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Orthogonal basis selection method for robust partial eigenvalue assignment problem in second-order control systems

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Abstract

In this paper, we study the robust partial eigenvalue assignment problem using state feedback control in second-order control systems, which means that the feedback control matrices not only assign specific eigenvalues to the second-order closed-loop system, but also that the system is robust, or insensitive to perturbations in the coefficient matrices. Some measures of robustness of the closed-loop system are discussed and a numerical method is proposed such that it aims to minimize one measure of the conditioning of the closed-loop system. Numerical examples show that the method is convergent, and often leads to better conditioned closed-loop system.

1. Introduction

Consider the time-invariant second-order control system

$$M\ddot{z} + C\dot{z} + Kz = Bu, \tag{1}$$

where $z = z(t) \in \mathbf{R}^n$, $u = u(t) \in \mathbf{R}^m$, $M, C, K \in \mathbf{R}^{n\times n}$ are symmetric matrices with M positive definite and K nonsingular, and $B \in \mathbf{R}^{n\times m}$ with $\mathrm{rank}(B) = m$. This kind of systems arise naturally in variety of applications, including, for example, the control of large flexible space structure, earthquake engineering, the control of mechanical multi-body systems, stabilization of damped systems and robotics. One of the important control problems is to design a proportional and derivative state feedback controller of the form

$$u = F^{\mathrm{T}}\dot{z} + G^{\mathrm{T}}z + v,$$

where $F, G \in \mathbf{R}^{n\times m}$, such that the closed-loop system

$$M\ddot{z} + (C - BF^{\mathrm{T}})\dot{z} + (K - BG^{\mathrm{T}})z = Bv \tag{2}$$

has desired properties.

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It is well known that the behavior of the second-order system (1) is governed by the eigenstructure of its associated quadratic pencil

$$Q(\lambda) = \lambda^2 M + \lambda C + K. \tag{3}$$

The desired properties of the closed-loop system can therefore be achieved by selecting the feedback gain matrices F and G to assign specified eigenvalues to the quadratic pencil associated with the closed-loop system

$$Q_c(\lambda) = \lambda^2 M + \lambda(C - BF^{\mathrm{T}}) + (K - BG^{\mathrm{T}}). \tag{4}$$

The problem of finding matrices F and G such that the quadratic pencil (4) has the specified eigenvalues is called the eigenvalue assignment problem in second-order control systems. In most practical situations, however, only a few eigenvalues of the open-loop system (1) are undesirable, so it makes more sense to alter only those undesirable eigenvalues, while keeping the rest of the spectrum invariant. This leads to the following problem, known as the partial eigenvalue assignment problem in second-order control systems:

Problem PEA. Given $n \times n$ real symmetric matrices M, C, K with M positive definite and K nonsingular, the $n \times m$ real control matrix B, the self-conjugate subset $\{\lambda_1, \lambda_2, \ldots, \lambda_p\}$ $(p<n)$ of the open-loop spectrum $\{\lambda_1, \ldots, \lambda_p; \lambda_{p+1}, \ldots, \lambda_{2n}\}$ and the corresponding eigenvector set $\{x_1, x_2, \ldots, x_p\}$, and given a self-conjugate set $\{\mu_1, \mu_2, \ldots, \mu_p\}$ of numbers, find $n \times m$ real feedback matrices F and G such that the spectrum of the quadratic pencil (4) is $\{\mu_1, \ldots, \mu_p, \lambda_{p+1}, \ldots, \lambda_{2n}\}$.

It is well known (see Ref. [1]) that the system modelled by Eq. (1) is completely controllable if and only if

$$\mathrm{rank}([\lambda^2 M + \lambda C + K, B]) = n$$

for every eigenvalue λ of the quadratic pencil (3). Completely controllability is a necessary and sufficient condition for the existence of F and G such that the quadratic pencil (4) has a spectrum that can be arbitrarily assigned. However, if the system is only partially controllable, that is, if

$$\mathrm{rank}([\lambda^2 M + \lambda C + K, B]) = n$$

only for the p eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_p$, then only those eigenvalues can be arbitrarily assigned by an appropriate choice of F and G. Furthermore, when $m>1$, the solution to Problem PEA is essentially undetermined, with many degrees of freedom. Therefore, the question arises as to how this freedom is to be parameterized and how it is to be exploited in practice, such that the eigenvalues of the closed-loop quadratic pencil (4) should be insensitive to perturbations in matrices $M_c = M$, $C_c = C - BF^{\mathrm{T}}$ and $K_c = K - BG^{\mathrm{T}}$. This leads to the following robust partial eigenvalue assignment problem in second-order control systems:

Problem RPEA. Find a solution (F, G) to Problem PEA, such that the closed-loop system is robust, in the sense that the eigenvalues of the quadratic pencil $Q_c(\lambda)$ in Eq. (4) are as insensitive to perturbations in the matrices M_c, C_c and K_c as possible.

Problem PEA is first proposed by B.N. Datta, S. Elhay and Y.M. Ram [2]. In that paper, three orthogonality relations between the eigenvectors of a symmetric definite quadratic pencil are derived, and then a numerical method is proposed based on these relations. Different from some other algorithms before, the method in Ref. [2] works directly with the data matrices M, C and K of the second-order system, rather than the $2n \times 2n$ first-order linearization of the second-order system. This allows the exploitation of matrix structural properties, such as symmetry, sparsity and bandedness. Furthermore, the method does not require knowledge of the unchanged eigenvalues and their corresponding eigenvectors of the open-loop pencil. However, the robustness issue is not considered there, and so it usually cannot give better conditioned closed-loop systems, which can be seen from the numerical examples in Section 4. Some related work can be also found in Ref. [12–14].

In our former paper [3], we give a numerical method for finding a solution to Problem RPEA, where eigenvectors are chosen in certain subspaces such that each vector is as orthogonal as possible to the space spanned by the remaining vectors. Although numerical experiments show that the method proposed in Ref. [3] does often lead to better conditioned closed-loop systems, but that method can not be guaranteed to converge theoretically. In this paper, we propose another numerical method for Problem RPEA. The new method is to minimize one measure of conditioning of the closed-loop system. It uses similar techniques as described in Ref. [4], where eigenvectors are chosen in certain subspaces such that some measure of the distance between the eigenvectors and some orthogonal basis of a certain subspace is minimized. A critical advantage of the present method over the method proposed in Ref. [3] is that it is convergent, and the numerical experiments in Section 4 also show that it gives rise to significant improvement in the conditioning of the closed-loop systems.

Throughout this paper, the following notations will be used. The $2n$ eigenvalues of the open-loop pencil $Q(\lambda)$ are $\lambda_1, \lambda_2, \ldots, \lambda_{2n}$, and corresponding eigenvectors are $x_1, x_2, \ldots, x_{2n}$, and we let

$\Lambda = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_{2n})$, whose diagonal elements are eigenvalues of $Q(\lambda)$,
$\Lambda_1 = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_p)$, whose diagonal elements are the eigenvalues to be altered,
$\Lambda_2 = \mathrm{diag}(\lambda_{p+1}, \ldots, \lambda_{2n})$, whose diagonal elements are the eigenvalues kept unchanged,
$X = [x_1, x_2, \ldots, x_{2n}]$, whose columns are corresponding eigenvectors of $Q(\lambda)$,
$X_1 = [x_1, x_2, \ldots, x_p]$,
$X_2 = [x_{p+1}, x_{p+2}, \ldots, x_{2n}]$,
$D = \mathrm{diag}(\mu_1, \mu_2, \ldots, \mu_p)$, whose diagonal elements are eigenvalues of $Q_c(\lambda)$,
$Y = [y_1, y_2, \ldots, y_p]$, whose columns are corresponding eigenvectors of $Q_c(\lambda)$.

In addition, for any matrix $A \in \mathbf{C}^{m\times n}$, A^{T} denotes the transpose of A, $\bar{A}$ the conjugate of A, and A^* the conjugate transpose of A. The range and the null space of A are denoted by $\mathscr{R}(A)$ and $\mathscr{N}(A)$, respectively, that is

$$\mathscr{R}(A) = \{y \in \mathbf{C}^m : y = Ax \text{ for some } x \in \mathbf{C}^n\},$$

$$\mathscr{N}(A) = \{x \in \mathbf{C}^n : Ax = 0\}.$$

Given any subset $\mathscr{S} \subset \mathbf{C}^n$, the orthogonal complement of $\mathscr{S}$ is denoted by $\mathscr{S}^{\perp}$, that is

$$\mathscr{S}^{\perp} = \{x \in \mathbf{C}^n : x^* y = 0 \text{ forall } y \in \mathscr{S}\}.$$

The symbol $\|\cdot\|_F$ stands for the Frobenius norm, and $\|\cdot\|_2$ the spectral norm and the Euclidean vector norm.

2. Robust partial eigenvalue assignment

Clearly, Problem RPEA is only a qualitative description for the robust eigenvalue assignment problem in second-order control systems. In order to do further study, we must precisely formulate this problem in quantitative form. To this end, we should first show how solutions to Problem PEA are parameterized.

