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# Iterative method for dynamic condensation combined with substructuring scheme

Dongsoo Choi[a,1], Hyungi Kim[a,b,2], Maenghyo Cho[a,*]

[a]School of Mechanical and Aerospace Engineering, Seoul National University, San 56-1, Shillim-dong, Kwanak-gu, Seoul 151-742, Republic of Korea

[b]Korea Aerospace Research Institute, 115 Gwahangno Yuseong-gu, Daejeon 305-333, Republic of Korea

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**Abstract**

An iterated improved reduced system (IIRS) procedure combined with substructuring scheme for both undamped and nonclassically damped structures is presented. Iterated IIRS method is an efficient reduction technique because the highly accurate eigenproperties from the repeatedly updated condensed matrices can be obtained without consuming expensive computational cost. However, single domain direct approach of this method to large structures requires much computational resources and even makes analysis intractable in the case only limited computer storage is available. These problems can be overcome by combining the substructuring scheme with IIRS procedure. The newly developed IIRS method combined with a substructuring scheme can provide an efficient methodology for large-scale eigenvalue problems. The validation of the present method and the evaluation of computational efficiency are demonstrated through the numerical examples.


## 1. Introduction

Modern structural dynamics using finite element method requires computational models having a very large number of degrees of freedom if the structural engineers are to accurately evaluate response of structures under the detailed models. Eigenvalue problems of such structures need a large amount of computing time. Although modern supercomputers can solve more than several million degrees of freedom problems, the analysis cost is very high and they are not easily accessible by most design and analysis engineers who work for daily design and analysis jobs. Therefore, many researchers have been interested in solving large-scale eigenvalue problem with limited computer storage and speed. One of the ways to resolve these problems is to reduce the size of the problem. This way is to truncate the higher modes from the given full system or eliminate the unimportant degrees of freedom. The researches on constructing reduced models have been proceeded in the two different ways. One is a reduced-order method, which constructs a reduced system with a few modes

*Corresponding author. Tel.: +82 2 880 1693; fax: +82 2 886 1693.
*E-mail addresses:* rycaon70@snu.ac.kr (D. Choi), shotgun1@kari.re.kr (H. Kim), mhcho@snu.ac.kr (M. Cho).
[1]Tel.: +82 2 880 1695.
[2]Tel.: +82 42 870 3531.

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dominating the response of a structure. The other is a condensation method in which the reduced matrices are constructed with the master degrees of freedom by transformation matrix. The former has an advantage of simplicity in constructing reduced system and do not require much computational resources. But the truncation of higher modes leads to increase the errors of eigenvalues and eigenvectors. On the other hand, the latter can calculate more accurate eigenproperties than the former but this method requires much computation cost because of the construction of transformation matrix. Therefore, the condensation method can be computationally efficient reduction techniques if the transformation matrix is constructed without consuming much computational cost.

For the last several decades various approximate techniques have been developed to calculate eigenproperties by the dynamic condensation method. The condensation technique was first proposed by Guyan [1] and Irons [2] in 1965. These methods involve elimination of the degrees of freedom, which do not give any significant influence on the solution field. But the accuracy of their methods was very low because the inertia effects were not considered when constructing the condensation matrices. O'Callahan [3] improved Guyan's method by considering the first-order approximation terms in the transformation formula of the slave degrees of freedom. Although O'Callahan's method provides a better result than that of Guyan, it may have a nonpositive definite mass matrix by the wrong selection of the master degrees of freedom. Godis [4] generated the transformation for the standard IRS (Improved Reduced System) method by using a binomial series expansion in approximating the eigenvalue term. An iterative dynamic condensation method was proposed by Suarez and Singh [5]. In this method the eigensolution was obtained using the orthogonality conditions of the eigenvectors. Friswell et al. [6] proposed an iterated IRS (IIRS) technique, and the convergence of this method was proved later [7]. Recently, Qu [8] proposed an iterative method for condensation of viscously damped system. In this method, two governing equations for the dynamic condensation matrix, which relates the eigenvectors associated with the master and slave degrees of freedom in state space, were derived. Rivera [9] developed a dynamic condensation approach applicable to nonclassically damped structures as an extension of undamped systems of Suarez and Singh [5]. Qu [10] proposed an efficient method for dynamic condensation of nonclassically damped vibration systems. In this paper, a standard subspace iteration method for undamped models was extended to the nonclassically damped systems. Qu et al. [11,12] proposed various condensation methods for nonclassically damped systems defined in displacement space and state space. Recently, Xia and Lin [13] proposed an improved dynamic condensation technique by modifying the iterative transformation matrix and accelerated the convergence. Through this technique, the more accurate and efficient lowest eigensolution of structures was obtained in comparison with the IIRS method. Kim and Cho [15] proposed the two-level condensation scheme for undamped structural system and calculated the sensitivity from the reduced system. In this scheme the reduced matrices is constructed by the well-selected primary degrees of freedom through the element level energy estimation [14].

However, although these condensation techniques can reduce the size of the model drastically, it takes a large amount of computing time for the construction of the reduced system when the problem has a large number of degrees of freedom over several hundred thousands. One of the ways to overcome this problem is to apply a substructuring scheme. In static and dynamic problems, if the whole structure can be divided into substructures, then the problem can be solved more readily with limited memory. Craig and Bampton [16] employed component mode synthesis for dynamic analysis. In the 1990s, Aminpour et al. [17] performed the coupled analysis with the independent subdomains by hybrid interface formulation. Bouhaddi and Fillod [18,19] proposed the dynamic substructuring method using Guyan condensation method based on the important degrees of freedom in the matching system. Most recently, various efficient model reduction approaches for large eigenproblems over one million degrees of freedom are proposed, e.g. dual Craig–Bampton method [20] and automated multilevel substructuring method [21,22]. However these methods are mode-based reduction methods so that their accuracies are not better than those of the degree-of-freedom-based reduction methods. Kim and Cho [23] developed three-type subdomain schemes by combining two-level condensation scheme with substructuring scheme. Their method is degree-of-freedom-based reduction method (IRS) combined with substructuring scheme.

The objective of this study is to develop an iterated IRS method combined with substructuring scheme. The iterated IRS method has several merits. Firstly, the iterated IRS method can be applied to nonclassically damped models as well as to undamped models. Secondly, this method can reduce the eigenvalue analysis errors significantly through successive iterations. Thirdly, the analysis results are not sensitive to the selected master degrees of freedom in the reduced system. The outline of this paper is given as follows. Firstly, iterated IRS method for a single global system is reviewed briefly. And then the new algorithm of iterated IRS method combined with substructuring scheme for undamped and nonclassically damped systems is derived, respectively. After discussion on the convergence of the present method, finally, two numerical examples are provided to demonstrate the accuracy and efficiency of the present method.

## 2. Iterated IRS method

Iterated IRS method used in this study is based on Friswell's method [6,7] for undamped single domain system and Qu and Rivera's method [8,10] for nonclassically single structural system. In this section, iterated IRS schemes for both systems are introduced since they will be combined with the substructuring scheme later.

### 2.1. Undamped system

The dynamic equilibrium of an $n$ degrees of freedom system can be written in a matrix form as

$$\mathbf{M}\ddot{X}(t) + \mathbf{C}\dot{X}(t) + \mathbf{K}X(t) = \mathbf{f}(t) \tag{1}$$

where the mass matrix $\mathbf{M}$, damping matrix $\mathbf{C}$, and stiffness matrix $\mathbf{K}$ are assumed to be positive definite, positive semidefinite, and positive semidefinite, respectively. The corresponding eigenvalue problem of this system can be expressed in displacement space as

$$\mathbf{K\Phi} = \mathbf{M\Phi}\Lambda \tag{2}$$

where $\mathbf{\Phi}$ is the eigenvector, representing the vibrating mode corresponding to the eigenvalue $\Lambda$. To apply the dynamic condensation scheme, Eq. (2) can be rewritten in a partitioned form as

$$\begin{bmatrix} \mathbf{K}_{mm} & \mathbf{K}_{ms} \\ \mathbf{K}_{sm} & \mathbf{K}_{ss} \end{bmatrix} \begin{Bmatrix} \mathbf{\Phi}_{mm} \\ \mathbf{\Phi}_{sm} \end{Bmatrix} = \begin{bmatrix} \mathbf{M}_{mm} & \mathbf{M}_{ms} \\ \mathbf{M}_{sm} & \mathbf{M}_{ss} \end{bmatrix} \begin{Bmatrix} \mathbf{\Phi}_{mm} \\ \mathbf{\Phi}_{sm} \end{Bmatrix} \Lambda_{mm} \tag{3}$$

