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Comments on energy confinement in non-periodic structures

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Abstract

In this study, we investigate the possibility of mode localization occurrence in a non-periodic Pflüger's column model of a rocket with an intermediate concentrated mass at its middle point. We discuss the effects of varying the intermediate mass magnitude and its position and the resulting energy confinement for two cases. Free vibration analysis and the severity of mode localization are appraised, without decoupling the system, by considering as a solution basis the fundamental free response or dynamical solution. This allows for the reduction of the dimension of the algebraic modal equation that arises from satisfying the boundary and continuity conditions. By using the same methodology, we also consider the case of a cantilevered Plüger's column with rotational stiffness at the middle support instead of an intermediate concentrated mass.

1. Introduction

For weakly coupled quasi-periodic structural systems, mode localization is a phenomenon in which the free vibration modes will typically be spatially localized, resulting in confined regions of the structure where the vibration energy is concentrated. This can be used as a possible control technique. Typically, some modular structures may be very sensitive to small imperfections, that is, the mathematical model can differ from the real physical system, due to small variations, such as a manufacturing error, a geometrical irregularity, or a mistuned parameter, leading to energy confinement. Due to these reasons, the subject is worthy of theoretical studies leading to some guidance for engineering practice. See, for example, Refs. [1–4]. All these, and others previous works, are mainly concerned with disordered periodic structures. To our knowledge, the energy confinement phenomenon in *non-periodic structures* has not being considered.

In this work, we study the normal modes and the possibility of mode localization occurrence in a *non-periodic structure*. We discuss possible mode localization due to the effects of varying the magnitude or the position of an intermediate concentrated mass in a column model of a rocket. We consider a non-periodic

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cantilevered Plüger's column subjected (or not) to a tangential follower thrust provided by a solid fuel rocket motor at the bottom end of the column. The vibration modes of such a system are determined with the use of a fundamental free response. This allows for the reduction of the dimension of the algebraic modal equation that arises from satisfying the boundary and continuity conditions. Further, we work in the physical space of the problem, without using the so-called state space formulation. Using the same methodology, we also comment the case of a cantilevered Plüger's column with rotational stiffness at the middle support instead of an intermediate concentrated mass.

2. Problem statement

We consider the transverse vibrations of a two-span Plüger's column. The system is modelled as a cantilever column subjected to a tangential follower thrust provided by a solid fuel rocket motor at the bottom end of the column, and with an intermediate concentrated mass attached at a middle point in the column. The rocket motor is considered as a rigid body. A mathematical model of the column is shown in Fig. 1.

The equations of small motions of each span are given by

$$m_j \frac{\partial^2 w_j(t,x_j)}{\partial t^2} + P \frac{\partial^2 w_j(t,x_j)}{\partial x_j^2} + EI_j \frac{\partial^4 w_j(t,x_j)}{\partial x_j^4} = 0 \tag{1}$$

for $j = 1, 2$, where m_j are masses per unit length, EI_j are flexural stiffness products for each substructure and P is the tangential follower thrust. The column has a total length L. The rocket motor is considered as a rigid body with M mass.

The boundary conditions at $x = 0$ and $x = L$ are given by

$$w(t,0) = w'(t,0) = 0, \quad EIw''(t,L) = 0, \quad EIw'''(t,L) = M\ddot{w}(t,L) \tag{2}$$

and we have the intermediate compatibility conditions at $x = x_1$:

$$\begin{aligned} &w_1(t,x_1) = w_2(t,x_1), \quad w_1'(t,x_1) = w_2'(t,x_1), \\ &EI_1 w_1''(t,x_1) = EI_2 w_2''(t,x_1), \\ &\quad - M_c \ddot{w}_2(t,x_1) + EI_1 w_1'''(t,x_1) = EI_2 w_2'''(t,x_1). \end{aligned} \tag{3}$$

Here M_c is the magnitude of the intermediate concentrated mass.