2.1. Parameterized solutions

Note that in Problem PEA the eigenvalues $\lambda_{p+1}, \lambda_{p+2}, \ldots, \lambda_{2n}$ are kept unchanged, if the corresponding eigenvectors $x_{p+1}, x_{p+2}, \ldots, x_{2n}$ are also required to be kept unchanged, then it follows from the definition of eigenvalues of a quadratic pencil that (F, G) is a solution to Problem PEA if and only if there exist a $n \times p$ matrix $Y = [y_1, y_2, \ldots, y_p]$ satisfying

$$y_j \neq 0, \ j = 1, 2, \ldots, p \quad \text{and} \quad y_i = \bar{y}_k \quad \text{if } \mu_i = \bar{\mu}_k, \tag{5}$$

such that

$$MYD^2 + (C - BF^{\mathrm{T}})YD + (K - BG^{\mathrm{T}})Y = 0, \tag{6}$$

and

$$MX_2\Lambda_2^2 + (C - BF^{\mathrm{T}})X_2\Lambda_2 + (K - BG^{\mathrm{T}})X_2 = 0. \tag{7}$$

Note that Λ_2 and X_2 also satisfy that

$$MX_2\Lambda_2^2 + CX_2\Lambda_2 + KX_2 = 0,$$

Eq. (7) then becomes

$$B(F^{\mathrm{T}}X_2\Lambda_2 + G^{\mathrm{T}}X_2) = 0. \tag{8}$$

Because B is assumed to be of full column rank, Eq. (8) implies that

$$F^{\mathrm{T}}X_2\Lambda_2 + G^{\mathrm{T}}X_2 = 0,$$

that is,

$$\begin{bmatrix} G \\ F \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} X_2 \\ X_2\Lambda_2 \end{bmatrix} = 0. \tag{9}$$

From Ref. [2], we know that if the sets $\{\lambda_1, \lambda_2, \ldots, \lambda_p\}$ and $\{\lambda_{p+1}, \lambda_{p+2}, \ldots, \lambda_{2n}\}$ are disjoint, then

$$\Lambda_1 X_1^{\mathrm{T}} M X_2 \Lambda_2 - X_1^{\mathrm{T}} K X_2 = 0, \tag{10}$$

that is,

$$\begin{bmatrix} -KX_1 \\ MX_1\Lambda_1 \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} X_2 \\ X_2\Lambda_2 \end{bmatrix} = 0. \tag{11}$$

If we also assume that all eigenvalues of $Q(\lambda)$ are nondefective, that is, we assume that the matrix

$$\begin{bmatrix} X_1 & X_2 \\ X_1\Lambda_1 & X_2\Lambda_2 \end{bmatrix}$$

is nonsingular, and note that the sets $\{x_1, x_2, \ldots, x_p\}$ and $\{\lambda_1, \lambda_2, \ldots, \lambda_p\}$ are both self-conjugate, then Eq. (11) implies that

$$\begin{bmatrix} -KX_1 \\ MX_1\Lambda_1 \end{bmatrix}^{*} \begin{bmatrix} X_2 \\ X_2\Lambda_2 \end{bmatrix} = 0,$$

and hence we have

$$\mathscr{R}\left(\begin{bmatrix} X_2 \\ X_2\Lambda_2 \end{bmatrix}\right)^{\perp} = \mathscr{R}\left(\begin{bmatrix} -KX_1 \\ MX_1\Lambda_1 \end{bmatrix}\right). \tag{12}$$

This, together with Eq. (9), gives rise to that there must exists a $p \times m$ matrix U such that

$$\begin{bmatrix} G \\ F \end{bmatrix} = \begin{bmatrix} -KX_1 \\ MX_1\Lambda_1 \end{bmatrix} U,$$

which implies that

$$F = MX_1\Lambda_1 U, \quad G = -KX_1 U. \tag{13}$$

Substituting Eq. (13) into Eq. (6) gives

$$MYD^2 + CYD + KY = BU^{\mathrm{T}}(\Lambda_1 X_1^{\mathrm{T}} MYD - X_1^{\mathrm{T}} KY). \tag{14}$$

From Eq. (14), we can see that for each $y_j, j = 1, 2, \ldots, p$, there exist a v_j such that

$$Q(\mu_j)y_j = (\mu_j^2 M + \mu_j C + K)y_j = Bv_j,$$

that is

$$y_j = (Q(\mu_j))^{-1} B v_j,$$

and so y_j must satisfy that

$$y_j \in \mathscr{R}((Q(\mu_j))^{-1}B). \tag{15}$$

Together with condition (5), y_j then must satisfy

$$0 \neq y_j \in \mathscr{R}((Q(\mu_j))^{-1}B), \quad j = 1, 2, \ldots, p \quad \text{and} \quad y_i = \bar{y}_k \ \text{ if } \mu_i = \bar{\mu}_k. \tag{16}$$

Thus we have proved that if the eigenvectors $x_{p+1}, \ldots, x_{2n}$ are also required to be kept unchanged in Problem PEA, then F and G are a pair of solutions to Problem PEA if and only if F and G are in the form of Eq. (13), where y_j $(j = 1, 2, \ldots, p)$ satisfy condition (16), and U is determined by Eq. (14).

It is worthwhile to point out that for a given Y, where $Y = [y_1, y_2, \ldots, y_p]$ with y_j satisfying condition (16), Eq. (14) may have no solutions. The following is a simple example:

Example. Consider a second-order control system with

$$M = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} -3 & 0 \\ 0 & 3 \end{bmatrix}, \quad K = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Then it is easy to show that the 4 eigenvalues of its associated quadratic pencil are

$$\lambda_1 = 1, \quad \lambda_2 = 2, \quad \lambda_3 = -1, \quad \lambda_4 = -2,$$

and the corresponding eigenvectors are

$$x_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad x_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad x_3 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad x_4 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Suppose we are to change λ_1 and λ_2 to $\mu_1 = -3$ and $\mu_2 = -4$, while keep λ_3, λ_4 and x_3, x_4 unchanged. After some calculation, we have

$$(Q(\mu_1))^{-1}B = \begin{bmatrix} 1/20 & 0 \\ 0 & 1/2 \end{bmatrix}, \quad (Q(\mu_2))^{-1}B = \begin{bmatrix} 1/30 & 0 \\ 0 & 1/6 \end{bmatrix}.$$

Case 1: If y_1 and y_2 are taken as $y_1 = y_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, then Eq. (14) becomes

$$\begin{bmatrix} 20 & 30 \\ 0 & 0 \end{bmatrix} = U^{\mathrm{T}} \begin{bmatrix} -5 & -6 \\ -8 & -10 \end{bmatrix},$$

which has a unique solution.

Case 2: If y_1 and y_2 are taken as $y_1 = y_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, then Eq. (14) becomes

$$\begin{bmatrix} 0 & 0 \\ 2 & 6 \end{bmatrix} = U^{\mathrm{T}} \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},$$

which has no solution.

Next we try to give some necessary and sufficient conditions on Y such that Eq. (14) has solutions.

In fact, if $Y = [y_1, y_2, \ldots, y_p]$ with y_j satisfying condition (16), then there must exist a unique $m \times p$ matrix $V = [v_1, v_2, \ldots, v_p]$ such that

$$MYD^2 + CYD + KY = BV, \tag{17}$$

and if the QR decomposition of B is $B = [Q_1, \ Q_2]\begin{bmatrix} R \\ 0 \end{bmatrix}$, where $[Q_1, \ Q_2]$ is an orthogonal matrix with $Q_1 \in \mathbf{R}^{n\times m}$ and $Q_2 \in \mathbf{R}^{n\times(n-m)}$, and $R \in \mathbf{R}^{m\times m}$ is upper triangular and nonsingular, then we have

$$V = R^{-1}Q_1^{\mathrm{T}}(MYD^2 + CYD + KY). \tag{18}$$

On the other hand, substituting Eq. (17) into Eq. (14) gives

$$BV = BU^{\mathrm{T}}(\Lambda_1 X_1^{\mathrm{T}} MYD - X_1^{\mathrm{T}} KY). \tag{19}$$

This, together with $\mathrm{rank}(B) = m$, implies that

$$V = U^{\mathrm{T}}(\Lambda_1 X_1^{\mathrm{T}} MYD - X_1^{\mathrm{T}} KY). \tag{20}$$

It is well known that Eq. (20) has a solution if and only if

$$\mathrm{rank}\left(\begin{bmatrix} Z \\ V \end{bmatrix}\right) = \mathrm{rank}\,(Z), \tag{21}$$

where V is defined by Eq. (18), and

$$Z = \Lambda_1 X_1^{\mathrm{T}} MYD - X_1^{\mathrm{T}} KY. \tag{22}$$

In summary, we have proved the following theorem:

Theorem 2.1. *Assume that the sets* $\{\lambda_1, \ldots, \lambda_p\}$ *and* $\{\lambda_{p+1}, \ldots, \lambda_{2n}\}$ *are disjoint and all eigenvalues of* $Q(\lambda)$ *are nondefective. If the eigenvectors* $x_{p+1}, \ldots, x_{2n}$ *are also required to be kept unchanged in Problem PEA, then it has a solution if and only if there exists a matrix* $Y = [y_1, y_2, \ldots, y_p]$ *with* y_j *satisfying condition* (16) *such that equality* (21) *holds, where* Z *and* V *are defined by* Eqs. (22) *and* (18), *respectively*.

Lemma 2.1. *Suppose* $Z \in \mathbf{C}^{r\times k}$, $V \in \mathbf{C}^{s\times k}$, *then* $\operatorname{rank}\left(\begin{bmatrix} Z \\ V \end{bmatrix}\right) = \operatorname{rank}(Z)$ *if and only if* $\mathcal{N}(Z) \subseteq \mathcal{N}(V)$.

Proof. Suppose the singular value decomposition of Z is $Z = \tilde{U}\begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix}\tilde{V}^{\mathrm{T}}$, where $\tilde{U}$ and $\tilde{V}$ are unitary matrices, $\Sigma = \operatorname{diag}(\sigma_1, \ldots, \sigma_k)$, $\sigma_1 \geqslant \sigma_2 \geqslant \cdots \geqslant \sigma_k > 0$, and $k = \operatorname{rank}(Z)$, then we have

$$\begin{bmatrix} \tilde{U} & 0 \\ 0 & I \end{bmatrix}\begin{bmatrix} Z \\ V \end{bmatrix}\tilde{V}^{\mathrm{T}} = \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \\ B_1 & B_2 \end{bmatrix},$$

where $V\tilde{V}^{\mathrm{T}} = [B_1,\ B_2]$, from which it is easy to know that $\operatorname{rank}\left(\begin{bmatrix} Z \\ V \end{bmatrix}\right) = \operatorname{rank}(Z)$ if and only if $B_2 = 0$, which is equivalent to $\mathcal{N}(Z) \subseteq \mathcal{N}(V)$. Thus the lemma is proved. □

Combining Theorem 2.1 with Lemma 2.1 yields

Theorem 2.2. *Assume that the sets* $\{\lambda_1, \ldots, \lambda_p\}$ *and* $\{\lambda_{p+1}, \ldots, \lambda_{2n}\}$ *are disjoint and all eigenvalues of* $Q(\lambda)$ *are nondefective. If the eigenvectors* $x_{p+1}, \ldots, x_{2n}$ *are also required to be kept unchanged in Problem PEA, then it has a solution if and only if there exists a matrix* $Y = [y_1, y_2, \ldots, y_p]$ *with* y_j *satisfying condition* (16) *such that* $\mathcal{N}(Z) \subseteq \mathcal{N}(V)$, *where* Z *and* V *are defined by* Eqs. (22) *and* (18), *respectively*.