In above equation, the subscript $m$ indicates the master degrees of freedom which are kept in reduced system and $s$ represents the slave degrees of freedom which should be eliminated. To eliminate the slave degrees of freedom field, employ the second row of Eq. (3) and rearrange the results yields

$$\mathbf{\Phi}_{sm} = -\mathbf{K}_{ss}^{-1}\mathbf{K}_{sm}\mathbf{\Phi}_{mm} + \mathbf{K}_{ss}^{-1}(\mathbf{M}_{sm}\mathbf{\Phi}_{mm} + \mathbf{M}_{ss}\mathbf{\Phi}_{sm})\Lambda_{mm} \tag{4}$$

According to the definition of the transformation matrix, that is

$$\mathbf{\Phi}_{sm} = \mathbf{t}\mathbf{\Phi}_{mm} \tag{5}$$

Substituting Eq. (5) into Eq. (4) and rearranging it for the transformation matrix as

$$\mathbf{t} = -\mathbf{K}_{ss}^{-1}\mathbf{K}_{sm} + \mathbf{K}_{ss}^{-1}(\mathbf{M}_{sm} + \mathbf{M}_{ss}\mathbf{t})\mathbf{\Phi}_{mm}\Lambda_{mm}\mathbf{\Phi}_{mm}^{-1} \tag{6}$$

By this transformation matrix, the whole field can be expressed with only master degrees of freedom field as

$$\begin{bmatrix} \mathbf{\Phi}_{mm} \\ \mathbf{\Phi}_{sm} \end{bmatrix} = \begin{bmatrix} \mathbf{I}_{mm} \\ \mathbf{t} \end{bmatrix} \mathbf{\Phi}_{mm} = \mathbf{T\Phi}_{mm} \tag{7}$$

where $\mathbf{I}$ is the unit matrix of size $m \times m$. Substituting Eq. (7) into Eq. (3) and premultiplying $\mathbf{T}^{\mathrm{T}}$ on the left of this equation, we can obtain the reduced system matrices as

$$\mathbf{K}_R = \mathbf{T}^{\mathrm{T}}\mathbf{K}\mathbf{T} = \mathbf{T}^{\mathrm{T}} \begin{bmatrix} \mathbf{K}_{mm} & \mathbf{K}_{ms} \\ \mathbf{K}_{sm} & \mathbf{K}_{ss} \end{bmatrix} \mathbf{T}$$

$$\mathbf{M}_R = \mathbf{T}^{\mathrm{T}}\mathbf{M}\mathbf{T} = \mathbf{T}^{\mathrm{T}} \begin{bmatrix} \mathbf{M}_{mm} & \mathbf{M}_{ms} \\ \mathbf{M}_{sm} & \mathbf{M}_{ss} \end{bmatrix} \mathbf{T} \tag{8}$$

Through the above-reduced matrices, we can construct a reduced eigenvalue problem of size $m \times m$ as

$$\mathbf{K}_R \mathbf{\Phi}_{mm} = \mathbf{M}_R \mathbf{\Phi}_{mm} \Lambda_{mm} \tag{9}$$

From Eq. (9), we get an approximation eigenvalue as

$$\mathbf{\Phi}_{mm} \Lambda_{mm} \mathbf{\Phi}_{mm}^{-1} = \mathbf{M}_R^{-1} \mathbf{K}_R \tag{10}$$

Substituting Eq. (10) into Eq. (6), we get a transformation matrix as

$$\mathbf{t} = -\mathbf{K}_{ss}^{-1} \mathbf{K}_{sm} + \mathbf{K}_{ss}^{-1} (\mathbf{M}_{sm} + \mathbf{M}_{ss} \mathbf{t}) \mathbf{M}_R^{-1} \mathbf{K}_R \tag{11}$$

Since this equation is nonlinear, the iterative form of it is given by

$$\mathbf{t}^{(k)} = -\mathbf{K}_{ss}^{-1} \mathbf{K}_{sm} + \mathbf{K}_{ss}^{-1} \left( \mathbf{M}_{sm} + \mathbf{M}_{ss} \mathbf{t}^{(k-1)} \right) \left( \mathbf{M}_R^{(k-1)} \right)^{-1} \mathbf{K}_R^{(k-1)} \tag{12}$$

Using the above equation, the iterative form of reduced matrices can be constructed as

$$\mathbf{K}_R^{(k)} = (\mathbf{T}^{(k)})^{\mathrm{T}} \mathbf{K} \mathbf{T} = (\mathbf{T}^{(k)})^{\mathrm{T}} \begin{bmatrix} \mathbf{K}_{mm} & \mathbf{K}_{ms} \\ \mathbf{K}_{sm} & \mathbf{K}_{ss} \end{bmatrix} \mathbf{T}^{(k)}$$
$$\mathbf{M}_R^{(k)} = (\mathbf{T}^{(k)})^{\mathrm{T}} \mathbf{M} \mathbf{T} = (\mathbf{T}^{(k)})^{\mathrm{T}} \begin{bmatrix} \mathbf{M}_{mm} & \mathbf{M}_{ms} \\ \mathbf{M}_{sm} & \mathbf{M}_{ss} \end{bmatrix} \mathbf{T}^{(k)} \tag{13}$$

Therefore, the lowest $m$ eigenvalues and the associated eigenvectors after $(k-1)$th iteration are estimated by solving the generalized eigenproblem as

$$\mathbf{K}_R^{(k)} \mathbf{\Phi}_{mm}^{(k)} = \mathbf{M}_R^{(k)} \mathbf{\Phi}_{mm}^{(k)} \Lambda_{mm}^{(k)} \tag{14}$$

### 2.2. Nonclassically damped system

There are lots of situations in which the classical damping assumptions are invalid. Examples of such cases are the structures made up of materials with different damping characteristics in different parts, structures equipped with passive and active control system, and structures with layers of damping materials [9,10]. In the nonclassically damped system, the damping matrix cannot be assumed as a linear combination of mass and stiffness matrices. To solve a differential equation of motion with a nonclassically damped matrix, the state vector which is a combination of velocity and displacement vectors should be used to convert second-order differential equations to the first-order equations. And the solution of such equations results in complex eigenvalues, eigenvectors, frequencies and damping ratios. Therefore, the Eq. (1) can be converted to

$$\mathbf{A} \dot{Y}(t) + \mathbf{B} Y(t) = \mathbf{q}(t) \tag{15}$$

where the state vector $Y(t)$ and the system matrices which are real and symmetric $\mathbf{A}$ and $\mathbf{B}$ are defined as

$$Y(t) = \begin{Bmatrix} \dot{X}(t) \\ X(t) \end{Bmatrix}, \quad \mathbf{A} = \begin{bmatrix} \mathbf{K} & \mathbf{0} \\ \mathbf{0} & -\mathbf{M} \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} -\mathbf{C} & -\mathbf{M} \\ -\mathbf{M} & \mathbf{0} \end{bmatrix} \tag{16}$$

Thus, by considering $Y(t) = \tilde{\Psi} \mathrm{e}^{\tilde{\Omega} t}$, the eigenvalue problem for nonclassically damped system can be expressed as

$$\mathbf{A} \tilde{\mathbf{\Psi}} = \mathbf{B} \tilde{\mathbf{\Psi}} \tilde{\mathbf{\Omega}} \tag{17}$$

where the complex conjugate eigenvector matrix $\tilde{\mathbf{\Psi}}$ and the eigenvalue or spectral matrix $\tilde{\mathbf{\Omega}}$ has forms as

$$\tilde{\mathbf{\Psi}} = \begin{bmatrix} \mathbf{\Psi} & \mathbf{\Psi}^* \\ \mathbf{\Psi\Omega} & \mathbf{\Psi}^* \mathbf{\Omega}^* \end{bmatrix}, \quad \tilde{\mathbf{\Omega}} = \begin{bmatrix} \mathbf{\Omega} & \mathbf{0} \\ \mathbf{0} & \mathbf{\Omega}^* \end{bmatrix} \tag{18}$$

Here the $\tilde{\mathbf{\Omega}}$ is arranged in an ascending order and the $\tilde{\mathbf{\Psi}}$ is assumed to be normalized as

$$\tilde{\mathbf{\Psi}}^{\mathrm{T}} \mathbf{A} \tilde{\mathbf{\Psi}} = \tilde{\mathbf{\Omega}}, \quad \tilde{\mathbf{\Psi}}^{\mathrm{T}} \mathbf{B} \tilde{\mathbf{\Psi}} = \mathbf{I} \tag{19}$$

