\frac{c_0 e_1}{2}t+b_9(t,e_1,\alpha,e_2,\beta)\right)\varepsilon+O(\varepsilon^2), \\
&E_2\left(t,\varepsilon,\sqrt{\frac{2e_1}{\omega_1}}\sin\alpha,\sqrt{\frac{2e_1}{\omega_1}}\cos\alpha,\sqrt{m_1}e_2 cn\beta,\sqrt{m_1}e_2^2 cn'\beta\right) \\
&= m_1\frac{e_2^4}{4}+m_1 e_2^3\left(-\frac{c_0 e_2 k}{2m_1}t+b_{10}(t,e_1,\alpha,e_2,\beta)\right)\varepsilon+O(\varepsilon^2). \qquad (32)
\end{aligned}$$

Then, for all $t \in [T_1, T]$, T_1 adequately big,

$$-\frac{c_0 e_1}{2}t + b_9(t, e_1, \alpha, e_2, \beta) < 0, \quad -\frac{c_0 e_2 k}{2m_1}t + b_{10}(t, e_1, \alpha, e_2, \beta) < 0.$$

And taking $\varepsilon < \varepsilon_0 = \varepsilon_0(\sqrt{(2e_1/\omega_1)}\sin\alpha, \sqrt{(2e_1/\omega_1)}\cos\alpha, \sqrt{m_1}e_2 cn\beta, \sqrt{m_1}e_2^2 cn'\beta, T)$, where $\varepsilon_0 > 0$ is adequately small, it follows from Eq. (32) that $E_1 < e_1 = E_1(0)$ and $E_2 < m_1 e_2^4/4 = E_2(0)$. Hence, the condition given in Eq. (3) is not satisfied. So, there is no energy transfer in the point $(e_1, \alpha, e_2, \beta)$ of the phase space, since the condition given by Eq. (31) is satisfied.

3.1.2. The resonant case

In view of Eq. (31), there are three possible cases for resonance. Here we consider only one. It is assumed that for some $m_0 \in \mathbb{N}$, the following condition is satisfied:

$$\begin{aligned} &\text{(a) } e_2 = \frac{2K}{(2m_0+1)\pi}, \\ &\text{(b) } \omega_1 = \omega. \end{aligned} \tag{33}$$

Remark 2. If there is no external excitation, condition $(33)_1$ means that the unperturbed system, obtained making $\varepsilon = 0$ at Eq. (9), has a periodic solution. Indeed, the above condition can be interpreted as a $(2m_0+1)$:1 internal resonance of the unperturbed system in order to get a periodic motion. Note that in this system there are two oscillators, a linear one and a nonlinear one. For each oscillator there are periodic solutions given by $x = A\sin t, y = B\,cn(Bt)$. In order to get $t \to (x(t), y(t))$ to be periodic, the condition given at Eq. (33) is sufficient. If $m_0 = 0$ and $\alpha = \pi/2, \beta = 0$ in Eq. (16) then that periodic solution is a nonlinear normal mode in the sense of Rosenberg [8].

From Eqs. (27), (30), (23) and (33), one obtains

$$\begin{aligned} &E_1\left(t, \varepsilon, \sqrt{\frac{2e_1}{\omega_1}}\sin\alpha, \sqrt{\frac{2e_1}{\omega_1}}\cos\alpha, \sqrt{m_1}e_2 cn\beta, \sqrt{m_1}e_2^2 cn'\beta\right) \\ &= e_1 + \left(\begin{array}{l}\left(-\frac{c_0 e_1}{2} + a_{m_0} e_2\sqrt{\frac{m_1 e_1}{2\omega}}\cos\left(\alpha - \frac{\omega\beta}{e_2}\right) + A\sqrt{\frac{e_1}{2\omega}}\sin\alpha\right)t \\ +b_7(t, e_1, \alpha, e_2, \beta)\end{array}\right)\varepsilon + O(\varepsilon^2), \\ &E_2\left(t, \varepsilon, \sqrt{\frac{2e_1}{\omega_1}}\sin\alpha, \sqrt{\frac{2e_1}{\omega_1}}\cos\alpha, \sqrt{m_1}e_2 cn\beta, \sqrt{m_1}e_2^2 cn'\beta\right) \\ &= m_1\frac{e_2^4}{4} + m_1 e_2^3\left(\begin{array}{l}\left(-\frac{c_0 e_2 k}{2m_1} - \cos\left(\alpha - \frac{\omega\beta}{e_2}\right)\frac{(2m_0+1)\pi a_{m_0}}{4Ke_2}\sqrt{\frac{2e_1}{\omega m_1^3}}\right)t \\ +b_8(t, e_1, \alpha, e_2, \beta)\end{array}\right)\varepsilon + O(\varepsilon^2). \end{aligned} \tag{34}$$

From now on, in this subsection, it is assumed that $A = 0$. So, in Eq. (34), ω is replaced by ω_1.

As earlier, using similar steps as the ones used in the argument of the Section 3.1.1, the problem of energy transfer reduces itself to the analysis of the signal of the coefficients of t in Eq. (34). So, there are four cases to be considered. The computation of the details will only be made in the next item. The computation of the other cases are analogous.