We may see that it is somewhat difficult to verify whether the given data satisfies $\operatorname{rank}\left(\begin{bmatrix} Z \\ V \end{bmatrix}\right) = \operatorname{rank}(Z)$ or $\mathcal{N}(Z) \subseteq \mathcal{N}(V)$, since the matrix Y in Eqs. (18) and (22) is essentially undetermined. The following theorem gives some sufficient conditions, which are easy to use.

Theorem 2.3. *Assume that the sets* $\{\lambda_1, \ldots, \lambda_p\}$ *and* $\{\lambda_{p+1}, \ldots, \lambda_{2n}\}$ *are disjoint and all eigenvalues of* $Q(\lambda)$ *are nondefective. If*

$$\operatorname{span}\left\{\begin{pmatrix} y_j \\ \mu_j y_j \end{pmatrix} : y_j \in \mathcal{R}(Q(\mu_j)^{-1}B),\ j = 1, 2, \ldots, p\right\} \cap \mathcal{R}\left(\begin{bmatrix} X_2 \\ X_2\Lambda_2 \end{bmatrix}\right) = \{0\}, \tag{23}$$

and there exist a matrix $Y = [y_1, y_2, \ldots, y_p]$ *with* y_j *satisfying condition* (16) *such that the matrix* $\begin{bmatrix} Y \\ YD \end{bmatrix}$ *is of full column rank, then* $Z = \Lambda_1 X_1^{\mathrm{T}} MYD - X_1^{\mathrm{T}} KY$ *is nonsingular, and hence Problem PEA has at least one solution.*

Proof. Let $x \in \mathbf{C}^p$ such that $Zx = 0$, that is,

$$\begin{bmatrix} -KX_1 \\ MX_1\Lambda_1 \end{bmatrix}^{\mathrm{T}}\begin{bmatrix} Y \\ YD \end{bmatrix}x = 0.$$

This, together with Eq. (11), implies that

$$\begin{bmatrix} Y \\ YD \end{bmatrix}x \in \mathcal{R}\left(\begin{bmatrix} X_2 \\ X_2\Lambda_2 \end{bmatrix}\right).$$

Obviously, we have

$$\begin{bmatrix} Y \\ YD \end{bmatrix} x \in \operatorname{span}\left\{ \begin{pmatrix} y_j \\ \mu_j y_j \end{pmatrix} : y_j \in \mathscr{R}(Q(\mu_j)^{-1}B),\ j = 1, 2, \ldots, p \right\}.$$

Thus, it follows from Eq. (23) that $\begin{bmatrix} Y \\ YD \end{bmatrix} x = 0$, and so $x = 0$, since we assume that $\begin{bmatrix} Y \\ YD \end{bmatrix}$ has full column rank. This show that Z must be nonsingular, which implies that equality (21) must hold, and hence, by Theorem 2.1, Problem PEA has at least one solution. □

In what follows we assume that Problem PEA does have solutions, assume that the sets $\{\lambda_1, \ldots, \lambda_p\}$ and $\{\lambda_{p+1}, \ldots, \lambda_{2n}\}$ are disjoint, assume that all eigenvalues of $Q(\lambda)$ are nondefective, and let

$$\mathscr{Y} = \{Y = [y_1, y_2, \ldots, y_p]: y_j \text{ satisfies conditions (16) and (21)}\}.$$

Then it is easy to see that solving Problem RPEA is equivalent to selecting $Y \in \mathscr{Y}$ such that the corresponding closed-loop system is robust, in the sense that the eigenvalues of the closed-loop quadratic pencil are as insensitive to perturbations in the coefficient matrices as possible.

2.2. Measures of robustness

To formulate Problem RPEA in quantitative form, another key problem is how to choose a proper measure of robustness. From the numerical analysis point of view the robustness problem of a closed-loop system is essentially the conditioning problem of the eigenproblem corresponding to the closed-loop quadratic pencil. Thus let us first recall some basic results on the conditioning of matrix eigenproblems. For a $2n \times 2n$ matrix $\tilde{A}$, if $\tilde{A}$ is nondefective, that is, $\tilde{A}$ has $2n$ linearly independent eigenvectors, then it can be shown [4–6] and [14]) that the sensitivity of the eigenvalue λ_j to perturbation in the components of $\tilde{A}$ depends on the magnitude of the condition number c_j, where

$$c_j = \|\tilde{z}_j\|_2 \|\tilde{y}_j\|_2 / |\tilde{z}_j^* \tilde{y}_j| \geqslant 1, \tag{24}$$

and $\tilde{y}_j$, $\tilde{z}_j$ are the right and left eigenvectors of $\tilde{A}$ corresponding to λ_j. In the case of multiple eigenvalues, a particular choice of eigenvectors is assumed. More precisely, if a perturbation $O(\varepsilon)$ is made in $\tilde{A}$, then the corresponding first-order perturbation in the eigenvalue λ_j of $\tilde{A}$ is of the order of $\varepsilon n c_j$. If $\tilde{A}$ is defective, then the corresponding perturbation in some eigenvalue is at least an order of magnitude worse in ε, and therefore, system matrices which are defective are necessarily less robust than those which are nondefective.

Note that the left eigenvectors $\tilde{z}_j$ can be chosen as

$$\tilde{Z}^* \equiv [\tilde{z}_1, \tilde{z}_2, \ldots, \tilde{z}_{2n}]^* = \tilde{Y}^{-1}, \tag{25}$$

where $\tilde{Y} = [\tilde{y}_1, \tilde{y}_2, \ldots, \tilde{y}_{2n}]$, and if the right eigenvectors $\tilde{y}_j$ are normalized such that $\|\tilde{y}_j\|_2 = 1$, then the condition numbers are given by

$$c_j = \|\tilde{z}_j\|_2 \geqslant 1, \tag{26}$$

and we have

$$\|\tilde{Y}\|_F = (2n)^{1/2}, \quad \|\tilde{Y}^{-1}\|_F = \|\tilde{Z}^*\|_F = \left(\sum_{j=1}^{2n} c_j^2\right)^{1/2}. \tag{27}$$

We can then define the measure v by

$$v = (2n)^{-1/2}\|c\|_2 = (2n)^{-1/2}\|\tilde{Y}^{-1}\|_F, \tag{28}$$

where $c = [c_1, c_2, \ldots, c_{2n}]$. Note that $v = (2n)^{-1}\|\tilde{Y}\|_F \|\tilde{Y}^{-1}\|_F \equiv (2n)^{-1}\kappa_F(\tilde{Y})$, and then minimizing the measure v is equivalent to minimizing the F-condition number of the matrix of eigenvectors $\tilde{Y}$.

As for the closed-loop quadratic pencil (4), it is well known that when M is nonsingular, the quadratic eigenvalue problem corresponding to Eq. (4) is equivalent to the following standard eigenvalue problem:

$$\tilde{A}\begin{pmatrix} x \\ \lambda x \end{pmatrix} \equiv \begin{bmatrix} 0 & I \\ M^{-1}(K - BG^{\mathrm{T}}) & M^{-1}(C - BF^{\mathrm{T}}) \end{bmatrix}\begin{pmatrix} x \\ \lambda x \end{pmatrix} = \lambda\begin{pmatrix} x \\ \lambda x \end{pmatrix}, \tag{29}$$

and in this case the matrix of eigenvectors $\tilde{Y}$ is

$$\tilde{Y} = \begin{bmatrix} Y & X_2 \\ YD & X_2\Lambda_2 \end{bmatrix}, \tag{30}$$

where the matrices X_2 and Λ_2 are to remain unaltered, and D is given. Thus it is natural to choose

$$\kappa_F\left(\begin{bmatrix} Y & X_2 \\ YD & X_2\Lambda_2 \end{bmatrix}\right) \tag{31}$$

as the measure of robustness of the closed-loop system.

2.3. Formulation

Now, based on the discussion above, it is natural to formulate Problem RPEA in quantitative form as finding $Y \in \mathscr{Y}$ such that the minimum value of the function

$$\phi(Y) = \kappa_F\left(\begin{bmatrix} Y & X_2 \\ YD & X_2\Lambda_2 \end{bmatrix}\right) = \left\|\begin{bmatrix} Y & X_2 \\ YD & X_2\Lambda_2 \end{bmatrix}\right\|_F \left\|\begin{bmatrix} Y & X_2 \\ YD & X_2\Lambda_2 \end{bmatrix}^{-1}\right\|_F \tag{32}$$

is achieved. However, it is often very difficult to minimize $\phi(Y)$ numerically, and even when such a solution can be found, usually the cost is very expensive. So, instead of finding the optimal solution, it is more advisable to find a way to compute a better solution without costing too much. To this end, we first prove the following theorem.