In the dynamic condensation technique, the total degrees of freedom $2n$ of the full model are usually divided into the master degrees of freedom $2m$, which will be retained in the reduced model, and the slave degrees of freedom $2s$, which will be omitted. Based on this division, Eq. (17) can be rewritten in a partitioned form as

$$\begin{bmatrix} \mathbf{A}_{mm} & \mathbf{A}_{ms} \\ \mathbf{A}_{sm} & \mathbf{A}_{ss} \end{bmatrix} \begin{bmatrix} \tilde{\mathbf{\Psi}}_{mm} \\ \tilde{\mathbf{\Psi}}_{sm} \end{bmatrix} = \begin{bmatrix} \mathbf{B}_{mm} & \mathbf{B}_{ms} \\ \mathbf{B}_{sm} & \mathbf{B}_{ss} \end{bmatrix} \begin{bmatrix} \tilde{\mathbf{\Psi}}_{mm} \\ \tilde{\mathbf{\Psi}}_{sm} \end{bmatrix} \tilde{\mathbf{\Omega}}_{mm} \tag{20}$$

In Eq. (20), the submatrices are given by

$$\mathbf{A}_{mm} = \begin{bmatrix} \mathbf{K}_{mm} & \mathbf{0} \\ \mathbf{0} & -\mathbf{M}_{mm} \end{bmatrix}, \quad \mathbf{A}_{ms} = \begin{bmatrix} \mathbf{K}_{ms} & \mathbf{0} \\ \mathbf{0} & -\mathbf{M}_{ms} \end{bmatrix}, \quad \mathbf{A}_{ss} = \begin{bmatrix} \mathbf{K}_{ss} & \mathbf{0} \\ \mathbf{0} & -\mathbf{M}_{ss} \end{bmatrix}$$
$$\mathbf{B}_{mm} = \begin{bmatrix} -\mathbf{C}_{mm} & -\mathbf{M}_{mm} \\ -\mathbf{M}_{mm} & \mathbf{0} \end{bmatrix}, \quad \mathbf{B}_{ms} = \begin{bmatrix} -\mathbf{C}_{ms} & -\mathbf{M}_{ms} \\ -\mathbf{M}_{ms} & \mathbf{0} \end{bmatrix}, \quad \mathbf{B}_{ss} = \begin{bmatrix} -\mathbf{C}_{ss} & -\mathbf{M}_{ss} \\ -\mathbf{M}_{ss} & \mathbf{0} \end{bmatrix}$$
$$\tilde{\mathbf{\Psi}}_{mm} = \begin{bmatrix} \mathbf{\Psi}_{mm} & \mathbf{\Psi}^*_{mm} \\ \mathbf{\Psi}_{mm}\mathbf{\Omega}_{mm} & \mathbf{\Psi}^*_{mm}\mathbf{\Omega}^*_{mm} \end{bmatrix}, \quad \tilde{\mathbf{\Psi}}_{sm} = \begin{bmatrix} \mathbf{\Psi}_{sm} & \mathbf{\Psi}^*_{sm} \\ \mathbf{\Psi}_{sm}\mathbf{\Omega}_{mm} & \mathbf{\Psi}^*_{sm}\mathbf{\Omega}^*_{mm} \end{bmatrix}, \quad \tilde{\mathbf{\Omega}}_{mm} = \begin{bmatrix} \mathbf{\Omega}_{mm} & \mathbf{0} \\ \mathbf{0} & \mathbf{\Omega}^*_{mm} \end{bmatrix} \tag{21}$$

The main procedure of the iterated IRS method for nonclassically damped system is same as the undamped system except all system matrices are defined in state space. Thus, with the same condensation procedure of Eqs. (4)–(11), the iterative form of transformation matrix in state space can be constructed as

$$\mathbf{t}^{(k)} = -\mathbf{A}_{ss}^{-1}\mathbf{A}_{sm} + \mathbf{A}_{ss}^{-1}\left(\mathbf{B}_{sm} + \mathbf{B}_{ss}\mathbf{t}^{(k-1)}\right)\left(\mathbf{B}_R^{(k-1)}\right)^{-1}\mathbf{A}_R^{(k-1)} \tag{22}$$

Using Eq. (22), the iterative form of reduced matrices can be constructed as

$$\mathbf{A}_R^{(k)} = (\mathbf{T}^{(k)})^{\mathrm{T}}\mathbf{A}\mathbf{T} = (\mathbf{T}^{(k)})^{\mathrm{T}} \begin{bmatrix} \mathbf{A}_{mm} & \mathbf{A}_{ms} \\ \mathbf{A}_{sm} & \mathbf{A}_{ss} \end{bmatrix} \mathbf{T}^{(k)}$$
$$\mathbf{B}_R^{(k)} = (\mathbf{T}^{(k)})^{\mathrm{T}}\mathbf{B}\mathbf{T} = (\mathbf{T}^{(k)})^{\mathrm{T}} \begin{bmatrix} \mathbf{B}_{mm} & \mathbf{B}_{ms} \\ \mathbf{B}_{sm} & \mathbf{B}_{ss} \end{bmatrix} \mathbf{T}^{(k)} \tag{23}$$

Consequently, the lowest $2m$ eigenvalues and the associated eigenvectors after $(k-1)$th iteration are estimated by solving the generalized eigenproblem in state space of the reduced system as

$$\mathbf{A}_R^{(k)}\tilde{\mathbf{\Psi}}_{mm}^{(k)} = \mathbf{B}_R^{(k)}\tilde{\mathbf{\Psi}}_{mm}^{(k)}\tilde{\mathbf{\Omega}}_{mm}^{(k)} \tag{24}$$

## 3. Selection of master degrees of freedom

In dynamic condensation, how to select the master degrees of freedom may have much effect on the accuracy of eigenproperties. One of the commonly used criterions is to select the degrees of freedom with the lowest stiffness to mass ratio that is $K_{ii}/M_{ii}$ in the system matrices. But this method is not reliable because some missing eigenvalues in the lower eigenmodes may appear when all master degrees of freedom are selected in the one coordinate direction. The other method is Shah and Raymund's scheme [24]. Though this scheme provides better results, it is computationally inefficient. Another method is Cho and Kim's [14] element-based node selection method. This method has an advantage of better selection of masters and being computationally inexpensive. However, it can be applied only for undamped structural system. For nonclassically damped structural systems, it cannot guarantee the reliability. In this study, node-based arbitrary selection method by random function generation is used. The reason for this is that the accuracy of the eigensolutions can be guaranteed by successive iterations in the iterative form of condensation methods.

Moreover, this method not only can provide well-distributed master nodes but also requires no computing time in selection procedure.

## 4. Present method

### 4.1. Basic idea of a substructuring scheme

The main point of the condensation technique is to eliminate the slave degrees of freedom by over 90% and construct a reduced system with master degrees of freedom by less than 10% of total degrees of freedom using transformation matrix. Therefore, the construction of transformation matrix is very important. And whether the calculation of inverse of the slave degrees of freedom submatrix to build the transformation matrix is possible or not, is the pivotal point to construct a reduced system. Unfortunately, many condensation techniques mentioned in the previous section are just for a single domain system and not suitable to be applied to the practical problems. Therefore, it is the natural extension that substructuring scheme in dynamic condensation should be combined with these condensation methods.

The basic idea of the substructuring scheme is that if the whole structure can be divided into several (or more) substructures and the transformation matrices can be constructed in each subsystem, it will be a very efficient condensation technique. Because the transformation matrix is constructed in each subsystem, the size of transformation matrix will be reduced to that of each substructure. Thus, the reduced system can be constructed without much computational cost. Fig. 1 shows the basic schematic of the substructuring technique. The full system is divided into three kinds of degrees of freedom, i.e. master, slave, and interface degrees of freedom. The interface degrees of freedom are required to connect each subsystem. Furthermore, especially for undamped system in displacement space, the further condensation is possible when the size of retained degrees of freedom is over 10% of the full system or when it is necessary. Through sufficient iterations, the final reduced matrices can be constructed with the reliable eigenproperties. But this further condensation procedure is not allowable for nonclassically damped system because the system matrices are fully populated in state space.

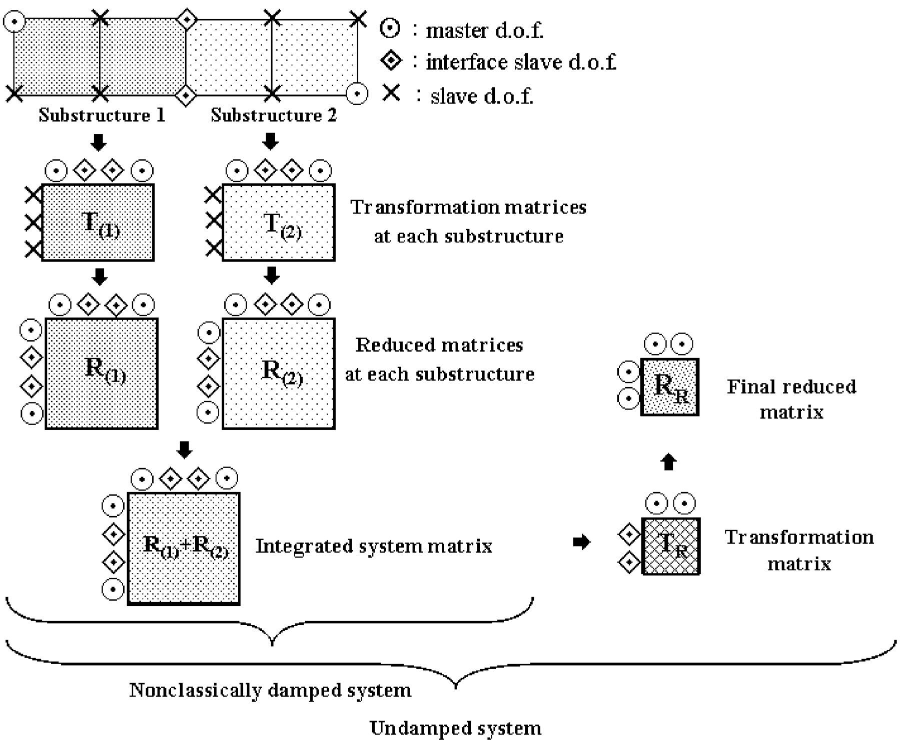


Fig. 1. Basic idea of a substructuring scheme.