(a) The linear oscillator loses energy and the energy of the nonlinear one increases.
 A sufficient condition for this is obtained in the following way. Suppose that

$$-\frac{c_0 e_1}{2} + a_{m_0} e_2\sqrt{\frac{m_1 e_1}{2\omega_1}}\cos\left(\alpha - \frac{\omega_1\beta}{e_2}\right) < 0,$$

$$-\frac{c_0 e_2 k}{2m_1} - \cos\left(\alpha - \frac{\omega_1 \beta}{e_2}\right)\frac{(2m_0+1)\pi a_{m_0}}{4Ke_2}\sqrt{\frac{2e_1}{\omega_1 m_1^3}} > 0. \tag{35}$$

Then, there is $T_0 > 0$, such that if $T_1 > T_0$ and $t \in [T_1, T]$ then

$$\left(\begin{array}{l}\left(-\frac{c_0 e_1}{2} + a_{m_0} e_2\sqrt{\frac{m_1 e_1}{2\omega}}\cos\left(\alpha - \frac{\omega\beta}{e_2}\right)\right)t \\ +b_7(t, e_1, \alpha, e_2, \beta)\end{array}\right) < 0,$$

$$\left(\begin{array}{l}\left(-\frac{c_0 e_2 k}{2m_1} - \cos\left(\alpha - \frac{\omega\beta}{e_2}\right)\frac{(2m_0+1)\pi a_{m_0}}{4Ke_2}\sqrt{\frac{2e_1}{\omega m_1^3}}\right)t \\ +b_8(t, e_1, \alpha, e_2, \beta)\end{array}\right) > 0.$$

Hence, there is $\varepsilon_0 > 0$, $\varepsilon_0 = \varepsilon_0(\sqrt{(2e_1/\omega_1)}\sin\alpha, \sqrt{(2e_1/\omega_1)}\cos\alpha, \sqrt{m_1}e_2\, cn\beta, \sqrt{m_1}e_2^2 cn'\beta, T)$ such that if $0 < \varepsilon < \varepsilon_0$ then $E_1 < e_1 = E_1(0)$ and $E_2 > m_1 e_2^4/4 = E_2(0)$. So from the definition given in Eq. (3) there is an energy transfer from the linear oscillator to the nonlinear one, in the point $(\sqrt{(2e_1/\omega_1)}\sin\alpha, \sqrt{(2e_1/\omega_1)}\cos\alpha, \sqrt{m_1}e_2 cn\beta, \sqrt{m_1}e_2^2 cn'\beta)$ of the phase space of the Eq. (9) when $A = 0$. Besides, from Eq. $(33)_1$, it is obtained that Eq. (35) is equivalent to

$$\cos\left(\alpha - \frac{\omega_1\beta}{e_2}\right) < -\frac{4c_0 k K^3\sqrt{2\omega_1 m_1}}{\pi^3(2m_0+1)^3 a_{m_0}\sqrt{e_1}}. \tag{36}$$

This case can be interpreted exactly as energy pumping.

(b) The linear oscillator gains energy and the nonlinear loses energy.
This happens if the following condition holds:

$$-\frac{c_0 e_1}{2} + a_{m_0} e_2\sqrt{\frac{m_1 e_1}{2\omega_1}}\cos\left(\alpha - \frac{\omega_1\beta}{e_2}\right) > 0,$$
$$-\frac{c_0 e_2 k}{2m_1} - \cos\left(\alpha - \frac{\omega_1\beta}{e_2}\right)\frac{(2m_0+1)\pi a_{m_0}}{4Ke_2}\sqrt{\frac{2e_1}{\omega_1 m_1^3}} < 0.$$

Hence, from Eq. $(33)_1$

$$\cos\left(\alpha - \frac{\omega_1\beta}{e_2}\right) > \frac{\pi c_0(2m_0+1)}{4a_{m_0}K}\sqrt{\frac{2e_1\omega_1}{m_1}}. \tag{37}$$

(c) It follows from Eq. (34) that it is not possible for both oscillators to gain energy. Obviously, this is in accordance with our physical intuition.

(d) Both oscillators lose energy.
Then

$$-\frac{c_0 e_1}{2} + a_{m_0} e_2\sqrt{\frac{m_1 e_1}{2\omega_1}}\cos\left(\alpha - \frac{\omega_1\beta}{e_2}\right) < 0,$$
$$-\frac{c_0 e_2 k}{2m_1} - \cos\left(\alpha - \frac{\omega_1\beta}{e_2}\right)\frac{(2m_0+1)\pi a_{m_0}}{4Ke_2}\sqrt{\frac{2e_1}{\omega_1 m_1^3}} < 0.$$

Or

$$-\frac{4c_0 k K^3\sqrt{2\omega_1 m_1}}{\pi^3(2m_0+1)^3 a_{m_0}\sqrt{e_1}} < \cos\left(\alpha - \frac{\omega_1\beta}{e_2}\right) < \frac{\pi c_0(2m_0+1)}{4Ka_{m_0}}\sqrt{\frac{2e_1\omega_1}{m_1}}. \tag{38}$$

In this case, the loss of energy of both oscillators is due to the dissipative effect that is modelled by the term c_0x' in Eq. (4).

Let

$$\Gamma = \frac{8c_0kK^3\sqrt{m_1}}{\pi^3(2m_0+1)^3a_{m_0}}, \qquad \Lambda = \frac{\pi c_0(2m_0+1)}{4a_{m_0}K}\sqrt{\frac{1}{m_1}}. \tag{39}$$

From Eq. (17), it follows that relations (36), (38) and (37) can, respectively, be written as

$$p_1(0)\cos\left(\frac{\omega_1\beta}{e_2}\right) + q_1(0)\sin\left(\frac{\omega_1\beta}{e_2}\right) < -\Gamma,$$

$$-\Gamma < p_1(0)\cos\left(\frac{\omega_1\beta}{e_2}\right) + q_1(0)\sin\left(\frac{\omega_1\beta}{e_2}\right) < \Lambda((q_1(0))^2 + (p_1(0))^2),$$

$$p_1(0)\cos\left(\frac{\omega_1\beta}{e_2}\right) + q_1(0)\sin\left(\frac{\omega_1\beta}{e_2}\right) > \Lambda((q_1(0))^2 + (p_1(0))^2).$$