3. Modal analysis

By inserting a displacement function of the form

$$w_j(t,x_j) = e^{\lambda t} X_j(x_j) \tag{4}$$

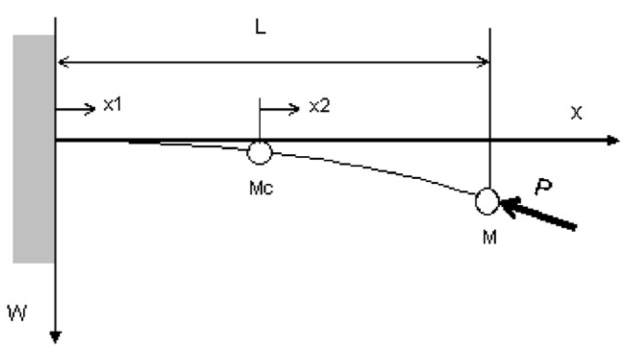

Fig. 1. Cantilevered Plüger's column.

and introducing the non-dimensional variables

$$\xi_j = \frac{x_j}{L}, \quad \eta_j = \frac{w_j}{L}, \quad \tau_j = \frac{t}{L^2}\sqrt{\frac{EI_j}{m_j}},$$
$$\rho_j = \frac{PL^2}{EI_j}, \quad \mu_j = \frac{M}{m_j L}, \tag{5}$$

Eqs. (1)–(3) are converted into the following non-dimensional boundary value problem:

$$X_j^{(\mathrm{iv})}(\xi_j) + \rho_j X_j''(\xi_j) + \lambda_j^2 X_j(\xi_j) = 0, \tag{6}$$

$$X_1(0) = X_1'(0) = 0, \quad X_2''(1) = 0, \quad X_2'''(1) = \mu_2 \lambda_2^2 X_2(1) \tag{7}$$

with the following compatibility conditions:

$$X_1(\xi_1) = X_2(\xi_1), \quad X_1'(\xi_1) = X_2'(\xi_1),$$
$$X_1''(\xi_1) = X_2''(\xi_1), \quad X_2'''(\xi_1) = X_1'''(\xi_1) - \mu_c X_1(\xi_1). \tag{8}$$

The general solutions of Eq. (6) can be written as

$$X_j(\xi) = A_j h(\xi_j) + B_j h'(\xi_j) + C_j h''(\xi_j) + D_j h'''(\xi_j), \tag{9}$$

where we choose as a solution basis the function $h(\xi_j)$ and its first three derivatives $h'(\xi_j)$, $h''(\xi_j)$, $h'''(\xi_j)$. This function is usually referred in literature as *dynamical basis* or *fundamental solution* and it is generated by the solution of the initial value problem [5]:

$$h^{(\mathrm{iv})}(\xi_j) + \rho_j L^2 h''(\xi_j) + \lambda_j^2 L^4 h(\xi_j) = 0,$$
$$h(0) = 0, \quad h'(0) = 0, \quad h''(0) = 0, \quad h'''(0) = 1. \tag{10}$$

In terms of the traditional spectral basis, the fundamental solution $h(\xi_j)$ has the following representation:

$$h(\xi_j) = \frac{\delta_j \sinh(\varepsilon_j \xi_j) - \varepsilon_j \sin(\delta_j \xi_j)}{\varepsilon_j \delta_j (\varepsilon_j^2 + \delta_j^2)}, \tag{11}$$

where $s_j = \varepsilon_j, -\varepsilon_j, i\delta_j, -i\delta_j$ are the roots of the characteristic polynomial

$$p_j(s) = s_j^4 + \rho_j L^2 s_j^2 - \lambda_j^2 L^4 \tag{12}$$

and

$$\varepsilon_j = \left[\frac{-\rho_j L^2}{2} + \left(\lambda_j^2 L + \frac{\rho_j L^4}{4}\right)^{1/2}\right]^{1/2}, \quad \delta_j = (\varepsilon_j^2 + \rho_j L^2)^{1/2}. \tag{13}$$

The fundamental response $h(\xi_j)$, has the same shape for each segment, but depends on different values of the involved physical parameters such as the flexural stiffness and the masses per unit length of each substructure.