### 4.2. Formulation of the substructuring scheme

#### 4.2.1. Undamped system

The following formulation is the substructuring scheme for undamped structural system. To employ the basic substructuring procedure, the single structure of $n$ degrees of freedom of Eq. (1) is divided into two substructures. And, to apply the dynamic condensation scheme in each substructure, the eigenvalue problem can also be constructed by the unit of substructure. Thus, the eigenvalue problem for substructure one can be expressed in a partitioned form as

$$\begin{bmatrix} \mathbf{K}_{ss}^{(1)} & \mathbf{K}_{sm}^{(1)} \\ \mathbf{K}_{ms}^{(1)} & \mathbf{K}_{mm}^{(1)} \end{bmatrix} \begin{bmatrix} \mathbf{\Phi}_{sm}^{(1)} \\ \mathbf{\Phi}_{mm} \end{bmatrix} = \begin{bmatrix} \mathbf{M}_{ss}^{(1)} & \mathbf{M}_{sm}^{(1)} \\ \mathbf{M}_{ms}^{(1)} & \mathbf{M}_{mm}^{(1)} \end{bmatrix} \begin{bmatrix} \mathbf{\Phi}_{sm}^{(1)} \\ \mathbf{\Phi}_{mm} \end{bmatrix} \Lambda_{mm} \tag{25a}$$

With the same manner, the eigenproblem for substructure two can also be described as

$$\begin{bmatrix} \mathbf{K}_{mm}^{(2)} & \mathbf{K}_{ms}^{(2)} \\ \mathbf{K}_{sm}^{(2)} & \mathbf{K}_{ss}^{(2)} \end{bmatrix} \begin{bmatrix} \mathbf{\Phi}_{mm} \\ \mathbf{\Phi}_{sm}^{(2)} \end{bmatrix} = \begin{bmatrix} \mathbf{M}_{mm}^{(2)} & \mathbf{M}_{ms}^{(2)} \\ \mathbf{M}_{sm}^{(2)} & \mathbf{M}_{ss}^{(2)} \end{bmatrix} \begin{bmatrix} \mathbf{\Phi}_{mm} \\ \mathbf{\Phi}_{sm}^{(2)} \end{bmatrix} \Lambda_{mm} \tag{25b}$$

In Eqs. (25a) and (25b), the system matrices can be assembled into one global system as

$$\begin{bmatrix} \mathbf{K}_{ss}^{(1)} & \mathbf{K}_{sm}^{(1)} & \\ \mathbf{K}_{ms}^{(1)} & \mathbf{K}_{mm} & \mathbf{K}_{ms}^{(2)} \\ & \mathbf{K}_{sm}^{(2)} & \mathbf{K}_{ss}^{(2)} \end{bmatrix} \begin{bmatrix} \mathbf{\Phi}_{sm}^{(1)} \\ \mathbf{\Phi}_{mm} \\ \mathbf{\Phi}_{sm}^{(2)} \end{bmatrix} = \begin{bmatrix} \mathbf{M}_{ss}^{(1)} & \mathbf{M}_{sm}^{(1)} & \\ \mathbf{M}_{ms}^{(1)} & \mathbf{M}_{mm} & \mathbf{M}_{ms}^{(2)} \\ & \mathbf{M}_{sm}^{(2)} & \mathbf{M}_{ss}^{(2)} \end{bmatrix} \begin{bmatrix} \mathbf{\Phi}_{sm}^{(1)} \\ \mathbf{\Phi}_{mm} \\ \mathbf{\Phi}_{sm}^{(2)} \end{bmatrix} \Lambda_{mm} \tag{26}$$

where $\mathbf{K}_{mm} = \mathbf{K}_{mm}^{(1)} + \mathbf{K}_{mm}^{(2)}$ and $\mathbf{M}_{mm} = \mathbf{M}_{mm}^{(1)} + \mathbf{M}_{mm}^{(2)}$ including the interface degrees of freedom. To eliminate the slave degrees of freedom field in each substructure, employ the first and the third rows of Eq. (26) as

$$\begin{aligned} \mathbf{K}_{ss}^{(1)}\mathbf{\Phi}_{sm}^{(1)} + \mathbf{K}_{sm}^{(1)}\mathbf{\Phi}_{mm} &= \left(\mathbf{M}_{ss}^{(1)}\mathbf{\Phi}_{sm}^{(1)} + \mathbf{M}_{sm}^{(1)}\mathbf{\Phi}_{mm}\right)\Lambda_{mm} \\ \mathbf{K}_{sm}^{(2)}\mathbf{\Phi}_{mm} + \mathbf{K}_{ss}^{(2)}\mathbf{\Phi}_{sm}^{(2)} &= \left(\mathbf{M}_{sm}^{(2)}\mathbf{\Phi}_{mm} + \mathbf{M}_{ss}^{(2)}\mathbf{\Phi}_{sm}^{(2)}\right)\Lambda_{mm} \end{aligned} \tag{27}$$

Through Eq. (27) the transformation relation of the master degrees of freedom field and the slave degrees of freedom field in each substructure is obtained. Rearranging Eq. (27) for the slave degrees of freedom field as

$$\begin{aligned} \mathbf{\Phi}_{sm}^{(1)} &= -\left(\mathbf{K}_{ss}^{(1)}\right)^{-1}\mathbf{K}_{sm}^{(1)}\mathbf{\Phi}_{mm} + \left(\mathbf{K}_{ss}^{(1)}\right)^{-1}\left(\mathbf{M}_{sm}^{(1)}\mathbf{\Phi}_{mm} + \mathbf{M}_{ss}^{(1)}\mathbf{\Phi}_{sm}^{(1)}\right)\Lambda_{mm} \\ \mathbf{\Phi}_{sm}^{(2)} &= -\left(\mathbf{K}_{ss}^{(2)}\right)^{-1}\mathbf{K}_{sm}^{(2)}\mathbf{\Phi}_{mm} + \left(\mathbf{K}_{ss}^{(2)}\right)^{-1}\left(\mathbf{M}_{sm}^{(2)}\mathbf{\Phi}_{mm} + \mathbf{M}_{ss}^{(2)}\mathbf{\Phi}_{sm}^{(2)}\right)\Lambda_{mm} \end{aligned} \tag{28}$$

According to the definition of the transformation matrices in each subsystem, that are,

$$\begin{aligned} \mathbf{\Phi}_{sm}^{(1)} &= \mathbf{t}_{(1)}\mathbf{\Phi}_{mm} \\ \mathbf{\Phi}_{sm}^{(2)} &= \mathbf{t}_{(2)}\mathbf{\Phi}_{mm} \end{aligned} \tag{29}$$

Substituting Eq. (29) into Eq. (28) and rearranging them,

$$\begin{aligned} \mathbf{t}_{(1)} &= -\left(\mathbf{K}_{ss}^{(1)}\right)^{-1}\mathbf{K}_{sm}^{(1)} + \left(\mathbf{K}_{ss}^{(1)}\right)^{-1}\left(\mathbf{M}_{sm}^{(1)} + \mathbf{M}_{ss}^{(1)}\mathbf{t}_{(1)}\right)\mathbf{\Phi}_{mm}\Lambda_{mm}\mathbf{\Phi}_{mm}^{-1} \\ \mathbf{t}_{(2)} &= -\left(\mathbf{K}_{ss}^{(2)}\right)^{-1}\mathbf{K}_{sm}^{(2)} + \left(\mathbf{K}_{ss}^{(2)}\right)^{-1}\left(\mathbf{M}_{sm}^{(2)} + \mathbf{M}_{ss}^{(2)}\mathbf{t}_{(2)}\right)\mathbf{\Phi}_{mm}\Lambda_{mm}\mathbf{\Phi}_{mm}^{-1} \end{aligned} \tag{30}$$