Taking $A = \cos(\omega_1\beta/e_2)$, $B = \sin(\omega_1\beta/e_2)$ and after some algebraic manipulation, the above relations define three regions in the plane of the phase variables $q_1(0), p_1(0)$:

$$\begin{cases} \text{Region 1: } Ap_1(0) + Bq_1(0) < -\Gamma, \\ \text{Region 2: } -\Gamma < Ap_1(0) + Bq_1(0) \text{ and} \\ \left(q_1(0) - \dfrac{B}{2\Lambda}\right)^2 + \left(p_1(0) - \dfrac{A}{2\Lambda}\right)^2 > \left(\dfrac{1}{2\Lambda}\right)^2, \\ \text{Region 3: } \left(q_1(0) - \dfrac{B}{2\Lambda}\right)^2 + \left(p_1(0) - \dfrac{A}{2\Lambda}\right)^2 < \left(\dfrac{1}{2\Lambda}\right)^2. \end{cases} \tag{40}$$

And, in this plane, Fig. 2 is obtained

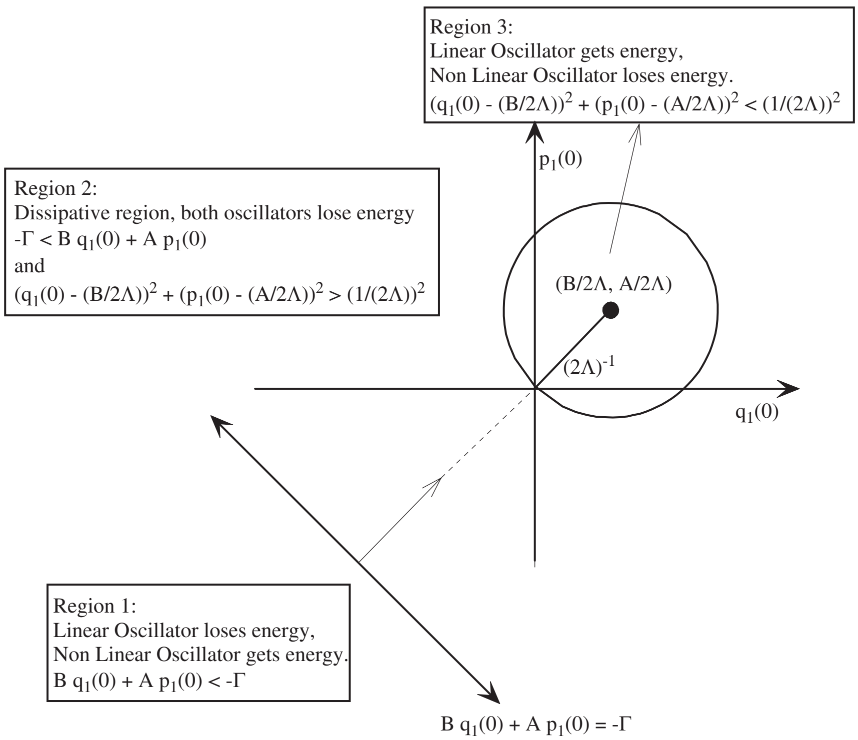

Fig. 2. Energy transfer regions in the presence of nonlinear spring.

In Region 1 there is energy transfer from the linear oscillator to the nonlinear one. In Region 2, both oscillators lose energy. Note that if $c_0 = 0$ then $\Gamma = \Lambda = 0$ and Region 2 collapses. So, that is the dissipative region. And, in Region 3, the energy transfer happens from the nonlinear to the linear one. Obviously, Fig. 2 is the projection of the four-dimensional phase space, with coordinates (q_1, p_1, q_2, p_2), onto the two-dimensional phase space, with coordinates (q_1, p_1). From a mechanical point of view, an interpretation of Fig. 2 is the following one: given the initial state $(q_2(0), p_2(0)) = (\sqrt{m_1} e_2\, cn\beta, \sqrt{m_1} e_2^2 cn'\beta)$ of the nonlinear oscillator, three regions are obtained, in the phase plane of the initial conditions of the linear oscillator, which are the above Regions 1, 2 and 3.

Note that, if one wants to get energy transfer from the linear to the non linear oscillator, or at least, to assure its loss of energy, a reasonable aim is to shrink Region 3. This can be done taking $\Lambda \gg 1$. From Eq. (39) there are two ways to get this:

(I) The nonlinear oscillator has a very small mass, or, $m_1 \ll 1$.
In this case we also have $\Gamma \ll 1$. So, except for a very small disc in the plane of the parameters $q_1(0), p_1(0)$, for all points in this plane the linear oscillator loses energy. Essentially, this plane is divided by Regions 1 and 2. In one of them the nonlinear oscillator absorbs energy. In the other, both oscillators lose energy due to dissipation.

(II) There is a very high resonance, or, $m_0 \gg 1$.
From Eq. (12) it follows that

$$\lim_{m_0 \to \infty} \frac{(2m_0 + 1)}{a_{m_0}} = \infty, \quad \lim_{m_0 \to \infty} (2m_0 + 1)^3 a_{m_0} = 0.$$

Hence, besides $\Lambda \gg 1$, $\Gamma \gg 1$. So, the straight line $Ap_1(0) + Bq_1(0) = -\Gamma$ will be placed far from the origin. Then, near the origin, except for a very small disc, that is Region 3, both oscillators lose energy due to dissipation.

Another interesting consequence of the above analysis is that if

$$\sqrt{m_1} < \frac{\pi c_0 (2m_0 + 1)\sqrt{2e_1\omega_1}}{4Ka_{m_0}} \tag{41}$$

then the *linear oscillator always loses energy due to energy transfer or dissipation*. Analogous inequalities are valid in other situations, as for example $A \neq 0$ and $\alpha = 0$.