Theorem 2.4. *Consider a nonsingular matrix $S = [S_1,\ S_2]$, where $S_1 \in \mathbf{C}^{2n\times p}$ can be chosen, $S_2 \in \mathbf{C}^{2n\times(2n-p)}$ is fixed, and each column of S is of unit length. Suppose the QR decomposition of S_2 is $S_2 = [Q_2,\ Q_1]\begin{bmatrix} R \\ 0 \end{bmatrix}$, where $[Q_2,\ Q_1]$ is unitary, and R is upper triangular and nonsingular, then the minimum value of the F-condition number of S is achieved by setting $S_1 = Q_1U$, where U is any unitary matrix.*

Proof. Writing $S_1 = [Q_2,\ Q_1]\begin{bmatrix} S_{12} \\ S_{11} \end{bmatrix}$, we have

$$S = [S_1,\ S_2] = [Q_2,\ Q_1]\begin{bmatrix} S_{12} & R \\ S_{11} & 0 \end{bmatrix} = [Q_1,\ Q_2]\begin{bmatrix} S_{11} & 0 \\ S_{12} & R \end{bmatrix},$$

and so it follows that

$$S^{-1} = \begin{bmatrix} S_{11} & 0 \\ S_{12} & R \end{bmatrix}^{-1}\begin{bmatrix} Q_1^* \\ Q_2^* \end{bmatrix} = \begin{bmatrix} S_{11}^{-1} & 0 \\ -R^{-1}S_{12}S_{11}^{-1} & R^{-1} \end{bmatrix}\begin{bmatrix} Q_1^* \\ Q_2^* \end{bmatrix}.$$

Since each column of S is of unit length, we have

$$\|S\|_F^2 = 2n \quad \text{and} \quad \|S^{-1}\|_F^2 = \left\|\begin{bmatrix} S_{11}^{-1} \\ -R^{-1}S_{12}S_{11}^{-1} \end{bmatrix}\right\|_F^2 + \|R^{-1}\|_F^2.$$

Note that $[S_{11}^{-1}\ \ 0]\begin{bmatrix} S_{11} \\ S_{12} \end{bmatrix} = I$ implies that $\|S_{11}^{-1}\|_F^2 \geqslant p$, then we obtain

$$(\kappa_F(S))^2 = \|S\|_F^2 \|S^{-1}\|_F^2 \geqslant 2n(\|S_{11}^{-1}\|_F^2 + \|R^{-1}\|_F^2) \geqslant 2n(p + \|R^{-1}\|_F^2),$$

and $\kappa_F(S)$ achieves its minimum value when S_{11} is a unitary matrix and $S_{12} = 0$. Thus the theorem is proved. □

Theorem 2.4 tells us that when the columns of S_1 form an orthogonal basis of $\mathscr{R}(S_2)^\perp$, the F-condition number of S is minimized. So, in essence, the aim of robust eigenvalue assignment problem is, therefore, to select $Y \in \mathscr{Y}$ satisfying $\|\tilde{y}_j\|_2 \equiv \left\| \begin{pmatrix} y_j \\ \mu_j y_j \end{pmatrix} \right\|_2 = 1$, such that matrix $\tilde{Y}_1 = \begin{bmatrix} Y \\ YD \end{bmatrix}$ is as close as possible to some orthogonal basis of $\mathscr{R}\left(\begin{bmatrix} X_2 \\ X_2 \Lambda_2 \end{bmatrix}\right)^\perp$. We next consider how to measure the distance between the matrix $\tilde{Y}_1 = [\tilde{y}_1, \tilde{y}_2, \ldots, \tilde{y}_p]$ and some matrix $\tilde{X}$ whose columns are an orthogonal basis of $\mathscr{R}\left(\begin{bmatrix} X_2 \\ X_2 \Lambda_2 \end{bmatrix}\right)^\perp$. Obviously, for a given $\tilde{Y}_1$, it is also very difficult to find a $\tilde{X}$ such that the distance between $\tilde{Y}_1$ and $\tilde{X}$ is minimized. However, for any given $\tilde{X}$, the distance can be minimized by setting $\tilde{Y}_1$ such that each column $\tilde{y}_j$ of $\tilde{Y}_1$ is the normalized orthogonal projection of the column $\tilde{x}_j$ of $\tilde{X}$ into the subspaces

$$\mathscr{W}_j = \mathscr{R}\left(\begin{bmatrix} Q(\mu_j)^{-1}B \\ \mu_j Q(\mu_j)^{-1}B \end{bmatrix}\right), \quad j = 1, 2, \ldots, p. \tag{33}$$

Suppose the columns of W_j form an orthogonal basis of $\mathscr{W}_j$, then $\tilde{y}_j$ is determined by

$$\tilde{y}_j = W_j W_j^* \tilde{x}_j / \| W_j^* \tilde{x}_j \|_2, \tag{34}$$

and we can use the following number as a measure of the distance between $\tilde{Y}_1$ and $\tilde{X}$

$$v_1 = \sum_{j=1}^{p} \sin^2 \theta_j, \tag{35}$$

where θ_j is the angle between $\tilde{y}_j$ and $\tilde{x}_j$, that is, $\cos\theta_j = \| W_j^* \tilde{x}_j \|_2$. If we define

$$f(\tilde{X}) = \sum_{j=1}^{p} \| W_j^* \tilde{x}_j \|_2^2, \tag{36}$$

then we have

$$v_1 = \sum_{j=1}^{p} \sin^2 \theta_j = p - \sum_{j=1}^{p} \cos^2 \theta_j = p - f(\tilde{X}).$$

This shows that minimizing v_1 is equivalent to maximizing $f(\tilde{X})$, and we hence turn to consider the following problem:

Problem A. Find a $\tilde{X}$, whose columns form an orthogonal basis of $\mathscr{R}\left(\begin{bmatrix} X_2 \\ X_2 \Lambda_2 \end{bmatrix}\right)^\perp$, such that the maximum value of $f(\tilde{X})$ is achieved.

Once we have found a solution to Problem A, $\tilde{Y}_1 = \begin{bmatrix} Y \\ YD \end{bmatrix}$ is then determined by Eq. (34). If such Y makes equality (21) hold, then U can be obtained from Eq. (20), and we may expect that F and G given by Eq. (13) are able to make the corresponding closed-loop system be better conditioned.

3. Orthogonal basis selection method

In this section, we describe a numerical approach to compute a $\tilde{X}$ to maximize $f(\tilde{X})$, where the columns of $\tilde{X}$ form an orthogonal basis of $\mathscr{R}\left(\begin{bmatrix} X_2 \\ X_2\Lambda_2 \end{bmatrix}\right)^{\perp}$, so we call it orthogonal basis selection (OBS) method. Firstly, we need to construct an orthogonal basis for the space $\mathscr{W}_j$ for each $j = 1, 2, \ldots, p$. In fact, it is not necessary to compute an orthogonal basis for the space $\mathscr{W}_j$ directly. In stead of that we can first compute an orthogonal basis for the space $\mathscr{R}(Q(\mu_j)^{-1}B)$, comprised by the columns of matrix $\tilde{W}_j$, then the columns of the matrix

$$W_j = \frac{1}{\sqrt{1 + |\mu_j|^2}} \begin{bmatrix} \tilde{W}_j \\ \mu_j \tilde{W}_j \end{bmatrix} \tag{37}$$

form an orthogonal basis for the space $\mathscr{W}_j$. As for the computation of $\tilde{W}_j$, we here use QR decomposition of $Q(\mu_j)^{-1}B$, that is, if the QR decomposition of $Q(\mu_j)^{-1}B$ is

$$Q(\mu_j)^{-1}B = [\tilde{W}_j, \quad \hat{W}_j] \begin{bmatrix} R_j \\ 0 \end{bmatrix} \tag{38}$$

with $[\tilde{W}_j, \quad \hat{W}_j]$ unitary and R_j upper triangular and nonsingular, then $\tilde{W}_j$ is the required matrix.

To solve Problem A, an iterative method is used, where vectors $\tilde{x}_j$ are determined iteratively by applying plane rotations to an initial matrix, such that each rotation increases $f(\tilde{X})$ by an optimal quantity. Naturally, we can choose an orthogonal basis of

$$\mathscr{R}\left(\begin{bmatrix} X_2 \\ X_2\Lambda_2 \end{bmatrix}\right)^{\perp} = \mathscr{R}\left(\begin{bmatrix} -KX_1 \\ MX_1\Lambda_1 \end{bmatrix}\right)$$

as the initial matrix. For the conveniency of computing plane rotations, we hope the matrix keep real, so we are to compute a real orthogonal basis as the initial matrix. Assume that the diagonal elements of Λ_1 have been ordered as

$$\Lambda_1 = \operatorname{diag}\{\lambda_1, \lambda_2, \ldots, \lambda_{2l-1}, \lambda_{2l}, \lambda_{2l+1}, \ldots, \lambda_p\},$$

where $\lambda_{2j-1} = \alpha_j + \mathrm{i}\beta_j, \lambda_{2j} = \alpha_j - \mathrm{i}\beta_j, \alpha_j, \beta_j \in \mathbf{R}, \beta_j \neq 0$, for $j = 1, 2, \ldots, l$, and $\lambda_j \in \mathbf{R}$, for $j = 2l+1, \ldots, p$, and correspondingly

$$X_1 = [x_1, x_2, \ldots, x_{2l-1}, x_{2l}, x_{2l+1}, \ldots, x_p],$$

where $x_{2j-1} = u_j + \mathrm{i}v_j, x_{2j} = u_j - \mathrm{i}v_j, u_j, v_j \in \mathbf{R}^n$, for $j = 1, 2, \ldots, l$, and $x_j \in \mathbf{R}^n$, for $j = 2l+1, \ldots, p$. Then we have

$$X_1 = X_{1R}H, \quad X_1\Lambda_1 = X_{1R}\Lambda_{1R}H,$$

where

$$X_{1R} = [u_1, v_1, \ldots, u_l, v_l, x_{2l+1}, \ldots, x_p],$$

$$\Lambda_{1R} = \operatorname{diag}\left(\begin{bmatrix} \alpha_1 & \beta_1 \\ -\beta_1 & \alpha_1 \end{bmatrix}, \ldots, \begin{bmatrix} \alpha_l & \beta_l \\ -\beta_l & \alpha_l \end{bmatrix}, \lambda_{2l+1}, \ldots, \lambda_p\right),$$

$$H = \operatorname{diag}\left(\begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix}, \ldots, \begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix}, 1, \ldots, 1\right),$$

and hence it follows that

$$\begin{bmatrix} -KX_1 \\ MX_1\Lambda_1 \end{bmatrix} = \begin{bmatrix} -KX_{1R}H \\ MX_{1R}\Lambda_{1R}H \end{bmatrix} = \begin{bmatrix} -KX_{1R} \\ MX_{1R}\Lambda_{1R} \end{bmatrix} H.$$

Therefore, if the QR decomposition of $\begin{bmatrix} -KX_{1R} \\ MX_{1R}\Lambda_{1R} \end{bmatrix}$ is

$$\begin{bmatrix} -KX_{1R} \\ MX_{1R}\Lambda_{1R} \end{bmatrix} = [\tilde{S}_1, \ \ \tilde{S}_2]\begin{bmatrix} R \\ 0 \end{bmatrix}, \tag{39}$$

with $[\tilde{S}_1, \ \tilde{S}_2]$ unitary, R upper triangular and nonsingular, then the columns of $\tilde{S}_1$ form an orthogonal basis of $\mathscr{R}\left(\begin{bmatrix} -KX_1 \\ MX_1\Lambda_1 \end{bmatrix}\right)$, and so we can select $\tilde{S}_1$ as the initial matrix $\tilde{X}_0$, which is real as required.