From Eq. (30), we get two transformation matrices. By these two transformation matrices, the whole degrees of freedom field can be reduced to the one with only master degrees of freedom field as

$$\begin{bmatrix} \mathbf{\Phi}_{sm}^{(1)} \\ \mathbf{\Phi}_{mm} \\ \mathbf{\Phi}_{sm}^{(2)} \end{bmatrix} = \begin{bmatrix} \mathbf{t}_{(1)} \\ \mathbf{I}_{mm} \\ \mathbf{t}_{(2)} \end{bmatrix} \mathbf{\Phi}_{mm} = \mathbf{T}\mathbf{\Phi}_{mm} \tag{31}$$

where $\mathbf{I}$ is a unit matrix of size $m \times m$ and $\mathbf{T}$ is a combined form of transformation matrix. Substituting Eq. (31) into Eq. (26) and premultiplying $\mathbf{T}^{\mathrm{T}}$ on the left of the equation, we can obtain the reduced system matrices as

$$\mathbf{K}_R = \mathbf{T}^{\mathrm{T}}\mathbf{K}\mathbf{T} = \mathbf{T}^{\mathrm{T}}\begin{bmatrix} \mathbf{K}_{ss}^{(1)} & \mathbf{K}_{sm}^{(1)} & \\ \mathbf{K}_{ms}^{(1)} & \mathbf{K}_{mm} & \mathbf{K}_{ms}^{(2)} \\ & \mathbf{K}_{sm}^{(2)} & \mathbf{K}_{ss}^{(2)} \end{bmatrix}\mathbf{T}$$

$$\mathbf{M}_R = \mathbf{T}^{\mathrm{T}}\mathbf{M}\mathbf{T} = \mathbf{T}^{\mathrm{T}}\begin{bmatrix} \mathbf{M}_{ss}^{(1)} & \mathbf{M}_{sm}^{(1)} & \\ \mathbf{M}_{ms}^{(1)} & \mathbf{M}_{mm} & \mathbf{M}_{ms}^{(2)} \\ & \mathbf{M}_{sm}^{(2)} & \mathbf{M}_{ss}^{(2)} \end{bmatrix}\mathbf{T} \tag{32}$$

From Eq. (32), we can construct a reduced eigenvalue problem of size $m \times m$ as

$$\mathbf{K}_R\mathbf{\Phi}_{mm} = \mathbf{M}_R\mathbf{\Phi}_{mm}\Lambda_{mm} \tag{33a}$$

From Eq. (33a), we can obtain the approximate eigenvalue as

$$\mathbf{\Phi}_{mm}\Lambda_{mm}\mathbf{\Phi}_{mm}^{-1} = \mathbf{M}_R^{-1}\mathbf{K}_R \tag{33b}$$

Substituting Eq. (33b) into Eq. (30), we get two transformation matrices for dynamic condensation as

$$\begin{aligned} \mathbf{t}_{(1)} &= -\left(\mathbf{K}_{ss}^{(1)}\right)^{-1}\mathbf{K}_{sm}^{(1)} + \left(\mathbf{K}_{ss}^{(1)}\right)^{-1}\left(\mathbf{M}_{sm}^{(1)} + \mathbf{M}_{ss}^{(1)}\mathbf{t}_{(1)}\right)\mathbf{M}_R^{-1}\mathbf{K}_R \\ \mathbf{t}_{(2)} &= -\left(\mathbf{K}_{ss}^{(2)}\right)^{-1}\mathbf{K}_{sm}^{(2)} + \left(\mathbf{K}_{ss}^{(2)}\right)^{-1}\left(\mathbf{M}_{sm}^{(2)} + \mathbf{M}_{ss}^{(2)}\mathbf{t}_{(2)}\right)\mathbf{M}_R^{-1}\mathbf{K}_R \end{aligned} \tag{34}$$

Since these equations are nonlinear, the iterative forms of these two governing equations for $k = 1,2,3, \ldots,$ are given by

$$\begin{aligned} \mathbf{t}_{(1)}^{(k)} &= -\left(\mathbf{K}_{ss}^{(1)}\right)^{-1}\mathbf{K}_{sm}^{(1)} + \left(\mathbf{K}_{ss}^{(1)}\right)^{-1}\left(\mathbf{M}_{sm}^{(1)} + \mathbf{M}_{ss}^{(1)}\mathbf{t}_{(1)}^{(k-1)}\right)\left(\mathbf{M}_R^{(k-1)}\right)^{-1}\mathbf{K}_R^{(k-1)} \\ \mathbf{t}_{(2)}^{(k)} &= -\left(\mathbf{K}_{ss}^{(2)}\right)^{-1}\mathbf{K}_{sm}^{(2)} + \left(\mathbf{K}_{ss}^{(2)}\right)^{-1}\left(\mathbf{M}_{sm}^{(2)} + \mathbf{M}_{ss}^{(2)}\mathbf{t}_{(2)}^{(k-1)}\right)\left(\mathbf{M}_R^{(k-1)}\right)^{-1}\mathbf{K}_R^{(k-1)} \end{aligned} \tag{35}$$

However, to obtain the initial approximate eigenvalue in Eq. (33b), the transformation matrices for static condensation in each substructure should be used. Thus, only the first terms of Eq. (35) are used as

$$\begin{aligned} \mathbf{t}_{(1)}^{(0)} &= -\left(\mathbf{K}_{ss}^{(1)}\right)^{-1}\mathbf{K}_{sm}^{(1)} \\ \mathbf{t}_{(2)}^{(0)} &= -\left(\mathbf{K}_{ss}^{(2)}\right)^{-1}\mathbf{K}_{sm}^{(2)} \end{aligned}, \quad \mathbf{T}^{(0)} = \begin{bmatrix} \mathbf{t}_{(1)}^{(0)} \\ \mathbf{I}_{mm} \\ \mathbf{t}_{(2)}^{(0)} \end{bmatrix} \tag{36}$$

Therefore, the Guyan reduction matrices are obtained as follows:

$$\begin{aligned} \mathbf{K}_{\mathrm{Guyan}} &= \begin{bmatrix} \left(\mathbf{t}_{(1)}^{(0)}\right)^{\mathrm{T}} & \mathbf{I}_{mm} & \left(\mathbf{t}_{(2)}^{(0)}\right)^{\mathrm{T}} \end{bmatrix} \begin{bmatrix} \mathbf{K}_{ss}^{(1)} & \mathbf{K}_{sm}^{(1)} & \\ \mathbf{K}_{ms}^{(1)} & \mathbf{K}_{mm} & \mathbf{K}_{ms}^{(2)} \\ & \mathbf{K}_{sm}^{(2)} & \mathbf{K}_{ss}^{(2)} \end{bmatrix} \begin{bmatrix} \mathbf{t}_{(1)}^{(0)} \\ \mathbf{I}_{mm} \\ \mathbf{t}_{(2)}^{(0)} \end{bmatrix} \\ &= \left(\mathbf{t}_{(1)}^{(0)}\right)^{\mathrm{T}}\mathbf{K}_{ss}^{(1)}\mathbf{t}_{(1)}^{(0)} + \mathbf{K}_{ms}^{(1)}\mathbf{t}_{(1)}^{(0)} + \left(\mathbf{t}_{(1)}^{(0)}\right)^{\mathrm{T}}\mathbf{K}_{sm}^{(1)} + \mathbf{K}_{mm} + \left(\mathbf{t}_{(2)}^{(0)}\right)^{\mathrm{T}}\mathbf{K}_{sm}^{(2)} + \mathbf{K}_{ms}^{(2)}\mathbf{t}_{(2)}^{(0)} + \left(\mathbf{t}_{(2)}^{(0)}\right)^{\mathrm{T}}\mathbf{K}_{ss}^{(2)}\mathbf{t}_{(2)}^{(0)} \end{aligned} \tag{37a}$$

$$\mathbf{M}_{\text{Guyan}} = \begin{bmatrix} \left(\mathbf{t}_{(1)}^{(0)}\right)^{\text{T}} & \mathbf{I}_{mm} & \left(\mathbf{t}_{(2)}^{(0)}\right)^{\text{T}} \end{bmatrix} \begin{bmatrix} \mathbf{M}_{ss}^{(1)} & \mathbf{M}_{sm}^{(1)} & \\ \mathbf{M}_{ms}^{(1)} & \mathbf{M}_{mm} & \mathbf{M}_{ms}^{(2)} \\ & \mathbf{M}_{sm}^{(2)} & \mathbf{M}_{ss}^{(2)} \end{bmatrix} \begin{bmatrix} \mathbf{t}_{(1)}^{(0)} \\ \mathbf{I}_{mm} \\ \mathbf{t}_{(2)}^{(0)} \end{bmatrix}$$

$$= \left(\mathbf{t}_{(1)}^{(0)}\right)^{\text{T}} \mathbf{M}_{ss}^{(1)} \mathbf{t}_{(1)}^{(0)} + \mathbf{M}_{ms}^{(1)} \mathbf{t}_{(1)}^{(0)} + \left(\mathbf{t}_{(1)}^{(0)}\right)^{\text{T}} \mathbf{M}_{sm}^{(1)} + \mathbf{M}_{mm} + \left(\mathbf{t}_{(2)}^{(0)}\right)^{\text{T}} \mathbf{M}_{sm}^{(2)} + \mathbf{M}_{ms}^{(2)} \mathbf{t}_{(2)}^{(0)} + \left(\mathbf{t}_{(2)}^{(0)}\right)^{\text{T}} \mathbf{M}_{ss}^{(2)} \mathbf{t}_{(2)}^{(0)} \quad (37b)$$