3.1.3. *Resonance condition and nonlinear normal modes*

Again, in this subsection it is assumed that $A = 0$ and besides $m_1 = 1$.

The unperturbed system is given by

$$\begin{cases} x'' + x = 0, \\ y'' + y^3 = 0. \end{cases} \tag{42}$$

Consider the following initial conditions:

$$x(0) = e_1 \sin\alpha, x'(0) = e_1 \cos\alpha, y(0) = e_2 cn(\beta), y'(0) = e_2^2 cn'\beta. \tag{43}$$

Then the curve $(x(t), y(t))$, where $x(t) = e_1 \sin(t + \alpha)$, $y(t) = e_2 cn(e_2 t + \beta)$, is the solution of Eqs. (42) and (43). Since all real zeros of $cn(t)$ are given by $(2p + 1)K$, $p \in \mathbb{Z}$, the conditions

$$\begin{aligned} &\text{(a) } e_2 = \frac{2Km}{\pi n}, \ m, n \in \mathbb{N}, m, n \neq 0, \\ &\text{(b) } \alpha - \frac{\omega_1 \beta}{e_2} = p_1 \pi - \frac{(2p+1)\pi n}{2m}, \ p_1 \in \mathbb{Z} \end{aligned} \tag{44}$$

are necessary for $(x(t), y(t))$ to be a nonlinear normal mode in the sense of Rosenberg [8].

So, if e_2 does not satisfy Eq. (33), then there is no energy transfer. If e_2 satisfies Eq. $(33)_1$, it follows from Eq. (44) that $\cos(\alpha - (\omega_1 \beta / e_2)) = 0$. Hence, it is concluded in case (d) that there is no energy transfer again.

Anyway, it is obtained that *for perturbations of nonlinear normal modes of the unperturbed system*, *there is no transfer of energy at all.*

3.2. Comments on cubic interaction

If the potential V is given by Eq. (8), similar results can be obtained by using exactly the earlier approach. The nonresonance condition is given by

$$\begin{aligned}&\text{(a) } e_2 \neq \frac{6K\omega_1}{\pi(2m+1)} \quad \text{for all } m \in \mathbb{N},\\ &\text{(b) } \omega_1 \neq \omega.\end{aligned} \tag{45}$$

There are several possibilities for resonances. Particularly, if Eq. $(45)_1$ does not hold then there is $m_0 \in \mathbb{N}$ such that $e_2 = (6K\omega_1/\pi(2m_0+1))$. Then, one has to consider the cases $\gcd(2m_0+1,3)=1$ or $\gcd(2m_0+1,3)=3$. Anyway, although the computations are very long, the results are similar to those in Section 3.1.2. Moreover, figure analogous to Fig. 2 can be obtained and it involves cubic curves.

4. On linear anchor springs

Assume that $f(x) = \omega_1^2 x$, $g(y) = \omega_2^2 y$, $A = 0$ and $m_1 = 1$ in Eq. (4). From Eqs. (5) and (4) we have

$$\begin{cases} \dot{q}_1 = p_1, \\ \dot{p}_1 = -\omega_1^2 q_1 - \varepsilon\left(c_0 p_1 + \dfrac{\partial V}{\partial q_1}\right), \\ \dot{q}_2 = p_2, \\ \dot{p}_2 = -\omega_2^2 q_2 - \varepsilon\left(c_0 p_2 + \dfrac{\partial V}{\partial q_2}\right). \end{cases} \tag{46}$$

In this case, the unperturbed energies are given by

$$\begin{cases} H_1(q_1,p_1) = \dfrac{p_1^2 + \omega_1^2 q_1^2}{2}, \\ H_2(q_2,p_2) = \dfrac{p_2^2 + \omega_2^2 q_2^2}{2}. \end{cases} \tag{47}$$

Now, consider the following change of variables (action-angle variables)

$$\begin{aligned} q_1 &= \sqrt{\frac{2I}{\omega_1}}\sin\theta, \quad p_1 = \sqrt{2\omega_1 I}\cos\theta, \\ q_2 &= \sqrt{\frac{2J}{\omega_2}}\sin\varphi, \quad p_2 = \sqrt{2\omega_2 J}\cos\varphi. \end{aligned} \tag{48}$$

From this it follows, using Eqs. (47), (48) and (2), that $E_1 = \omega_1 I, E_2 = \omega_2 J$. Using Eq. (48) in Eq. (46), it yields that

$$\begin{pmatrix} \dot{I} \\ \dot{\theta} \\ \dot{J} \\ \dot{\varphi} \end{pmatrix} = \begin{pmatrix} 0 \\ \omega_1 \\ 0 \\ \omega_2 \end{pmatrix} + \varepsilon \begin{pmatrix} -2c_0 I\cos^2\theta - \sqrt{\dfrac{2I}{\omega_1}}(\cos\theta)\dfrac{\partial V}{\partial q_1}\left(\sqrt{\dfrac{2I}{\omega_1}}\sin\theta, \sqrt{\dfrac{2J}{\omega_2}}\sin\varphi\right) \\ c_0\sin\theta\cos\theta + \dfrac{\sin\theta}{\sqrt{2\omega_1 I}}\dfrac{\partial V}{\partial q_1}\left(\sqrt{\dfrac{2I}{\omega_1}}\sin\theta, \sqrt{\dfrac{2J}{\omega_2}}\sin\varphi\right) \\ -2c_0 J\cos^2\varphi - \sqrt{\dfrac{2J}{\omega_2}}\cos\varphi\dfrac{\partial V}{\partial q_2}\left(\sqrt{\dfrac{2I}{\omega_1}}\sin\theta, \sqrt{\dfrac{2J}{\omega_2}}\sin\varphi\right) \\ c_0\sin\varphi\cos\varphi + \dfrac{\sin\varphi}{\sqrt{2\omega_2 J}}\dfrac{\partial V}{\partial q_2}\left(\sqrt{\dfrac{2I}{\omega_1}}\sin\theta, \sqrt{\dfrac{2J}{\omega_2}}\sin\varphi\right) \end{pmatrix}. \tag{49}$$