Suppose we have had $\tilde{X}_k$, then we are to apply a series of rotations to $\tilde{X}_k$ to obtain $\tilde{X}_{k+1}$ iteratively. Let $\tilde{X}_k = [\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_p]$. If μ_j is real, then we can simply take $\tilde{y}_j$ as the normalized projection of $\tilde{x}_j$ into the subspace $\mathscr{W}_j$, that is,

$$\tilde{y}_j = W_j W_j^* \tilde{x}_j / \| W_j^* \tilde{x}_j \|_2, \tag{40}$$

and in this case the corresponding value to be maximized is

$$\cos^2\theta_j = \| W_j^* \tilde{x}_j \|_2^2,$$

where θ_j is the angle between $\tilde{y}_j$ and $\tilde{x}_j$. If μ_j is complex and $\mu_j = \bar{\mu}_k$, then the corresponding eigenvectors $\tilde{y}_j$ and $\tilde{y}_k$ should also be complex conjugate, and so in this case we take $\tilde{y}_j$ and $\tilde{y}_k$ as the normalized projection of $\frac{1}{\sqrt{2}}(\tilde{x}_j + \mathrm{i}\tilde{x}_k)$ and $\frac{1}{\sqrt{2}}(\tilde{x}_j - \mathrm{i}\tilde{x}_k)$ into the subspaces $\mathscr{W}_j$ and $\mathscr{W}_k$, respectively, that is,

$$\begin{aligned} \tilde{y}_j &= W_j W_j^* \left(\frac{1}{\sqrt{2}}(\tilde{x}_j + \mathrm{i}\tilde{x}_k)\right) \Big/ \left\| W_j^* \left(\frac{1}{\sqrt{2}}(\tilde{x}_j + \mathrm{i}\tilde{x}_k)\right)\right\|_2, \\ \tilde{y}_k &= W_k W_k^* \left(\frac{1}{\sqrt{2}}(\tilde{x}_j - \mathrm{i}\tilde{x}_k)\right) \Big/ \left\| W_k^* \left(\frac{1}{\sqrt{2}}(\tilde{x}_j - \mathrm{i}\tilde{x}_k)\right)\right\|_2, \end{aligned} \tag{41}$$

and then the corresponding value to be maximized is

$$\cos^2\theta_j = \cos^2\theta_k = \frac{1}{2}\| W_j^*(\tilde{x}_j + \mathrm{i}\tilde{x}_k)\|_2^2,$$

where θ_j is the angle between $\tilde{y}_j$ and $\frac{1}{\sqrt{2}}(\tilde{x}_j + \mathrm{i}\tilde{x}_k)$, and θ_k is the angle between $\tilde{y}_k$ and $\frac{1}{\sqrt{2}}(\tilde{x}_j - \mathrm{i}\tilde{x}_k)$.

In view of this case, at each step of the iteration we will select two indices $1 \leqslant j_1 < j_2 \leqslant p$ and update the two vectors $\tilde{x}_{j_1}, \tilde{x}_{j_2}$ of the current matrix $\tilde{X}_k$ by a rotation, which maintains their orthogonality and maximizes $\sum_{j=1}^{p}\cos^2\theta_j$, where $\cos\theta_j, j = 1, 2, \ldots, p$ are defined above. More precisely, for a certain j_1 and j_2, the updated vectors $\hat{x}_{j_1}$ and $\hat{x}_{j_2}$ are taken as

$$[\hat{x}_{j_1}, \ \ \hat{x}_{j_2}] = [\tilde{x}_{j_1}, \ \ \tilde{x}_{j_2}]\begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}, \tag{42}$$

where α is to be selected such that $\sum_{j=1}^{p}\cos^2\theta_j$ achieves its maximum value. In order to determine α, we should consider the following four different cases.

Case a: μ_{j_1} and μ_{j_2} are both real. In this case only $\cos\theta_{j_1}$ and $\cos\theta_{j_2}$ are changed, and

$$\cos^2\theta_{j_1} + \cos^2\theta_{j_2} = \| W_{j_1}^* \hat{x}_{j_1}\|_2^2 + \| W_{j_2}^* \hat{x}_{j_2}\|_2^2;$$

Case b: μ_{j_1} is real, while μ_{j_2} is complex with $\mu_{j_2} = \bar{\mu}_{j_2'}$. In this case $\cos\theta_{j_1}$, $\cos\theta_{j_2}$, and $\cos\theta_{j_2'}$ are changed, and

$$\cos^2\theta_{j_1} + \cos^2\theta_{j_2} + \cos^2\theta_{j_2'} = \| W_{j_1}^* \hat{x}_{j_1}\|_2^2 + \| W_{j_2}^*(\hat{x}_{j_2} + \mathrm{i}\tilde{x}_{j_2'})\|_2^2;$$

Case c: μ_{j_2} is real, while μ_{j_1} is complex with $\mu_{j_1} = \bar{\mu}_{j_1'}$. In this case $\cos\theta_{j_1}$, $\cos\theta_{j_1'}$, and $\cos\theta_{j_2}$ are changed, and

$$\cos^2\theta_{j_1} + \cos^2\theta_{j_1'} + \cos^2\theta_{j_2} = \| W_{j_1}^*(\hat{x}_{j_1} + \mathrm{i}\tilde{x}_{j_1'})\|_2^2 + \| W_{j_2}^* \hat{x}_{j_2}\|_2^2;$$

Case d: Both μ_{j_1} and μ_{j_2} are complex with $\mu_{j_1} = \bar{\mu}_{j_1'}$, $\mu_{j_2} = \bar{\mu}_{j_2'}$. In this case $\cos\theta_{j_1}$, $\cos\theta_{j_1'}$, $\cos\theta_{j_2}$, and $\cos\theta_{j_2'}$ are changed, and

$$\cos^2\theta_{j_1} + \cos^2\theta_{j_1'} + \cos^2\theta_{j_2} + \cos^2\theta_{j_2'} = \| W_{j_1}^*(\hat{x}_{j_1} + \mathrm{i}\tilde{x}_{j_1'})\|_2^2 + \| W_{j_2}^*(\hat{x}_{j_2} + \mathrm{i}\tilde{x}_{j_2'})\|_2^2.$$

For Case a it is enough to select α maximizing the function

$$\varphi(\alpha) = a_1\cos 2\alpha + a_2\sin 2\alpha + a_0, \tag{43}$$

where

$$a_1 = \tfrac{1}{2}(\| W_{j_1}^*\tilde{x}_{j_1}\|_2^2 + \| W_{j_2}^*\tilde{x}_{j_2}\|_2^2 - \| W_{j_1}^*\tilde{x}_{j_2}\|_2^2 - \| W_{j_2}^*\tilde{x}_{j_1}\|_2^2),$$

$$a_2 = \tilde{x}_{j_1}^{\mathrm{T}} W_{j_1} W_{j_1}^*\tilde{x}_{j_2} - \tilde{x}_{j_1}^{\mathrm{T}} W_{j_2} W_{j_2}^*\tilde{x}_{j_2},$$

$$a_0 = \frac{1}{2}(\| W_{j_1}^*\tilde{x}_{j_1}\|_2^2 + \| W_{j_2}^*\tilde{x}_{j_2}\|_2^2 + \| W_{j_1}^*\tilde{x}_{j_2}\|_2^2 + \| W_{j_2}^*\tilde{x}_{j_1}\|_2^2).$$

Now, let

$$\varphi'(\alpha) = 2(a_2\cos 2\alpha - a_1\sin 2\alpha) = 0. \tag{44}$$

Then it is easy to verify that when α satisfies that

$$\sin 2\alpha = \frac{a_2}{\sqrt{a_1^2 + a_2^2}} \equiv s_1 \quad \text{and} \quad \cos 2\alpha = \frac{a_1}{\sqrt{a_1^2 + a_2^2}} \equiv c_1, \tag{45}$$

the function $\varphi(\alpha)$ achieves its maximum value $\sqrt{a_1^2 + a_2^2} + a_0$. Thus it follows that

$$\sin\alpha = \sqrt{\frac{1 - c_1}{2}}, \quad \cos\alpha = \mathrm{sign}(s_1)\sqrt{\frac{1 + c_1}{2}}, \tag{46}$$

which are just the required values in Eq. (42).