As shown Eq. (37), the Guyan reduction matrices are constructed in the substructure level and these reduced matrices are assembled into global system. For the dynamic condensation, these Guyan reduction matrices become starting reduced system matrices for iteration as follows

$$\mathbf{K}_R^{(0)} = \mathbf{K}_{\text{Guyan}}$$
$$\mathbf{M}_R^{(0)} = \mathbf{M}_{\text{Guyan}} \quad (38)$$

Substituting Eq. (38) into Eq. (34), the initial transformation matrices, i.e. when $k = 1$, that is 0th iteration, are given by

$$\begin{aligned} \mathbf{t}_{(1)}^{(1)} &= -\left(\mathbf{K}_{ss}^{(1)}\right)^{-1} \mathbf{K}_{sm}^{(1)} + \left(\mathbf{K}_{ss}^{(1)}\right)^{-1} \left(\mathbf{M}_{ss}^{(1)} \mathbf{t}_{(1)}^{(0)} + \mathbf{M}_{sm}^{(1)}\right) \left(\mathbf{M}_R^{(0)}\right)^{-1} \mathbf{K}_R^{(0)} \\ \mathbf{t}_{(2)}^{(1)} &= -\left(\mathbf{K}_{ss}^{(2)}\right)^{-1} \mathbf{K}_{sm}^{(2)} + \left(\mathbf{K}_{ss}^{(2)}\right)^{-1} \left(\mathbf{M}_{ss}^{(2)} \mathbf{t}_{(2)}^{(0)} + \mathbf{M}_{sm}^{(2)}\right) \left(\mathbf{M}_R^{(0)}\right)^{-1} \mathbf{K}_R^{(0)} \end{aligned}, \quad \mathbf{T}^{(1)} = \begin{bmatrix} \mathbf{t}_{(1)}^{(1)} \\ \mathbf{I}_{mm} \\ \mathbf{t}_{(2)}^{(1)} \end{bmatrix} \quad (39)$$

Using Eq. (39), the reduced system matrices can be constructed as

$$\mathbf{K}_R^{(1)} = \begin{bmatrix} \left(\mathbf{t}_{(1)}^{(1)}\right)^{\text{T}} & \mathbf{I} & \left(\mathbf{t}_{(2)}^{(1)}\right)^{\text{T}} \end{bmatrix} \begin{bmatrix} \mathbf{K}_{ss}^{(1)} & \mathbf{K}_{sm}^{(1)} & \\ \mathbf{K}_{ms}^{(1)} & \mathbf{K}_{mm} & \mathbf{K}_{ms}^{(2)} \\ & \mathbf{K}_{sm}^{(2)} & \mathbf{K}_{ss}^{(2)} \end{bmatrix} \begin{bmatrix} \mathbf{t}_{(1)}^{(1)} \\ \mathbf{I} \\ \mathbf{t}_{(2)}^{(1)} \end{bmatrix}$$

$$= \left(\mathbf{t}_{(1)}^{(1)}\right)^{\text{T}} \mathbf{K}_{ss}^{(1)} \mathbf{t}_{(1)}^{(1)} + \mathbf{K}_{ms}^{(1)} \mathbf{t}_{(1)}^{(1)} + \left(\mathbf{t}_{(1)}^{(1)}\right)^{\text{T}} \mathbf{K}_{sm}^{(1)} + \mathbf{K}_{mm} + \left(\mathbf{t}_{(2)}^{(1)}\right)^{\text{T}} \mathbf{K}_{sm}^{(2)} + \mathbf{K}_{ms}^{(2)} \mathbf{t}_{(2)}^{(1)} + \left(\mathbf{t}_{(2)}^{(1)}\right)^{\text{T}} \mathbf{K}_{ss}^{(2)} \mathbf{t}_{(2)}^{(1)} \quad (40a)$$

$$\mathbf{M}_R^{(1)} = \begin{bmatrix} \left(\mathbf{t}_{(1)}^{(1)}\right)^{\text{T}} & \mathbf{I} & \left(\mathbf{t}_{(2)}^{(1)}\right)^{\text{T}} \end{bmatrix} \begin{bmatrix} \mathbf{M}_{ss}^{(1)} & \mathbf{M}_{sm}^{(1)} & \\ \mathbf{M}_{ms}^{(1)} & \mathbf{M}_{mm} & \mathbf{M}_{ms}^{(2)} \\ & \mathbf{M}_{sm}^{(2)} & \mathbf{M}_{ss}^{(2)} \end{bmatrix} \begin{bmatrix} \mathbf{t}_{(1)}^{(1)} \\ \mathbf{I} \\ \mathbf{t}_{(2)}^{(1)} \end{bmatrix}$$

$$= \left(\mathbf{t}_{(1)}^{(1)}\right)^{\text{T}} \mathbf{M}_{ss}^{(1)} \mathbf{t}_{(1)}^{(1)} + \mathbf{M}_{ms}^{(1)} \mathbf{t}_{(1)}^{(1)} + \left(\mathbf{t}_{(1)}^{(1)}\right)^{\text{T}} \mathbf{M}_{sm}^{(1)} + \mathbf{M}_{mm} + \left(\mathbf{t}_{(2)}^{(1)}\right)^{\text{T}} \mathbf{M}_{sm}^{(2)} + \mathbf{M}_{ms}^{(2)} \mathbf{t}_{(2)}^{(1)} + \left(\mathbf{t}_{(2)}^{(1)}\right)^{\text{T}} \mathbf{M}_{ss}^{(2)} \mathbf{t}_{(2)}^{(1)} \quad (40b)$$

From Eq. (40), it is clear that the reduced system matrices are also constructed by the unit of subsystem and combined into whole system.

For the first iteration, i.e. when $k = 2$, the reduced matrices $\mathbf{K}_R^{(1)}, \mathbf{M}_R^{(1)}$ and the transformation matrices $\mathbf{t}_{(1)}^{(1)}, \mathbf{t}_{(2)}^{(1)}$ obtained in the previous step are used in the next construction of transformation matrices as

$$\begin{aligned} \mathbf{t}_{(1)}^{(2)} &= -\left(\mathbf{K}_{ss}^{(1)}\right)^{-1} \mathbf{K}_{sm}^{(1)} + \left(\mathbf{K}_{ss}^{(1)}\right)^{-1} \left(\mathbf{M}_{ss}^{(1)} t_{(1)}^{(1)} + \mathbf{M}_{sm}^{(1)}\right) \left(\mathbf{M}_R^{(1)}\right)^{-1} \mathbf{K}_R^{(1)} \\ \mathbf{t}_{(2)}^{(2)} &= -\left(\mathbf{K}_{ss}^{(2)}\right)^{-1} \mathbf{K}_{sm}^{(2)} + \left(\mathbf{K}_{ss}^{(2)}\right)^{-1} \left(\mathbf{M}_{ss}^{(2)} t_{(2)}^{(1)} + \mathbf{M}_{sm}^{(2)}\right) \left(\mathbf{M}_R^{(1)}\right)^{-1} \mathbf{K}_R^{(1)} \end{aligned} \quad (41)$$