In this case, taking the initial conditions given in Eq. (16) and using the Regular Perturbation Theory as it was applied in the earlier section, terms I_1, J_1, in Eq. (22), can be obtained:

$$I_1(t, e_1, \alpha, e_2, \beta) = \int_0^t \begin{pmatrix} -2c_0 e_1 \cos^2(\omega_1 s + \alpha) \\ -\begin{pmatrix} \sqrt{\dfrac{2e_1}{\omega_1}}(\cos(\omega_1 s + \alpha)) \\ \times \dfrac{\partial V}{\partial q_1}\left(\sqrt{\dfrac{2e_1}{\omega_1}}\sin(\omega_1 s + \alpha), \sqrt{\dfrac{2e_2}{\omega_2}}\sin(\omega_2 s + \beta)\right) \end{pmatrix} \end{pmatrix} \mathrm{d}s,$$

$$J_1(t, e_1, \alpha, e_2, \beta) = \int_0^t \begin{pmatrix} -2c_0 e_2 \cos^2(\omega_2 s + \beta) \\ -\begin{pmatrix} \sqrt{\dfrac{2e_2}{\omega_2}}(\cos(\omega_2 s + \beta)) \\ \times \dfrac{\partial V}{\partial q_2}\left(\sqrt{\dfrac{2e_1}{\omega_1}}\sin(\omega_1 s + \alpha), \sqrt{\dfrac{2e_2}{\omega_2}}\sin(\omega_2 s + \beta)\right) \end{pmatrix} \end{pmatrix} \mathrm{d}s.$$

Under the potentials given in Eqs. (7) and (8) and assuming the non-resonance condition

$$\omega_1 \neq \omega_2, \tag{50}$$

it can be proved that there is no energy transfer in the sense of Eq. (3). This proof can be obtained by using steps similar to those used in the Section 3.1.1.

Now, assume the quadratic potential in Eq. (7) and the resonance condition

$$\omega_1 = \omega_2 = \omega_0. \tag{51}$$

Similarly to Section 3, we have

$$\begin{aligned} &E_1\left(t, \varepsilon, \sqrt{\frac{2e_1}{\omega_0}}\sin\alpha, \sqrt{2\omega_0 e_1}\cos\alpha, \sqrt{\frac{2e_2}{\omega_0}}\sin\beta, \sqrt{2\omega_0 e_2}\cos\beta\right) \\ &\quad = \omega_0\left(e_1 + \varepsilon\begin{pmatrix} \left(-c_0 e_1 + \dfrac{\sqrt{e_1 e_2}}{\omega_0}\sin(-\alpha+\beta)\right)t \\ +b_9(t, e_1, \alpha, e_2, \beta) \end{pmatrix}\right) \\ &\qquad + O(\varepsilon^2), \\ &E_2\left(t, \varepsilon, \sqrt{\frac{2e_1}{\omega_0}}\sin\alpha, \sqrt{2\omega_0 e_1}\cos\alpha, \sqrt{\frac{2e_2}{\omega_0}}\sin\beta, \sqrt{2\omega_0 e_2}\cos\beta\right) \\ &\quad = \omega_0\left(e_2 + \varepsilon\left(\left(-c_0 e_2 - \frac{\sqrt{e_1 e_2}}{\omega_0}\sin(-\alpha+\beta)\right)t + b_{10}(t, e_1, \alpha, e_2, \beta)\right)\right) \\ &\qquad + O(\varepsilon^2), \end{aligned} \tag{52}$$

The following conclusions are obtained using an argument similar to the one used earlier in Subsection 3.1.2.

(a) The oscillator 1 loses energy and the oscillator 2 gains energy.

Then

$$\begin{cases} -c_0e_1 + \dfrac{\sqrt{e_1e_2}}{\omega_0}\sin(-\alpha+\beta) < 0, \\ -c_0e_2 - \dfrac{\sqrt{e_1e_2}}{\omega_0}\sin(-\alpha+\beta) > 0, \end{cases}$$

or

$$\sin(-\alpha+\beta) < -\omega_0c_0\sqrt{\frac{e_2}{e_1}}. \tag{53}$$

(b) Both oscillators lose energy.
Then

$$\begin{cases} -c_0e_1 + \dfrac{\sqrt{e_1e_2}}{\omega_0}\sin(-\alpha+\beta) < 0, \\ -c_0e_2 - \dfrac{\sqrt{e_1e_2}}{\omega_0}\sin(-\alpha+\beta) < 0, \end{cases}$$

or

$$-\omega_0c_0\sqrt{\frac{e_2}{e_1}} < \sin(-\alpha+\beta) < \omega_0c_0\sqrt{\frac{e_1}{e_2}}. \tag{54}$$

(c) The oscillator 2 loses energy and the oscillator 1 gains energy.

Then

$$\begin{cases} -c_0e_1 + \dfrac{\sqrt{e_1e_2}}{\omega_0}\sin(-\alpha+\beta) > 0, \\ -c_0e_2 - \dfrac{\sqrt{e_1e_2}}{\omega_0}\sin(-\alpha+\beta) < 0, \end{cases}$$

or

$$\sin(-\alpha+\beta) > \omega_0c_0\sqrt{\frac{e_1}{e_2}}. \tag{55}$$

Clearly, an explicit solution of Eq. (46), with potential given by Eq. (7), can be obtained. But the above approach was chosen in order to use the same method in all this work.