For Cases b–d, the function to be maximized is

$$\varphi(\alpha) = a_1\cos 2\alpha + a_2\sin 2\alpha + a_3\cos\alpha + a_4\sin\alpha + a_0, \tag{47}$$

where

$$a_1 = \tfrac{1}{2}(\| W_{j_1}^*\tilde{x}_{j_1}\|_2^2 + \| W_{j_2}^*\tilde{x}_{j_2}\|_2^2 - \| W_{j_1}^*\tilde{x}_{j_2}\|_2^2 - \| W_{j_2}^*\tilde{x}_{j_1}\|_2^2),$$

$$a_2 = \tilde{x}_{j_1}^{\mathrm{T}} W_{j_1} W_{j_1}^*\tilde{x}_{j_2} - \tilde{x}_{j_1}^{\mathrm{T}} W_{j_2} W_{j_2}^*\tilde{x}_{j_2},$$

$$a_3 = \begin{cases} -2\mathrm{Im}(\tilde{x}_{j_2}^{\mathrm{T}} W_{j_2} W_{j_2}^*\tilde{x}_{j_2'}), & \text{Case b,} \\ -2\mathrm{Im}(\tilde{x}_{j_1}^{\mathrm{T}} W_{j_1} W_{j_1}^*\tilde{x}_{j_1'}), & \text{Case c,} \\ -2\mathrm{Im}(\tilde{x}_{j_1}^{\mathrm{T}} W_{j_1} W_{j_1}^*\tilde{x}_{j_1'} + \tilde{x}_{j_2}^{\mathrm{T}} W_{j_2} W_{j_2}^*\tilde{x}_{j_2'}), & \text{Case d,} \end{cases}$$

$$a_4 = \begin{cases} -2\mathrm{Im}(\tilde{x}_{j_2'}^{\mathrm{T}} W_{j_2} W_{j_2}^*\tilde{x}_{j_1}), & \text{Case b,} \\ -2\mathrm{Im}(\tilde{x}_{j_2}^{\mathrm{T}} W_{j_1} W_{j_1}^*\tilde{x}_{j_1'}), & \text{Case c,} \\ -2\mathrm{Im}(\tilde{x}_{j_2'}^{\mathrm{T}} W_{j_1} W_{j_1}^*\tilde{x}_{j_1} + \tilde{x}_{j_2}^{\mathrm{T}} W_{j_1} W_{j_1}^*\tilde{x}_{j_1'}), & \text{Case d,} \end{cases}$$

$$a_0 = \begin{cases} \tfrac{1}{2}(\| W_{j_1}^*\tilde{x}_{j_1}\|_2^2 + \| W_{j_2}^*\tilde{x}_{j_2}\|_2^2 + \| W_{j_1}^*\tilde{x}_{j_2}\|_2^2 + \| W_{j_2}^*\tilde{x}_{j_1}\|_2^2) + \| W_{j_2}^*\tilde{x}_{j_2'}\|_2^2, & \text{Case b,} \\ \tfrac{1}{2}(\| W_{j_1}^*\tilde{x}_{j_1}\|_2^2 + \| W_{j_2}^*\tilde{x}_{j_2}\|_2^2 + \| W_{j_1}^*\tilde{x}_{j_2}\|_2^2 + \| W_{j_2}^*\tilde{x}_{j_1}\|_2^2) + \| W_{j_1}^*\tilde{x}_{j_1'}\|_2^2, & \text{Case c,} \\ \tfrac{1}{2}(\| W_{j_1}^*\tilde{x}_{j_1}\|_2^2 + \| W_{j_2}^*\tilde{x}_{j_2}\|_2^2 + \| W_{j_1}^*\tilde{x}_{j_2}\|_2^2 + \| W_{j_2}^*\tilde{x}_{j_1}\|_2^2) + \| W_{j_1}^*\tilde{x}_{j_1'}\|_2^2 + \| W_{j_2}^*\tilde{x}_{j_2'}\|_2^2, & \text{Case d,} \end{cases}$$

and $\mathrm{Im}(\cdot)$ denotes the imaginary part of a complex number. Let

$$t = \tan\left(\frac{\alpha}{2}\right) \quad \text{then } \sin\alpha = \frac{2t}{1 + t^2} \quad \text{and} \quad \cos\alpha = \frac{1 - t^2}{1 + t^2},$$

and hence we have

$$\varphi(\alpha) = \frac{(a_1 - a_3)t^4 + (2a_4 - 4a_2)t^3 - 6a_1 t^2 + (4a_2 + 2a_4)t + a_1 + a_3}{(1 + t^2)^2} + a_0 \equiv \psi(t). \tag{48}$$

Then $\psi'(t) = 0$ gives rise to

$$(4a_2 - 2a_4)t^4 + (16a_1 - 4a_3)t^3 - 24a_2 t^2 - (16a_1 + 4a_3)t + 4a_2 + 2a_4 = 0. \tag{49}$$

Suppose all real solutions to Eq. (49) are $t_1, \ldots, t_s (s \leqslant 4)$, then we can easily find t_0 which satisfies that $\psi(t_0) = \max_{1 \leqslant i \leqslant s} \psi(t_i)$, and hence

$$\sin\alpha = \frac{2t_0}{1 + t_0^2}, \quad \cos\alpha = \frac{1 - t_0^2}{1 + t_0^2} \tag{50}$$

are the required values in Eq. (42).

The rotations are applied in a natural order in sweeps through the matrix $\tilde{X}_k$, and after a full sweep comprising $\frac{1}{2}p(p-1)$ rotations we obtain $\tilde{X}_{k+1}$. The sweeps are repeated until

$$|f(\tilde{X}_{k+1}) - f(\tilde{X}_k)| \leqslant \mathrm{tol}|f(\tilde{X}_k)|, \tag{51}$$

where tol is a given positive number. The projections $\tilde{y}_j$ of the resulting vectors into subspaces $\mathscr{W}_j$ are then determined by a similar way as Eq. (40) or Eq. (41). Let $\tilde{Y}_1 = [\tilde{y}_1, \tilde{y}_2, \ldots, \tilde{y}_p]$, then Y is just the first n rows of $\tilde{Y}_1$. In fact, to get Y, we do not need to form $\tilde{Y}_1$ explicitly. With Eq. (37), for real case, we have

$$y_j = \frac{1}{\sqrt{1 + |\mu_j|^2}} \tilde{W}_j W_j^* \tilde{x}_j / \| W_j^* \tilde{x}_j \|_2, \tag{52}$$

and for complex case, we have

$$y_j = \frac{1}{\sqrt{1 + |\mu_j|^2}} \tilde{W}_j W_j^* \left(\frac{1}{\sqrt{2}} (\tilde{x}_j + \mathrm{i}\tilde{x}_k) \right) \Big/ \left\| W_j^* \left(\frac{1}{\sqrt{2}} (\tilde{x}_j + \mathrm{i}\tilde{x}_k) \right) \right\|_2,$$

$$y_k = \frac{1}{\sqrt{1 + |\mu_j|^2}} \tilde{W}_j W_k^* \left(\frac{1}{\sqrt{2}} (\tilde{x}_j - \mathrm{i}\tilde{x}_k) \right) \Big/ \left\| W_k^* \left(\frac{1}{\sqrt{2}} (\tilde{x}_j - \mathrm{i}\tilde{x}_k) \right) \right\|_2. \tag{53}$$

Then we can compute Z by using Eq. (22). As for $V = [v_1, v_2, \ldots, v_p]$, straightforward calculation leads to

$$v_j = \frac{1}{\sqrt{1 + |\mu_j|^2}} R_j^{-1} W_j^* \tilde{x}_j / \| W_j^* \tilde{x}_j \|_2, \tag{54}$$

where R_j is defined in Eq. (38). Then if equality (21) holds, U is computed by solving Eq. (20), and then F and G are obtained by substituting U into Eq. (13).

In summarizing, we have the following algorithm.

Algorithm OBS

Input: $M, C, K \in \mathbf{R}^{n\times n}$, $B \in \mathbf{R}^{n\times m}$, $\Lambda_1 = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_p) \in \mathbf{C}^{p\times p}$,
$X_1 = [\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_p] \in \mathbf{C}^{n\times p}$, $D = \mathrm{diag}(\mu_1, \mu_2, \ldots, \mu_p) \in \mathbf{C}^{p\times p}$.

1. **For** $j = 1, 2, \ldots, p$
   compute the QR decomposition of $Q(\mu_j)^{-1}B$ as in Eq. (38) and compute W_j by Eq. (37),
   end
2. Form Λ_{1R} and X_{1R} and compute the QR decomposition of $\begin{bmatrix} -KX_{1R} \\ MX_{1R}\Lambda_{1R} \end{bmatrix}$ as in Eq. (39),
   and take $\tilde{S}_1$ as the initial matrix $\tilde{X}_0$, and set $k = 0$.
3. **For** $j_1 = 1, 2, \ldots, p, j_2 = j_1 + 1, \ldots, p$
   compute $\sin\alpha$ and $\cos\alpha$ by Eq. (46) or Eq. (50) and update $\tilde{x}_{j_1}$ and $\tilde{x}_{j_2}$ by Eq. (42),
   end

4. Set the resulted matrix as $\tilde{X}_{k+1}$.
5. If $|f(\tilde{X}_{k+1}) - f(\tilde{X}_k)| \leqslant \text{tol} \cdot |f(\tilde{X}_k)|$, then go to Step 6; otherwise, set $k = k + 1$ and return to Step 3.
6. Compute $Y = [y_1, y_2, \ldots, y_p]$ by Eqs. (52) and (53), and then compute Z by Eq. (22).
7. Compute $V = [v_1, v_2, \ldots, v_p]$ by Eq. (54) and solve Eq. (20) for U.
8. Compute F and G by Eq. (13).

Output: F and G.

In Algorithm OBS only the knowledge of the eigenvalues to be changed and their corresponding eigenvectors is required, and in each iteration of Step 3, $f(\tilde{X})$ is nondecreasing. Although at this point we do not have a complete convergence theory to support this algorithm, our numerical experiments show that it is convergent, and does lead to better conditioned closed-loop systems.

4. Numerical examples

To illustrate the performance of the OBS method, in this section we give some numerical examples, which were carried out using MATLAB 6.0 with machine epsilon $\varepsilon \approx 2.22 \times 10^{-16}$. And we set $\text{tol} = 10^{-6}$ in Algorithm OBS.