The substitution of Eq. (9) into de boundary and continuity conditions leads to a linear algebraic system

$$\mathscr{U}(\lambda)\mathbf{q} = \mathbf{0}, \tag{14}$$

for the vector $\mathbf{q}$ of order 8×1

$$\mathbf{q} = [A_1 \;\; B_1 \;\; C_1 \;\; D_1 \;\; A_2 \;\; B_2 \;\; C_2 \;\; D_2]^{\mathrm{T}}. \tag{15}$$

Here, the matrix $\mathscr{U}$ is of order 8×8 and it has the form $\mathscr{U} = \mathscr{B}\Phi$, where $\mathscr{B}$ is a matrix of order 8×16 formed with the coefficients associated to the boundary and continuity conditions. Φ is a matrix of order 16×8 whose components are values of the solution basis at the ends of the column and the conditions at the position of the intermediate concentrated mass. Observe that by using the initial values of the dynamical basis given by Eq. (10) then the system order is reduced to 4×4.

Non-zero solutions of Eq. (14) are obtained for frequency values that satisfy the characteristic equation:

$$\det(\mathscr{U}) = 0. \tag{16}$$

For determining the natural frequencies we solve the characteristic equation for ε_j and compute the corresponding parameter λ_j in order to find δ_j.

4. Results and discussion

Results for the first four eigensolutions are presented as functions of the system parameters. In order to investigate the effect of small disorders on the system dynamics, the ratio of the peak deflection amplitude in one span to that in the other span, Γ, is introduced. For each natural frequency, the corresponding amplitude ratio or degree of mode localization Γ between the two spans is defined as

$$\Gamma = \begin{cases} \dfrac{|X_1|}{|X_2|}, & \text{for } |X_1| \geqslant |X_2|, \\ \dfrac{|X_2|}{|X_1|}, & \text{for } |X_1| < |X_2|, \end{cases} \tag{17}$$

where X_1 and X_2 are the maximum amplitudes of displacements of the spans. If the vibration is confined to one of the substructures, Γ becomes large. The sign of this parameter indicates the substructure at which the vibration is confined: positive for vibration localization in the second substructure and negative for the first one. To be consistent with the terminology in Ref. [4], a mode is considered as localized when $|\Gamma| > 2$. In our numerical studies, a disorder or disturbance parameter, denoted by Δ, is chosen to be vary between 2% and 10%.

The values of the system parameters were chosen accordingly to the experimental setup described in Refs. [6,7]. In our simulations, we used $EI_1 = EI_2 = 58.8\,\mathrm{N\,m^2}$, $L = 1.0\,\mathrm{m}$, $M_c = 1.0\,\mathrm{kg}$ or $2.3\,\mathrm{kg}$, $M = 4.05\,\mathrm{kg}$, $m_1 = m_2 = 0.567\,\mathrm{kg/m}$, $P = 400\,\mathrm{N}$ and $x_1 = 0.50\,\mathrm{m}$. The maximum load is chosen to be smaller than the first buckling load, to ensure that ω_1 is real, according to Sugiyama et al. [7].

4.1. Effects of small disturbances in the intermediate mass

Figs. 2 and 3 show the effects on mode localization due to variations of the intermediate concentrated mass position and magnitude, respectively. In Fig. 2 only the mass $M_c = 1.0\,\mathrm{kg}$ is considered. The model localization factor for $M_c = 2.3$ and $1.0\,\mathrm{kg}$ is shown in Fig. 3.

In both figures, we observe that the maximum value of Γ is less than one for all modes. We observe also that as the disturbance increases the factor of localization decreases for all vibration modes. It is clear from the plots that it does not occur vibration energy confinement, according to our definition of localization.