Let $B = -p_2(0)$, $A = q_2(0)$. From Eqs. (48) and (16) it follows that the above results determine the following regions in the phase plane of the parameters $q_1(0), p_1(0)$

$$\begin{aligned} &\text{Region 1: } Ap_1(0) + Bq_1(0) < -2c_0e_2\omega, \\ &\text{Region 2: } -2c_0e_2\omega < Ap_1(0) + Bq_1(0) < 2c_0e_1\omega, \\ &\text{Region 3: } Ap_1(0) + Bq_1(0) > 2c_0e_1\omega, \end{aligned}$$

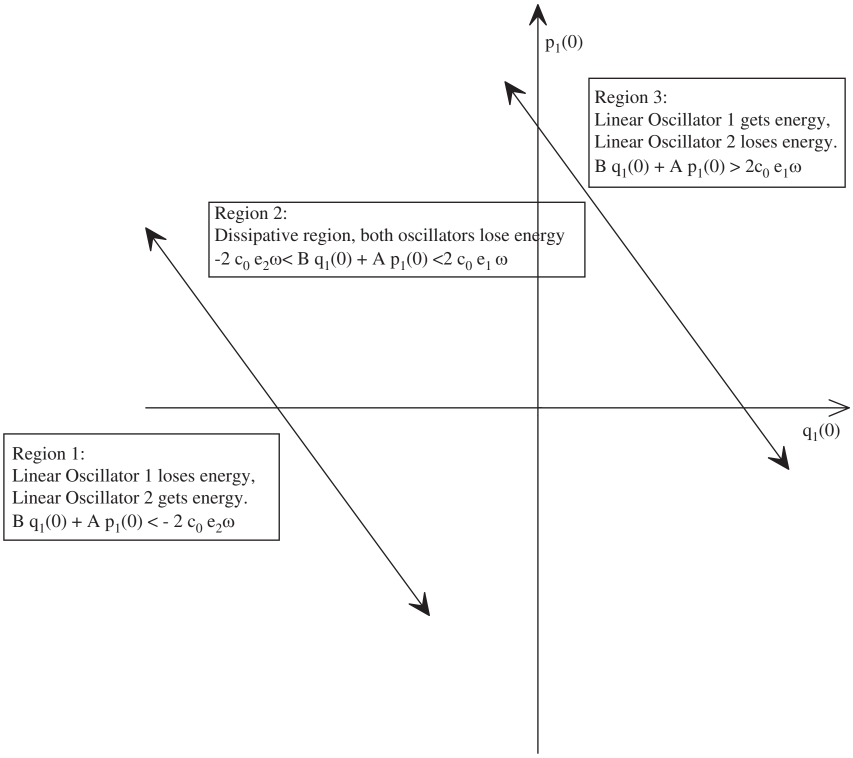

Fig. 3. Energy transfer regions in the presence of linear anchor spring and linear interaction.

and we note Region 1 corresponds to Case (a) above, Region 2 to Case (b) and Region 3 to Case (c). And this information is gathered in Fig. 3.

As earlier, the existence of Region 2 is due to dissipation.

If V is given by Eq. (8), the general approach is analogous to the earlier one.

Assume that $A=0$. It comes from the analysis of the perturbation expansion that there are three resonances: $\omega_1:\omega_2 = 1{:}1, 1{:}3$ and 3:1. Moreover, far from the resonances there is no energy transfer. In this paper only the 1:1 resonance will be taken into account. In this case, Eq. (51), an analogous equation to Eq. (52) is given by

$$E_1\left(t,\varepsilon,\sqrt{\frac{2e_1}{\omega_0}}\sin\alpha,\sqrt{2\omega_0 e_1}\cos\alpha,\sqrt{\frac{2e_2}{\omega_0}}\sin\beta,\sqrt{2\omega_0 e_2}\cos\beta\right)$$

$$= e_1+\varepsilon\left(\left(\begin{array}{l}-c_0e_1\\ \quad +\dfrac{3}{2}\dfrac{\left(\begin{array}{l}e_1e_2(\sin(-2\beta+2\alpha))\\ -\sqrt{e_1e_2}(e_1+e_2)(\sin(-\beta+\alpha))\end{array}\right)}{\omega_0^2}\end{array}\right)t \atop +b_{11}(t,e_1,\alpha,e_2,\beta)\right)+O(\varepsilon^2),$$

$$E_2\left(t,\varepsilon,\sqrt{\frac{2e_1}{\omega_0}}\sin\alpha,\ \sqrt{2\omega_0 e_1}\cos\alpha,\sqrt{\frac{2e_2}{\omega_0}}\sin\beta,\sqrt{2\omega_0 e_2}\cos\beta\right)$$

$$= e_2 + \varepsilon\left(\left(\begin{array}{l}-c_0 e_2 \\ -\dfrac{3}{2}\dfrac{\left(\begin{array}{l}e_1 e_2(\sin(-2\beta+2\alpha)) \\ -\sqrt{e_1 e_2}(e_1+e_2)(\sin(-\beta+\alpha))\end{array}\right)}{\omega_0^2}\end{array}\right)t \atop +b_{12}(t, e_1, \alpha, e_2, \beta)\right) + O(\varepsilon^2),$$

This expansion yields the following:

(a) The oscillator 1 loses energy and the oscillator 2 gains energy.
Then

$$\begin{cases} \left(\begin{array}{l}-c_0 e_1 \\ +\dfrac{3}{2}\dfrac{\left(\begin{array}{l}e_1 e_2(\sin(-2\beta+2\alpha)) \\ -\sqrt{e_1 e_2}(e_1+e_2)(\sin(-\beta+\alpha))\end{array}\right)}{\omega_0^2}\end{array}\right) < 0, \\ \left(\begin{array}{l}-c_0 e_2 \\ -\dfrac{3}{2}\dfrac{\left(\begin{array}{l}e_1 e_2(\sin(-2\beta+2\alpha)) \\ -\sqrt{e_1 e_2}(e_1+e_2)(\sin(-\beta+\alpha))\end{array}\right)}{\omega_0^2}\end{array}\right) > 0, \end{cases}$$

or

$$\frac{3}{2}\frac{e_1 e_2(\sin(-2\beta+2\alpha)) - \sqrt{e_1 e_2}(e_1+e_2)(\sin(-\beta+\alpha))}{\omega_0^2} < -c_0 e_2. \tag{56}$$

(b) Both oscillators lose energy.
Then

$$\begin{cases} \left(\begin{array}{l}-c_0 e_1 \\ +\dfrac{3}{2}\dfrac{\left(\begin{array}{l}e_1 e_2(\sin(-2\beta+2\alpha)) \\ -\sqrt{e_1 e_2}(e_1+e_2)(\sin(-\beta+\alpha))\end{array}\right)}{\omega_0^2}\end{array}\right) < 0, \\ \left(\begin{array}{l}-c_0 e_2 \\ +\dfrac{3}{2}\dfrac{\left(\begin{array}{l}e_1 e_2(\sin(-2\beta+2\alpha)) \\ -\sqrt{e_1 e_2}(e_1+e_2)(\sin(-\beta+\alpha))\end{array}\right)}{\omega_0^2}\end{array}\right) < 0, \end{cases}$$

or

$$-c_0 e_2 < \frac{3}{2}\frac{\left(\begin{array}{l}e_1 e_2(\sin(-2\beta+2\alpha)) \\ -\sqrt{e_1 e_2}(e_1+e_2)(\sin(-\beta+\alpha))\end{array}\right)}{\omega_0^2} < c_0 e_1. \tag{57}$$

(c) The oscillator 2 loses energy and the oscillator 1 gains energy.

Then

$$\begin{cases} \left(-c_0 e_1 + \dfrac{3}{2} \dfrac{\left(\begin{array}{l} e_1 e_2(\sin(-2\beta + 2\alpha)) \\ -\sqrt{e_1 e_2}(e_1 + e_2)(\sin(-\beta + \alpha)) \end{array} \right)}{\omega_0^2} \right) > 0, \\ \left(-c_0 e_2 - \dfrac{3}{2} \dfrac{\left(\begin{array}{l} e_1 e_2(\sin(-2\beta + 2\alpha)) \\ -\sqrt{e_1 e_2}(e_1 + e_2)(\sin(-\beta + \alpha)) \end{array} \right)}{\omega_0^2} \right) < 0, \end{cases}$$

or

$$\frac{3}{2} \frac{\left(\begin{array}{l} e_1 e_2(\sin(-2\beta + 2\alpha)) \\ -\sqrt{e_1 e_2}(e_1 + e_2)(\sin(-\beta + \alpha)) \end{array} \right)}{\omega_0^2} > c_0 e_1. \tag{58}$$

As earlier, there are three regions in the phase plane of the parameters $q_1(0), p_1(0)$. Using Eqs. (48) and (16) inequalities at Eqs. (56), (57) and (58) can be written, respectively, as

$$\begin{cases} (Ap_1(0) + Bq_1(0) + C)p_1(0) + Dq_1(0) + E < F, \\ F < (Ap_1(0) + Bq_1(0) + C)p_1(0) + Dq_1(0) + E < G, \\ (Ap_1(0) + Bq_1(0) + C)p_1(0) + Dq_1(0) + E > G, \end{cases} \tag{59}$$

where

$$\begin{aligned} &A = -\frac{q_2(0)p_2(0)}{\omega_0\sqrt{e_1 e_2}}, \quad B = -\frac{-p_2(0)^2 + \omega_0 e_2}{\omega_0\sqrt{e_1 e_2}}, \\ &C = \frac{q_2(0)(e_1 + e_2)}{2\sqrt{e_1 e_2}}, \quad D = -\frac{p_2(0)(e_1 + e_2)}{2\sqrt{e_1 e_2}}, \\ &E = \frac{\sqrt{e_1}q_2(0)p_2(0)}{\sqrt{e_2}}, \quad F = -\frac{2\omega_0^2 c_0\sqrt{e_2}}{3\sqrt{e_1}}, \\ &G = \frac{2\omega_0^2 c_0\sqrt{e_1}}{3\sqrt{e_2}}. \end{aligned}$$

Now, assuming $B \neq 0$ and making the following change of variables $u = Ap_1(0) + Bq_1(0) + C - (AD/B)$, $v = p_1(0) + (D/B)$ in Eq. (59), we obtain, respectively

$$\begin{cases} uv + \dfrac{AD^2}{B^2} - \dfrac{CD}{B} + E < F, \\ F < uv + \dfrac{AD^2}{B^2} - \dfrac{CD}{B} + E < G, \\ uv + \dfrac{AD^2}{B^2} - \dfrac{CD}{B} + E > G. \end{cases}$$

Define

$$U_0 = F - \left(\frac{AD^2}{B^2} - \frac{CD}{B} + E \right) \text{ and } V_0 = G - \left(\frac{AD^2}{B^2} - \frac{CD}{B} + E \right).$$

The Regions 1,2,3 have the same meaning as in the earlier case. When $U_0 < 0$ and $V_0 > 0$, the division of phase plane depicted in Fig. 4 is obtained.