We take the following four problems as our test examples:

Prob. 1. In this problem, $n = 3,\ m = 2,\ p = 2$. It is given in Refs. [7,8], and is defined by

$$M = 10I_3, \quad C = 0, \quad K = \begin{bmatrix} 40 & -40 & 0 \\ -40 & 80 & -40 \\ 0 & -40 & 80 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 2 \\ 3 & 2 \\ 3 & 4 \end{bmatrix}.$$

The system is undamped, and the eigenvalues of the quadratic pencil $Q(\lambda) = \lambda^2 M + \lambda C + K$ are

$$\{\pm 3.6039\text{i},\ \pm 2.49399\text{i},\ \pm 0.8901\text{i}\}.$$

We want to alter the first 2 eigenvalues to $-1, -2$, while keep others unchanged.

Prob. 2. In this problem, $n = 4, m = 2, p = 4$. It is from Ref. [9], and

$$M = I_4, \quad C = \begin{bmatrix} 0.5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.5 \end{bmatrix}, \quad K = \begin{bmatrix} 5 & -5 & 0 & 0 \\ -5 & 10 & -5 & 0 \\ 0 & -5 & 10 & -5 \\ 0 & 0 & -5 & 6 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

The 8 eigenvalues of the quadratic pencil $Q(\lambda)$ are

$$\{3.525, -3.559, -0.059 \pm 3.732\text{i}, -0.191 \pm 1.489\text{i}, -0.233 \pm 2.692\text{i}\}.$$

We are to alter the first 4 eigenvalues to $-1, -2, -3, -4$, while keep others unchanged.

Prob. 3. In this problem, $n = 4, m = 2, p = 2$. M, C, K and B are randomly chosen as

$$M = \begin{bmatrix} 1.4685 & 0.7177 & 0.4757 & 0.4311 \\ 0.7177 & 2.6938 & 1.2660 & 0.9676 \\ 0.4757 & 1.2660 & 2.7061 & 1.3918 \\ 0.4311 & 0.9676 & 1.3918 & 2.1876] \end{bmatrix}, \quad C = \begin{bmatrix} 1.3525 & 1.2695 & 0.7967 & 0.8160 \\ 1.2695 & 1.3274 & 0.9144 & 0.7325 \\ 0.7967 & 0.9144 & 0.9456 & 0.8310 \\ 0.8160 & 0.7325 & 0.8310 & 1.1536 \end{bmatrix},$$

$$K = \begin{bmatrix} 1.7824 & 0.0076 & -0.1359 & -0.7290 \\ 0.0076 & 1.0287 & -0.0101 & -0.0493 \\ -0.1359 & -0.0101 & 2.8360 & -0.2564 \\ -0.7290 & -0.0493 & -0.2564 & 1.9130 \end{bmatrix}, \quad B = \begin{bmatrix} 0.3450 & 0.4578 \\ 0.0579 & 0.7630 \\ 0.5967 & 0.9990 \\ 0.2853 & 0.3063 \end{bmatrix},$$

It is from Ref. [10], and the 8 eigenvalues of the quadratic pencil $Q(\lambda)$ are

$$\{-0.086 \pm 1.624\mathrm{i}, -0.102 \pm 0.888\mathrm{i}, -0.175 \pm 1.192\mathrm{i}, -0.448 \pm 0.247\mathrm{i}\}.$$

We are to alter the first 2 eigenvalues to $-1 \pm 1.624\mathrm{i}$, while keep others unchanged.

Prob. 4. In this problem, $n = 10$, $m = 3$, $p = 4$. It is from Ref. [11], and

$$M = I_{10}, \quad C = 0, \quad K = \begin{bmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ & & & -1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} I_3 \\ 0 \end{bmatrix}.$$

The 20 eigenvalues of the quadratic pencil $Q(\lambda)$ are

$$\{\pm 1.978\mathrm{i}, \pm 1.911\mathrm{i}, \pm 1.802\mathrm{i}, \pm 1.652\mathrm{i}, \pm 1.466\mathrm{i}, \pm 1.247\mathrm{i}, \pm \mathrm{i}, \pm 0.731\mathrm{i}, \pm 0.445\mathrm{i}, \pm 0.150\mathrm{i}\}.$$

We are to alter the first 4 eigenvalues to $-0.1, -0.2, -0.3, -0.4$, while keep others unchanged.

Based on the orthogonality relation in Ref. [2], the algorithm proposed by B.N. Datta, S. Elhay and Y.M. Ram' [2] for single-input systems can be easily extended to multi-input systems, that is, with randomly chosen Y satisfying Eq. (16), U is determined by Eq. (14), and F, G are then obtained by Eq. (13). For simplicity, we will write it as DER's algorithm. We then apply DER's algorithm, Algorithm OBS in this paper and the algorithm in Ref. [3] to these four problems, respectively. The algorithm in Ref. [3] is also an iterative method, and the convergence criterion is that the improvement of some measure of robustness is less than a specified tolerance. But the algorithm in Ref. [3] is not guaranteed to converge, and the criterion may not be satisfied, so to ensure the end of the iteration, a maximum number of allowed sweeps $k_{\max}$ is set (see Ref. [3] for details). Furthermore, since in DER's algorithm and the algorithm in Ref. [3], the initial matrices are randomly chosen, we run both these two algorithms 100 times for each problem.

To show the accuracy of the algorithms, we should have computed the differences between the computed solutions and the real solutions. But unfortunately, the real solutions are unknown for these four problems, so we compute the maximum differences between the computed eigenvalues of the closed-loop systems and the poles to be assigned instead, and list them in Table 1. For DER's algorithm and the algorithm in Ref. [3], the maximum differences are average maximum differences over 100 trials.

From Table 1, we can see that all the three algorithms are accurate enough in some sense, since the differences are all very small. To illustrate the performance of our algorithms, we also compute the F-condition numbers of the matrices of eigenvectors of the closed-loop systems generated by the three algorithms, and list them in Table 2. Similarly, for DER's algorithm and the algorithm in Ref. [3], the numbers are average F-condition numbers over 100 trials.

Table 1
Maximum differences

	DER's algorithm	Algorithm OBS	Algorithm in Ref. [3]
Prob. 1	2.158×10^{-13}	1.838×10^{-14}	1.776×10^{-14}
Prob. 2	5.019×10^{-14}	1.437×10^{-13}	1.688×10^{-14}
Prob. 3	2.037×10^{-14}	2.857×10^{-14}	2.842×10^{-15}
Prob. 4	2.902×10^{-11}	1.830×10^{-13}	2.737×10^{-13}

Table 2
F-condition numbers of the matrices of eigenvectors

	DER's algorithm	Algorithm OBS	Algorithm in Ref. [3]
Prob. 1	1717.91	16.896	16.896
Prob. 2	6952.19	1074.53	541.52
Prob. 3	101.62	22.75	29.01
Prob. 4	1.744×10^5	5.016×10^3	1.036×10^3

Table 3
Maximum changes of eigenvalues due to perturbations in coefficient matrices

	DER's algorithm	Algorithm OBS	Algorithm in Ref. [3]
Prob. 1	0.2779	0.0021	0.0025
Prob. 2	0.5871	0.2475	0.1668
Prob. 3	0.2881	0.0349	0.0361
Prob. 4	0.6327	0.3076	0.2853

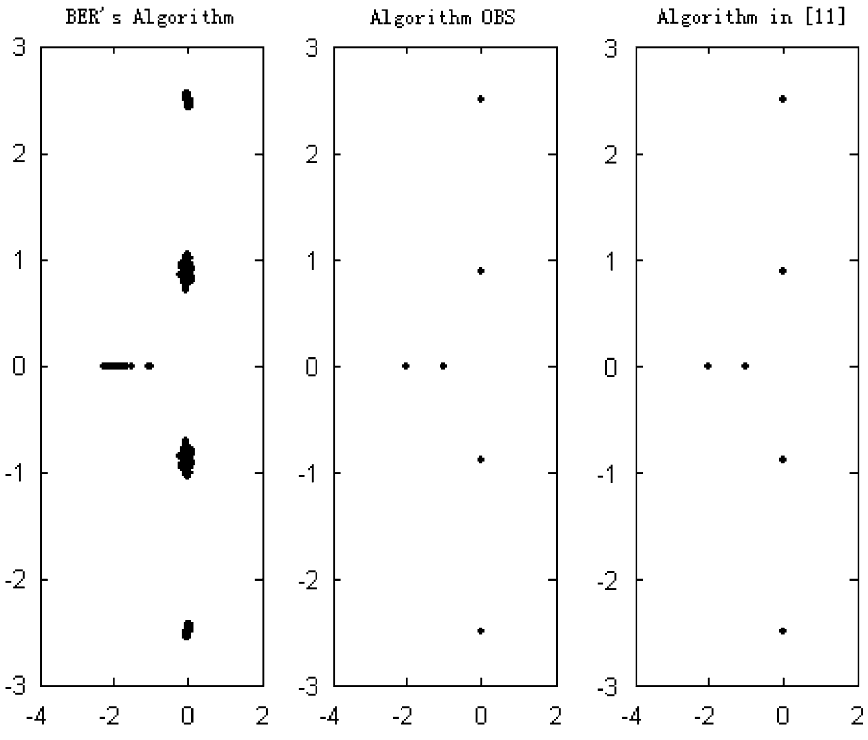

Fig. 1. The closed-loop eigenvalues under 100 random 1% perturbations on M_c, C_c and K_c for Prob. 1.