5. Column without tangential follower thrust

Interesting results arises when we do not consider the effect of the tangential follower thrust, that is, when we do $P = 0$ in Eq. (1). In this case, free vibrations can be determined by solving the differential equation for the amplitude distribution $X_j(\xi_j)$ which satisfies

$$X_j^{(\mathrm{iv})}(\xi_j) - \lambda_j^2 X_j(\xi_j) = 0, \quad j = 1, 2 \tag{18}$$

for each segment of the column. The fundamental basis $h(\xi_j)$ now is given by Claeyssen and Soder [8]

$$h(\xi_j) = \frac{\sinh(\lambda_j \xi_j) - \sin(\lambda_j \xi_j)}{2\lambda_j^3} \tag{19}$$

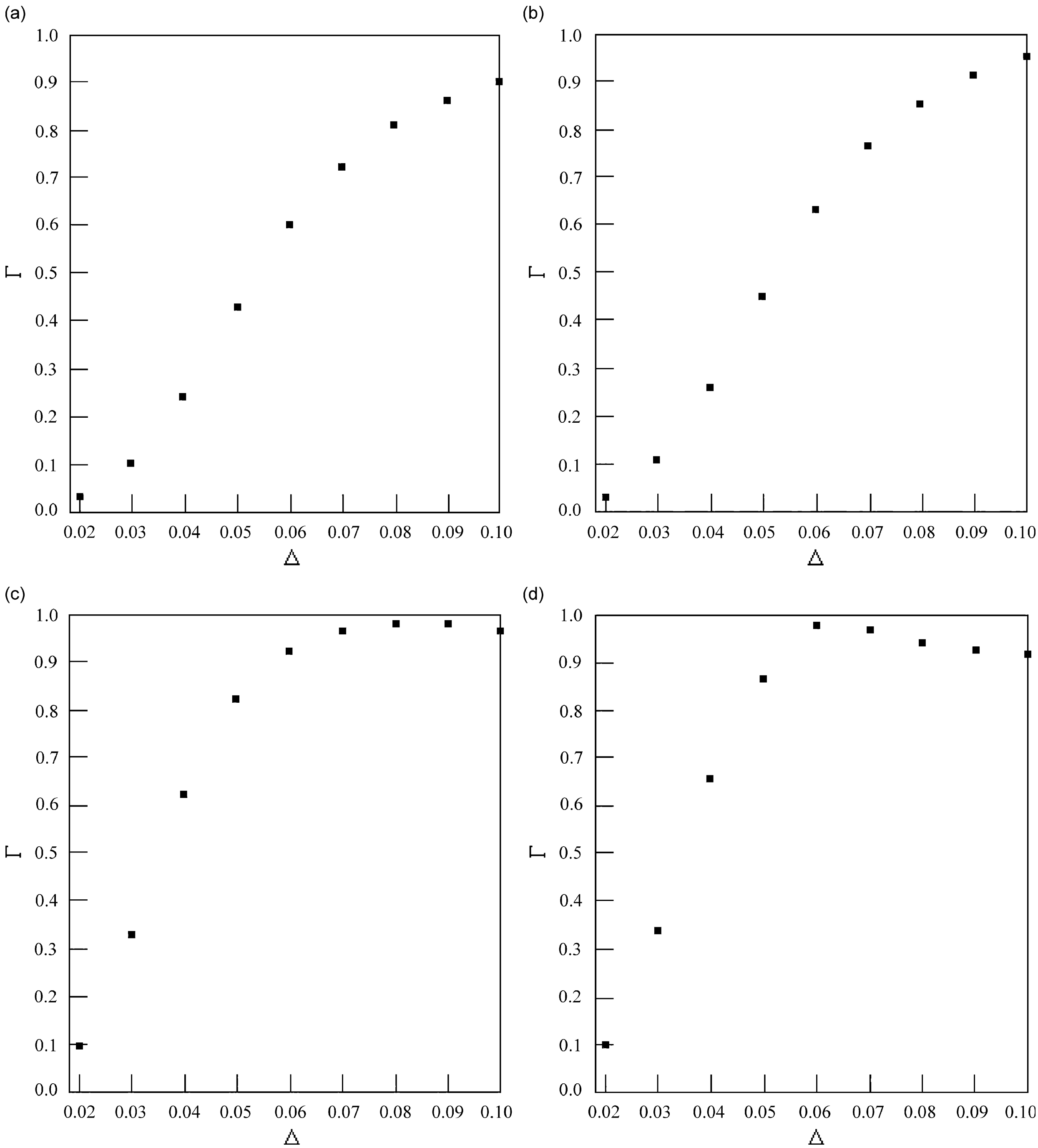

Fig. 2. Amplitude ratio Γ as a function of the disturbance parameter Δ for disturbs on M_c position: (a) first; (b) second; (c) third; (d) fourth mode.