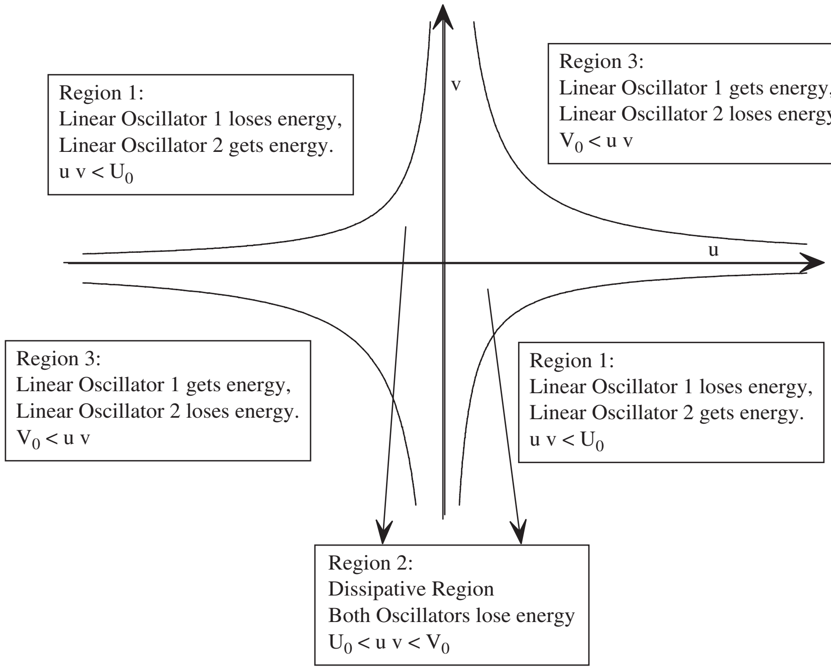

Fig. 4. Energy transfer regions in the presence of linear anchor spring and nonlinear interaction.

5. Conclusions

In this work, the problem of energy transfer in a dissipative mechanical system is analyzed. The approach presented here has been based on rigorous Regular Perturbation Theory. The definition of energy transfer, given in Section 2, was discussed through some examples. The main results can be summarized as follows:

(a) There is no energy transfer between the oscillators, if non-resonance conditions, such as Eqs. (31) and (45), hold.
(b) If resonance conditions such as Eqs. (36) and (37) hold, there is energy transfer between the oscillators. Regions on the phase space are obtained where such phenomenon happens, see for example, Eqs. (36) and (37) and Figs. 2–4.
(c) For perturbations of nonlinear normal modes of the system given by Eq. (42), there is no energy transfer.
(d) First-order approximation formulae are obtained giving explicity dependence of the energies E_1 and E_2 on the initial conditions, the mass m_1, the amplitude and the frequency of the external excitation. See for example, Eqs. (23) and (25).

In (a)–(c) absence of external excitation is assumed. In view of (d), similar results in the presence of external excitation can be obtained.

Of course, if one desires to make the best choice for a mechanical setting to obtain the energy loss of the linear oscillator 1, the best choice is the case given in Fig. 2, because the set of the initial conditions such that the nonlinear oscillator loses energy is a compact one. In all other cases, including the complete linear situation, such set is non compact, see Figs. 3 and 4. In fact, the nonlinearity of the spring acts as a compactification of Region 3.

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Appendix

Consider the following initial value problem:

$$\begin{cases} \dot{y}^2 = (1 - y^2)(1 - k^2 y^2), \\ y(0) = 0, \end{cases} \tag{60}$$

where k, $0<k<1$, is a parameter. The solution of Eq. (60) is a Jacobian elliptic function, which is denoted by $sn[t,k]$. Moreover, consider the following functions:

$$\begin{aligned} (cn[t,k])^2 &= 1 - (sn[t,k])^2, \quad cn[0,k] = 1, \\ (dn[t,k])^2 &= 1 - k^2(sn[t,k])^2, \quad dn[0,k] = 1. \end{aligned} \tag{61}$$

They are called Jacobian functions too. Let

$$K(k) = \int_0^1 (1 - t^2)^{-\frac{1}{2}}(1 - k^2 t^2)^{-\frac{1}{2}}\,\mathrm{d}t. \tag{62}$$

It can be proved, [10], that $sn[t,k]$, $cn[t,k]$ and $dn[t,k]$ are periodic functions whose periods are $4K(k), 4K(k)$ and $2K(k)$, respectively. Moreover, these functions are differentiable and

$$\begin{aligned} &sn'[t,k] = cn[t,k]\,\mathrm{d}n[t,k], \quad cn'[t,k] = -sn[t,k]\,\mathrm{d}n[t,k], \\ &dn'[t,k] = -k^2\, sn[t,k] cn[t,k]. \end{aligned}$$

From this, we obtain

$$cn''[t,k] = (-1 + 2k^2) cn[t,k] - 2k^2 (cn[t,k])^3. \tag{63}$$

Further, the Fourier expansion of $cn[t,k]$ is given by

$$cn\left(\frac{2K}{\pi}t\right) = \sum_{m=0}^{\infty} \frac{2\pi}{Kk} \frac{q^{m+\frac{1}{2}}}{1 + q^{2m-1}} \cos((2m+1)t), \tag{64}$$

where q is the only one solution in the interval $(0,1)$ of the following equation:

$$\left(\frac{2\sum_{l=0}^{\infty} q^{\left(l+\frac{1}{2}\right)^2}}{1 + 2\sum_{l=0}^{\infty} q^{l^2}}\right)^4 = k^2, \tag{65}$$

Ref. [10] pp. 511, 480, respectively.

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