Since DER's algorithm is proposed for Problem PEA, which means that it does not consider robust solutions to Problem PEA, we may expect that our algorithms should improve robustness of the closed-loop systems. From Table 2, we can see that Algorithm OBS and the algorithm in Ref. [3] do lead to better conditioned closed-loop systems. These are also illustrated in Table 3 and Figs. 1–4. We perturb the coefficient matrices $M_c = M, C_c = C - BF^{\mathrm{T}}$ and $K_c = K - BG^{\mathrm{T}}$ by $\Delta M_c, \Delta C_c$ and ΔK_c with

$$\|\Delta M_c\|_2 < 0.01\|M_c\|_2, \quad \|\Delta C_c\|_2 < 0.01\|C_c\|_2, \quad \|\Delta K_c\|_2 < 0.01\|K_c\|_2,$$

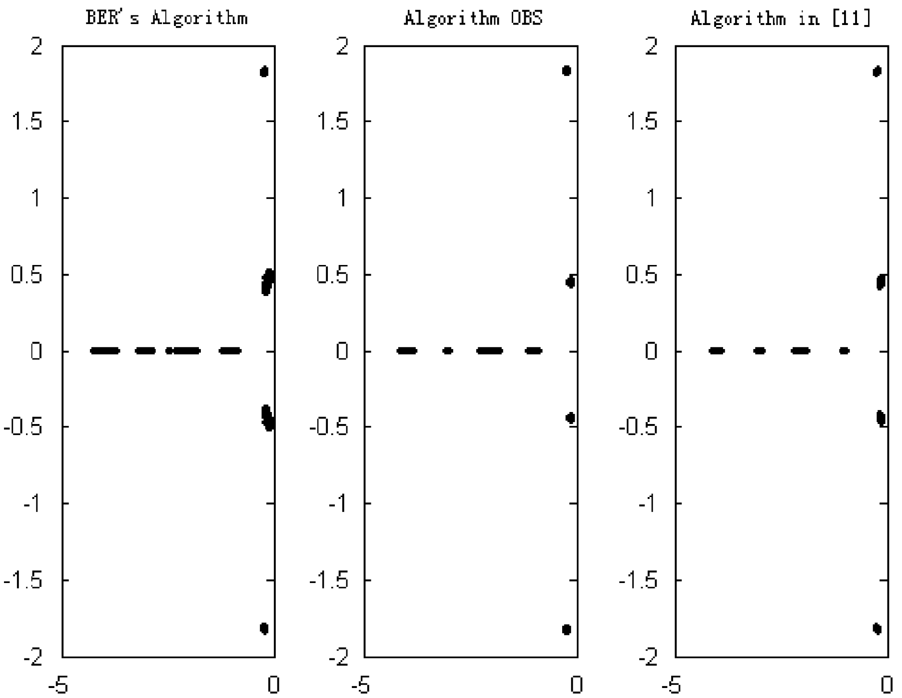

Fig. 2. The closed-loop eigenvalues under 100 random 1% perturbations on M_c, C_c and K_c for Prob. 2.

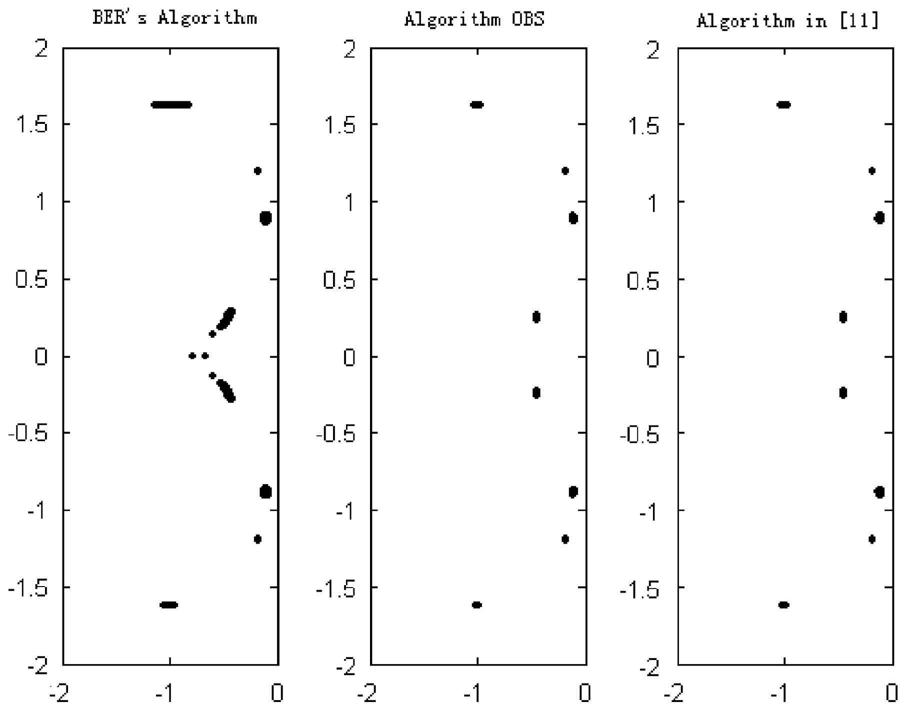

Fig. 3. The closed-loop eigenvalues under 100 random 1% perturbations on M_c, C_c and K_c for Prob. 3.

where F, G are computed by the three algorithms, respectively. For each problem we perturb it 100 times. The maximum differences between the eigenvalues of the closed-loop system and the eigenvalues of the perturbed closed-loop systems are listed in Table 3, and the eigenvalues of the perturbed closed-loop systems are plotted in Figs. 1–4.

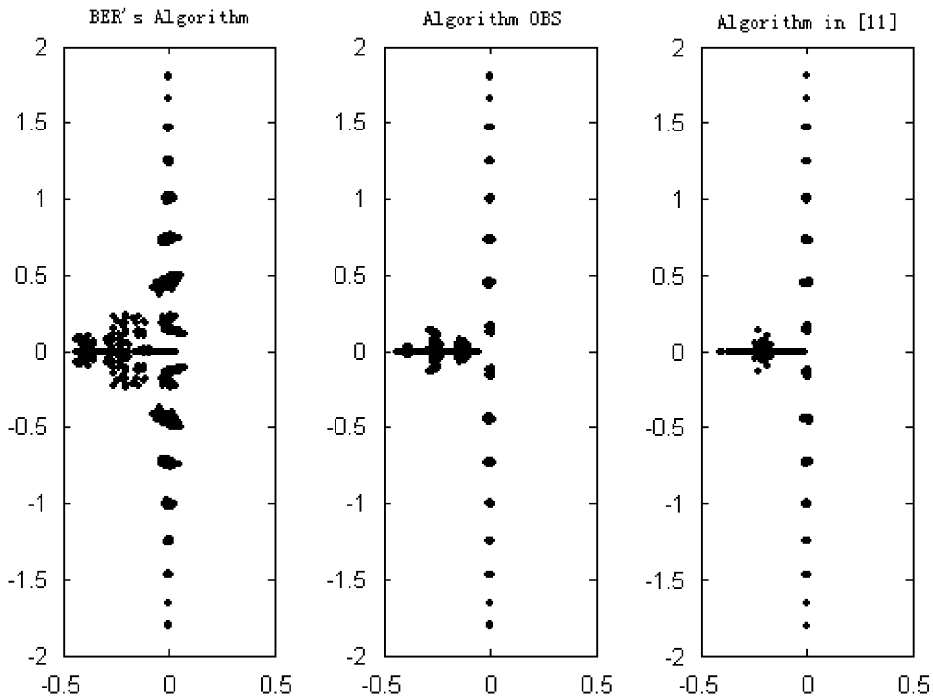

Fig. 4. The closed-loop eigenvalues under 100 random 1% perturbations on M_c, C_c and K_c for Prob. 4.

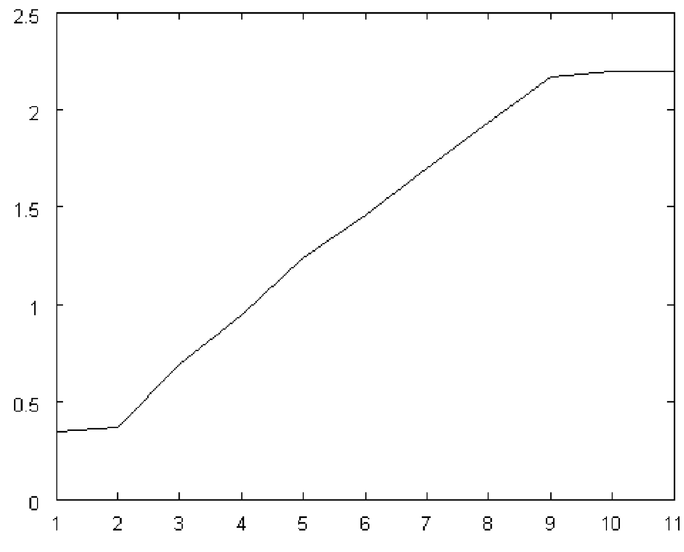

Fig. 5. The changes of $f(\tilde{X})$ with respect to the sweeps for Prob. 4.

For the four problems, Algorithm OBS converges after 2, 6, 2 and 10 sweeps, respectively. While for the algorithm in Ref. [3], sometimes the convergence criterion cannot be satisfied, and it stops just because the maximum number of allowed sweeps $k_{\max}$ is reached. Especially for Prob. 2 and Prob. 3, among the 100 trials to each of the two problems, the algorithm in Ref. [3] stops because $k_{\max}$ is reached for 94 and 96 times, respectively. Fig. 5 shows the changes of $f(\tilde{X})$ with respect to the sweeps, when Algorithm OBS is applied to Prob. 4.

5. Conclusions

In this paper, we have developed a new numerical method for the robust partial eigenvalue assignment problem in second-order control systems, which means that the feedback control matrices not only assign specific eigenvalues to the second-order closed-loop system, but also that the system is robust. It is to minimize one measure of conditioning of the closed-loop system, where eigenvectors are chosen in certain subspaces such that some measure of the distance between the eigenvectors and some orthogonal basis of a certain subspace is minimized. In this method, only the knowledge of the eigenvalues to be changed and their corresponding eigenvectors is required. Although we do not have a complete convergence theory to support this method, our numerical examples show that the present method is convergent, and often leads to better conditioned closed-loop system.

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