for $j = 1, 2$, and is the solution of the initial value problem

$$\begin{aligned} &h^{(\mathrm{iv})}(\xi_j) - \lambda_j^4 L^4 h(\xi_j) = 0, \\ &h(0) = 0, \quad h'(0) = 0, \quad h''(0) = 0, \quad h'''(0) = 1. \end{aligned} \tag{20}$$

Some applications are discussed in the sequel.

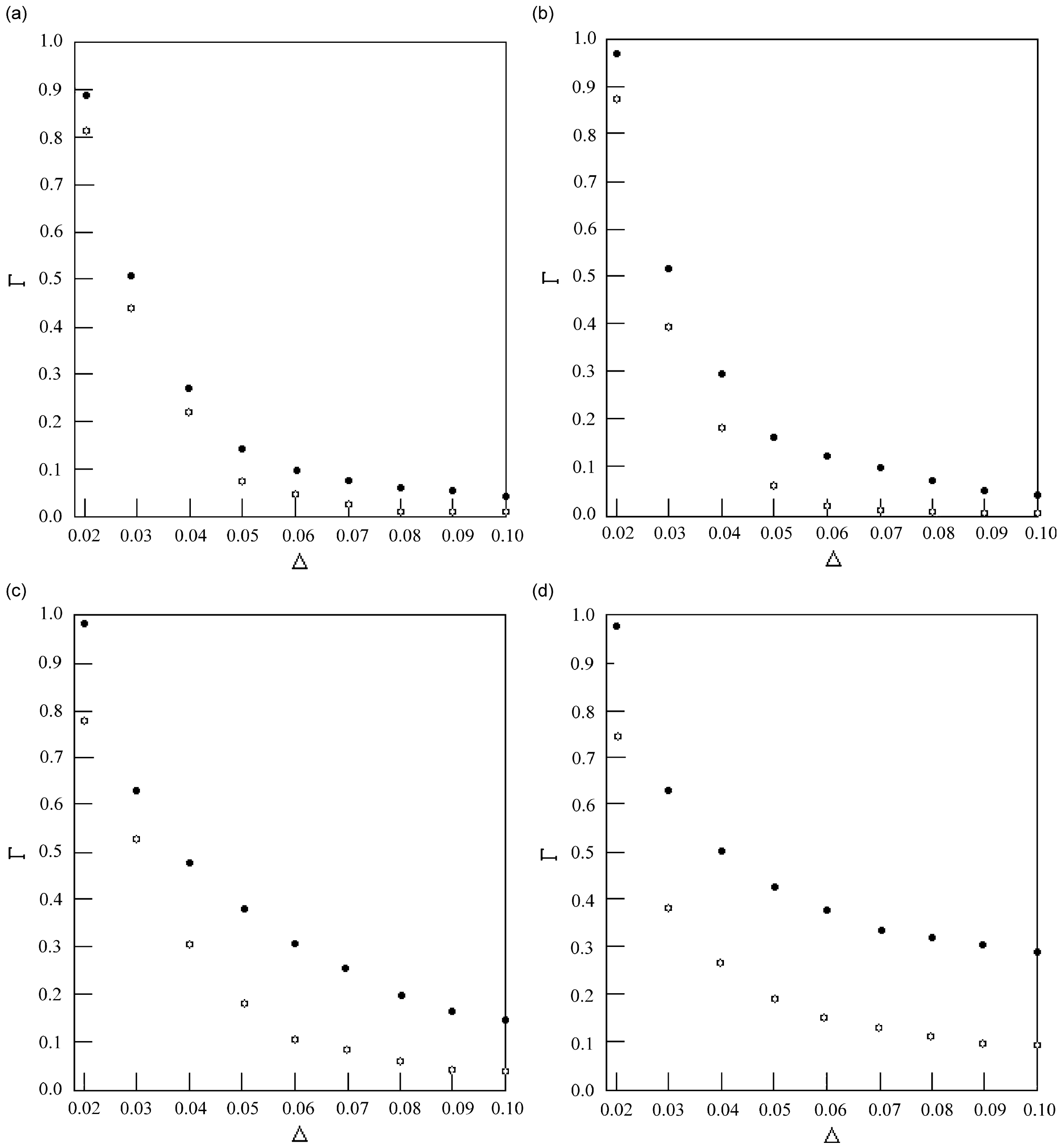

Fig. 3. Amplitude ratio Γ as a function of the disturbance parameter Δ for disturbs on M_c magnitude; (○) $M_c = 2.3$ kg; (●) $M_c = 1.0$ kg.

5.1. Cantilevered column with intermediate concentrated mass

For numerical simulations we consider the same material properties as before, except for intermediate concentrated mass M_c. Here we compute the amplitude ratio Γ for $M_c = 0.25$, 0.50 and 0.75 kg.

The simulations for small disturbances Δ on the concentrate intermediated mass magnitudes do not show mode localization occurrence. However, when we disturb its *position* we obtain the results shown in Fig. 4. Observe that all modes are localized to some degree for $M_c = 0.25$ and 0.50 kg. Nevertheless, for $M_c = 0.75$ kg