With Eq. (41), the system-reduced matrices of the first iteration are given by

$$\begin{aligned} \mathbf{K}_R^{(2)} &= \left(\mathbf{T}_{(1)}^{(2)}\right)^{\text{T}} \mathbf{K}_{ss}^{(1)} \mathbf{t}_{(1)}^{(2)} + \mathbf{K}_{ms}^{(1)} \mathbf{t}_{(1)}^{(2)} + \left(\mathbf{t}_{(1)}^{(2)}\right)^{\text{T}} \mathbf{K}_{sm}^{(1)} + \mathbf{K}_{mm} + \left(\mathbf{t}_{(2)}^{(2)}\right)^{\text{T}} \mathbf{K}_{sm}^{(2)} + \mathbf{K}_{ms}^{(2)} \mathbf{t}_{(2)}^{(2)} + \left(\mathbf{t}_{(2)}^{(2)}\right)^{\text{T}} \mathbf{K}_{ss}^{(2)} \mathbf{t}_{(2)}^{(2)} \\ \mathbf{M}_R^{(2)} &= \left(\mathbf{T}_{(1)}^{(2)}\right)^{\text{T}} \mathbf{M}_{ss}^{(1)} \mathbf{t}_{(1)}^{(2)} + \mathbf{M}_{ms}^{(1)} \mathbf{t}_{(1)}^{(2)} + \left(\mathbf{t}_{(1)}^{(2)}\right)^{\text{T}} \mathbf{M}_{sm}^{(1)} + \mathbf{M}_{mm} + \left(\mathbf{t}_{(2)}^{(2)}\right)^{\text{T}} \mathbf{M}_{sm}^{(2)} + \mathbf{M}_{ms}^{(2)} \mathbf{t}_{(2)}^{(2)} + \left(\mathbf{t}_{(2)}^{(2)}\right)^{\text{T}} \mathbf{M}_{ss}^{(2)} \mathbf{t}_{(2)}^{(2)} \end{aligned} \quad (42)$$

Through Eq. (42), the first iterated reduced system matrices are obtained. By these procedures, the iterative form of transformation matrix and reduced matrices are expressed as

$$\mathbf{T}^{(k)} = \begin{bmatrix} \mathbf{t}_{(1)}^{(k)} \\ \mathbf{I}_{mm} \\ \mathbf{t}_{(2)}^{(k)} \end{bmatrix} \tag{43a}$$

$$\mathbf{K}_R^{(k)} = (\mathbf{T}^{(k)})^{\mathrm{T}}\mathbf{K}\mathbf{T}^{(k)} = (\mathbf{T}^{(k)})^{\mathrm{T}} \begin{bmatrix} \mathbf{K}_{ss}^{(1)} & \mathbf{K}_{sm}^{(1)} & \\ \mathbf{K}_{ms}^{(1)} & \mathbf{K}_{mm} & \mathbf{K}_{ms}^{(2)} \\ & \mathbf{K}_{sm}^{(2)} & \mathbf{K}_{ss}^{(2)} \end{bmatrix} \mathbf{T}^{(k)}$$
$$\mathbf{M}_R^{(k)} = (\mathbf{T}^{(k)})^{\mathrm{T}}\mathbf{M}\mathbf{T}^{(k)} = (\mathbf{T}^{(k)})^{\mathrm{T}} \begin{bmatrix} \mathbf{M}_{ss}^{(1)} & \mathbf{M}_{sm}^{(1)} & \\ \mathbf{M}_{ms}^{(1)} & \mathbf{M}_{mm} & \mathbf{M}_{ms}^{(2)} \\ & \mathbf{M}_{sm}^{(2)} & \mathbf{M}_{ss}^{(2)} \end{bmatrix} \mathbf{T}^{(k)} \tag{43b}$$

Finally, the lowest $m$ eigenvalues and the associated eigenvectors after $(k-1)$th iteration are estimated by solving the generalized eigenproblem of the reduced system as

$$\mathbf{K}_R^{(k)}\mathbf{\Phi}_{mm}^{(k)} = \mathbf{M}_R^{(k)}\mathbf{\Phi}_{mm}^{(k)}\Lambda_{mm}^{(k)} \tag{44}$$

### 4.2.2. Nonclassically damped system

As mentioned in Section 2.2, the main procedure of the substructuring scheme for undamped structural system can also be applied to the formulation for nonclassically damped system. However, the size of all system matrices becomes doubled. With the same procedure in the previous section, the eigenvalue problem for two substructures can be expressed in a partitioned form in state space as

$$\begin{bmatrix} \mathbf{A}_{ss}^{(1)} & \mathbf{A}_{sm}^{(1)} \\ \mathbf{A}_{ms}^{(1)} & \mathbf{A}_{mm}^{(1)} \end{bmatrix} \begin{bmatrix} \tilde{\mathbf{\Psi}}_{sm}^{(1)} \\ \tilde{\mathbf{\Psi}}_{mm} \end{bmatrix} = \begin{bmatrix} \mathbf{B}_{ss}^{(1)} & \mathbf{B}_{sm}^{(1)} \\ \mathbf{B}_{ms}^{(1)} & \mathbf{B}_{mm}^{(1)} \end{bmatrix} \begin{bmatrix} \tilde{\mathbf{\Psi}}_{sm}^{(1)} \\ \tilde{\mathbf{\Psi}}_{mm} \end{bmatrix} \tilde{\mathbf{\Omega}}_{mm}$$
$$\begin{bmatrix} \mathbf{A}_{mm}^{(2)} & \mathbf{A}_{ms}^{(2)} \\ \mathbf{A}_{sm}^{(2)} & \mathbf{A}_{ss}^{(2)} \end{bmatrix} \begin{bmatrix} \tilde{\mathbf{\Psi}}_{mm} \\ \tilde{\mathbf{\Psi}}_{sm}^{(2)} \end{bmatrix} = \begin{bmatrix} \mathbf{B}_{mm}^{(2)} & \mathbf{B}_{ms}^{(2)} \\ \mathbf{B}_{sm}^{(2)} & \mathbf{B}_{ss}^{(2)} \end{bmatrix} \begin{bmatrix} \tilde{\mathbf{\Psi}}_{mm} \\ \tilde{\mathbf{\Psi}}_{sm}^{(2)} \end{bmatrix} \tilde{\mathbf{\Omega}}_{mm} \tag{45}$$

Returning to Eq. (26), the assembled form of nonclassically damped system matrix can be expressed as

$$\begin{bmatrix} \mathbf{A}_{ss}^{(1)} & \mathbf{A}_{sm}^{(1)} & \\ \mathbf{A}_{ms}^{(1)} & \mathbf{A}_{mm} & \mathbf{A}_{ms}^{(2)} \\ & \mathbf{A}_{sm}^{(2)} & \mathbf{A}_{ss}^{(2)} \end{bmatrix} \begin{bmatrix} \tilde{\mathbf{\Psi}}_{sm}^{(1)} \\ \tilde{\mathbf{\Psi}}_{mm} \\ \tilde{\mathbf{\Psi}}_{sm}^{(2)} \end{bmatrix} = \begin{bmatrix} \mathbf{B}_{ss}^{(1)} & \mathbf{B}_{sm}^{(1)} & \\ \mathbf{B}_{ms}^{(1)} & \mathbf{B}_{mm} & \mathbf{B}_{ms}^{(2)} \\ & \mathbf{B}_{sm}^{(2)} & \mathbf{B}_{ss}^{(2)} \end{bmatrix} \begin{bmatrix} \tilde{\mathbf{\Psi}}_{sm}^{(1)} \\ \tilde{\mathbf{\Psi}}_{mm} \\ \tilde{\mathbf{\Psi}}_{sm}^{(2)} \end{bmatrix} \tilde{\mathbf{\Omega}}_{mm} \tag{46}$$

In which $\mathbf{A}_{mm} = \mathbf{A}_{mm}^{(1)} + \mathbf{A}_{mm}^{(2)}$ and $\mathbf{B}_{mm} = \mathbf{B}_{mm}^{(1)} + \mathbf{B}_{mm}^{(2)}$. Through Eqs. (27)–(34), the iterative forms two transformation matrices of the nonclassically damped system for $k = 1,2,3, \ldots$, are given by

$$\mathbf{t}_{(1)}^{(k)} = -\left(\mathbf{A}_{ss}^{(1)}\right)^{-1}\mathbf{A}_{sm}^{(1)} + \left(\mathbf{A}_{ss}^{(1)}\right)^{-1}\left(\mathbf{B}_{sm}^{(1)} + \mathbf{B}_{ss}^{(1)}\mathbf{t}_{(1)}^{(k-1)}\right)\left(\mathbf{B}_R^{(k-1)}\right)^{-1}\mathbf{A}_R^{(k-1)}$$
$$\mathbf{t}_{(2)}^{(k)} = -\left(\mathbf{A}_{ss}^{(2)}\right)^{-1}\mathbf{A}_{sm}^{(2)} + \left(\mathbf{A}_{ss}^{(2)}\right)^{-1}\left(\mathbf{B}_{sm}^{(2)} + \mathbf{B}_{ss}^{(2)}\mathbf{t}_{(2)}^{(k-1)}\right)\left(\mathbf{B}_R^{(k-1)}\right)^{-1}\mathbf{A}_R^{(k-1)} \tag{47}$$