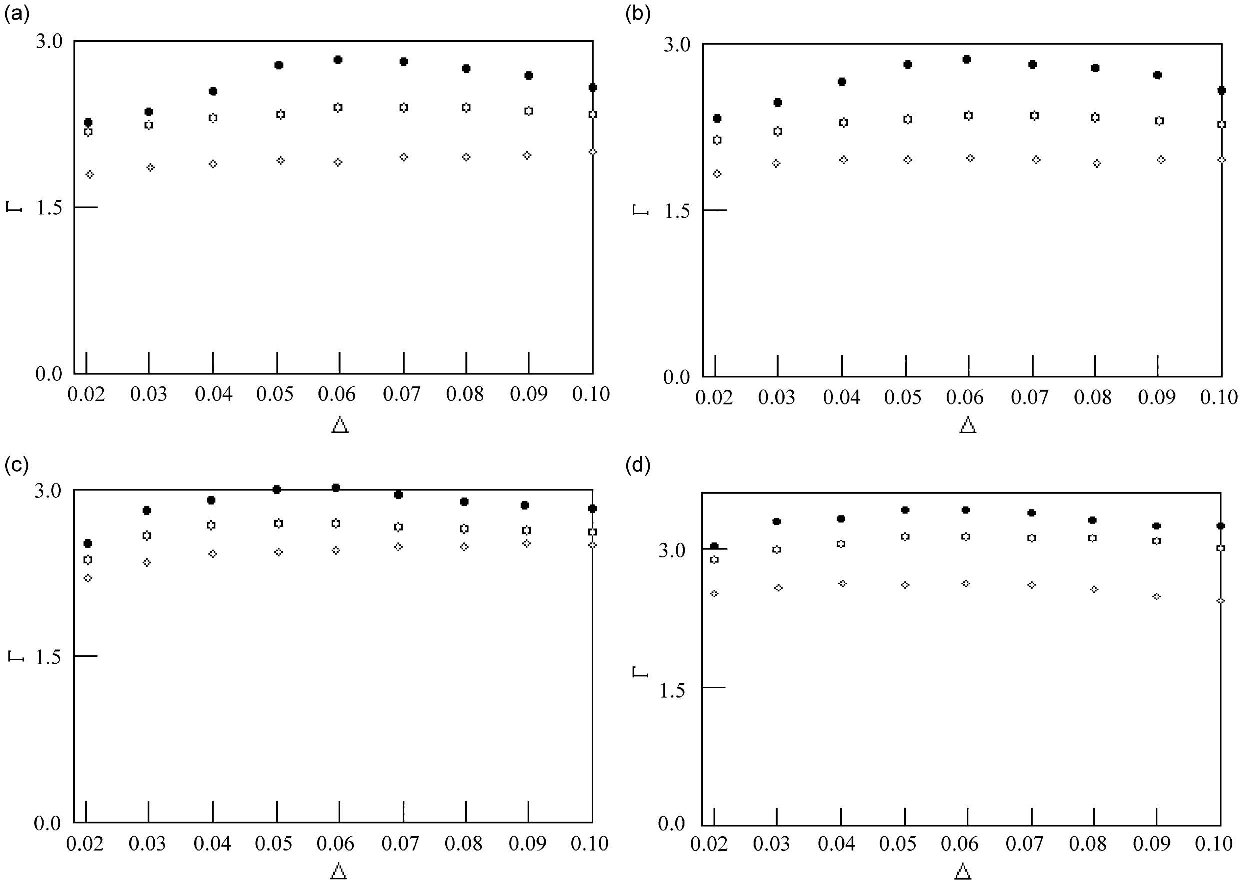

Fig. 4. Amplitude ratio Γ as a function of the disturbance parameter Δ for disturbs on M_c magnitude. (•) $M_c = 0.25$ kg, (○) $M_c = 0.50$ kg and (◇) $M_c = 0.75$ kg; (a) first; (b) second; (c) third; (d) fourth mode.

Table 1
Natural frequencies and degrees of mode localization for ordered system (f_o; Γ_o) and for disordered system (f_d; Γ_d)

Mode	f_o (Hz)	Z_o	f_d (Hz)	Z_d
1	681.42	0.000	681.48	1.015
2	2142.80	0.009	2197.52	−2.873
3	2249.00	0.021	2426.11	2.428
4	4615.12	0.005	4614.94	−2.531
5	7026.73	0.033	7703.70	2.061

the amplitude ratio is smaller than 2.0 for the first and second modes, indicating that mode localization does not occur. Note also that mode localization in the higher modes is more sensitive to the system parameters than in the lower ones, for all values of M_c. Tests were performed for intermediate concentrated masses higher than 0.75 kg, but for all those cases mode localization does not occur.

5.2. Cantilevered column with a rotational spring at the middle support

Now, we comment some results for a column with a rotational spring instead of a intermediate concentrated mass. The modal analysis for this case is done considering the boundary conditions given by Eq. (2) and the continuity conditions at $x = x_1$ given by $w_1(t, x_1) = 0$, $w_2(t, x_1) = 0$, $w_1'(t, x_1) = -w_2'(t, x_1)$ and

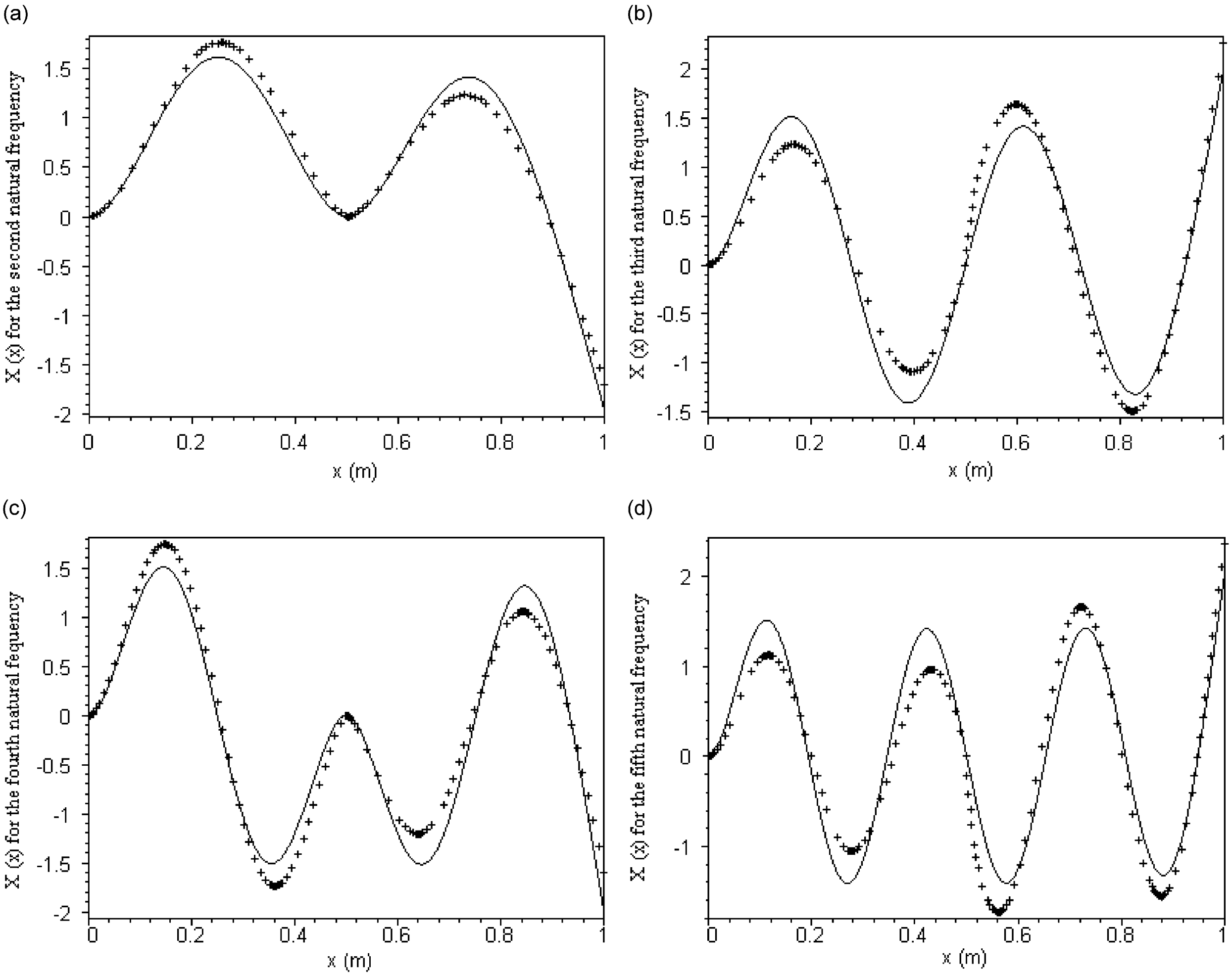

Fig. 5. First normal modes: (—) initial system; (+) disturbed system; (a) second; (b) third; (c) fourth; (d) fifth mode.

$w_1''(t,x_1) = w_2''(t,x_1) + kw_2'(t,x_1)$. The occurrence of mode localization phenomenon is examined introducing a disturbance on the length L_2 in -5%. For simulations we have used the following parameters (in the ordered case): $EI_1 = EI_2 = 2 \times 10^7\,\mathrm{N\,m^2}$, $L_1 = L_2 = 0.5\,\mathrm{m}$, $m_1 = m_2 = 20\,\mathrm{kg}$, $k = 2 \times 10^8\,\mathrm{N\,m}$.

The amount of mode localization is given in Table 1 for the first five frequencies, where Γ_o is the mode localization factor for the initial system and Γ_d represents the mode localization factor for the disturbed system. Selected normal modes are shown in Fig. 5: for the initial system, all the modes are not localized; on the other hand, for the disturbed system, all the modes, except the first one, are localized to some degree. Table 1 confirms these results. Note that weak coupling will cause the system to be sensitive to the disturbance, independently of the system periodicity.

From the results of this work, we propose that mode localization may occur in *non-periodic structures* as well as periodic ones. Nevertheless, further studies on the mode localization phenomenon for general non-periodic structures are required.

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