And through Eqs. (36)–(42), the iterative form of transformation matrix and reduced matrices are expressed as

$$\mathbf{T}^{(k)} = \begin{bmatrix} \mathbf{t}_{(1)}^{(k)} \\ \mathbf{I}_{mm} \\ \mathbf{t}_{(2)}^{(k)} \end{bmatrix} \tag{48a}$$

$$\mathbf{A}_R^{(k)} = (\mathbf{T}^{(k)})^{\mathrm{T}} \mathbf{A} \mathbf{T}^{(k)} = (\mathbf{T}^{(k)})^{\mathrm{T}} \begin{bmatrix} \mathbf{A}_{ss}^{(1)} & \mathbf{A}_{sm}^{(1)} & \\ \mathbf{A}_{ms}^{(1)} & \mathbf{A}_{mm} & \mathbf{A}_{ms}^{(2)} \\ & \mathbf{A}_{sm}^{(2)} & \mathbf{A}_{ss}^{(2)} \end{bmatrix} \mathbf{T}^{(k)}$$

$$\mathbf{B}_R^{(k)} = (\mathbf{T}^{(k)})^{\mathrm{T}} \mathbf{B} \mathbf{T}^{(k)} = (\mathbf{T}^{(k)})^{\mathrm{T}} \begin{bmatrix} \mathbf{B}_{ss}^{(1)} & \mathbf{B}_{sm}^{(1)} & \\ \mathbf{B}_{ms}^{(1)} & \mathbf{B}_{mm} & \mathbf{B}_{ms}^{(2)} \\ & \mathbf{B}_{sm}^{(2)} & \mathbf{B}_{ss}^{(2)} \end{bmatrix} \mathbf{T}^{(k)} \tag{48b}$$

Finally, the lowest $2m$ eigenvalues and the associated eigenvectors after $(k-1)$th iteration are estimated by solving the generalized eigenproblem of the reduced system as

$$\mathbf{A}_R^{(k)} \tilde{\mathbf{\Psi}}_{mm}^{(k)} = \mathbf{B}_R^{(k)} \tilde{\mathbf{\Psi}}_{mm}^{(k)} \tilde{\mathbf{\Omega}}_{mm}^{(k)} \tag{49}$$

And the solution of this eigenproblem results in the complex eigenproperties.

### 4.3. Further condensation only for undamped system

In dynamic condensation for undamped structure, further condensation is possible. The reason for the further condensation is that the unnecessary degrees of freedom exist in interfaces connecting each substructure. These degrees of freedom do not have significant effect on the accuracy of eigenproperties of condensation matrices. Thus these slave interface degrees of freedom can be eliminated by the further condensation. The procedure for this is identical to the steps for iterative IRS method. At this time the slave degrees of freedom in interface become the slave degrees of freedom. Through this further condensation the reduced matrices less than 10% of the full system can be constructed. After sufficient iterations the reliable eigenproperties can be obtained. But this is only for undamped system and the accuracy of the further condensation matrices cannot be higher than the results in the first condensation.

From Eq. (44), the reduced matrices **K** and **M** can be partitioned into the master and slave degrees of freedom again as

$$\begin{bmatrix} \mathbf{K}_{mm} & \mathbf{K}_{mi(s)} \\ \mathbf{K}_{i(s)m} & \mathbf{K}_{i(s)i(s)} \end{bmatrix} \begin{Bmatrix} \mathbf{\Phi}_{mm} \\ \mathbf{\Phi}_{i(s)m} \end{Bmatrix}_{(1)+(2)} = \begin{bmatrix} \mathbf{M}_{mm} & \mathbf{M}_{mi(s)} \\ \mathbf{M}_{i(s)m} & \mathbf{M}_{i(s)i(s)} \end{bmatrix} \begin{Bmatrix} \mathbf{\Phi}_{mm} \\ \mathbf{\Phi}_{i(s)m} \end{Bmatrix} \Lambda_{mm} \tag{50}$$

In Eq. (50), the subscript $i(s)$ indicates the slave degrees of freedom of interfaces. The iterative form of transformation matrix is

$$\mathbf{t}_{(1)+(2)}^{(k)} = -\mathbf{K}_{i(s)i(s)}^{-1} \mathbf{K}_{i(s)m} + \mathbf{K}_{i(s)i(s)}^{-1} \left( \mathbf{M}_{i(s)m} + \mathbf{M}_{i(s)i(s)} \mathbf{t}_{(1)+(2)}^{(k-1)} \right) \left( \mathbf{M}_R^{(k-1)} \right)^{-1} \mathbf{K}_R^{(k-1)} \tag{51}$$

From Eq. (51), the iterative form of further reduced matrices can be constructed as

$$\mathbf{K}_{RR}^{(k)} = \left(\mathbf{T}_{(1)+(2)}^{(k)}\right)^{\mathrm{T}} \mathbf{K}_R \mathbf{T}_{(1)+(2)}^{(k)} = \left(\mathbf{T}_{(1)+(2)}^{(k)}\right)^{\mathrm{T}} \begin{bmatrix} \mathbf{K}_{R_{mm}} & \mathbf{K}_{R_{ms}} \\ \mathbf{K}_{R_{sm}} & \mathbf{K}_{R_{ss}} \end{bmatrix} \mathbf{T}_{(1)+(2)}^{(k)}$$

$$\mathbf{M}_{RR}^{(k)} = \left(\mathbf{T}_{(1)+(2)}^{(k)}\right)^{\mathrm{T}} \mathbf{M}_R \mathbf{T}_{(1)+(2)}^{(k)} = \left(\mathbf{T}_{(1)+(2)}^{(k)}\right)^{\mathrm{T}} \begin{bmatrix} \mathbf{M}_{R_{mm}} & \mathbf{M}_{R_{ms}} \\ \mathbf{M}_{R_{sm}} & \mathbf{M}_{R_{ss}} \end{bmatrix} \mathbf{T}_{(1)+(2)}^{(k)} \tag{52}$$

### 4.4. Procedure for substructuring scheme

The main steps for the $(k-1)$th iterative substructuring reduction for two structural systems are as follows:

(1) Separate the finite element model into two (or more) substructures.
(2) Choose the master degrees of freedom using the node-based arbitrary selection method including interface degrees of freedom in each substructure and compute all the submatrices to be used in the following.
(3) Construct the Guyan reduction matrices in each substructure and assemble them into one by using Eqs. (36) and (37a), (37b).
(4) Calculate the approximate eigenvalue$(\mathbf{M}_R^{(k-1)})^{-1}\mathbf{K}_R^{(k-1)}$ by using Eq. (33b) and for nonclassically damped system, the approximate eigenvalue is $(\mathbf{B}_R^{(k-1)})^{-1}\mathbf{A}_R^{(k-1)}$.
(5) Construct the transformation matrices in each substructure using Eq. (39).
(6) Construct the reduced system matrices by using Eqs. (40a) and (40b).
(7) Solve for the eigenproblem of the reduced system by using Eq. (44).
(8) Check the convergence by using the following convergent criterion:

$$\frac{\left|\Lambda_i^{(k)} - \Lambda_i^{(k-1)}\right|}{\left|\Lambda_i^{(k)}\right|} \leqslant \varepsilon_1(\text{undamped}), \quad \frac{\left|\tilde{\Omega}_i^{(k)} - \tilde{\Omega}_i^{(k-1)}\right|}{\left|\tilde{\Omega}_i^{(k)}\right|} \leqslant \varepsilon_2(\text{nonclassically damped}), \quad i = 1, 2, \ldots, m \tag{53}$$

where $\varepsilon_1$ and $\varepsilon_2$ represents the relative errors.
(9) If $m$ eigenvalues converge, exit the iteration steps. If not converged, update the transformation matrices and approximate eigenvalue using Eqs. (39) and (40), and repeat steps (4)–(7) until the convergent criterion is satisfied.

## 5. Discussion on the convergence

The two substructuring schemes derived in Section 4 are exactly the same as the IIRS method of Friswell [6] and Qu [8], respectively. This implies that the eigenvalues and eigenvectors of the substructuring reduction scheme are the same as the eigensolutions obtained from the single domain reduction method if the selected master degrees of freedom are identical. Since the transformation matrix is the relation between the master and the slave degrees of freedom, the information of the slave degrees of freedom in each substructure can be transferred to the global master degrees of freedom. Thus, the present substructuring scheme can be identical to the previous iterated IRS methods of Section 2. Fig. 2 shows the simple cantilever beam structure and the

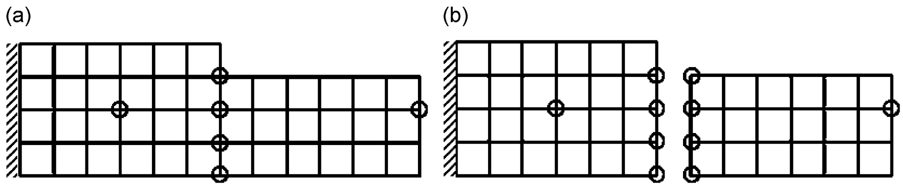


Fig. 2. A simple cantilever beam and the selection of master degrees of freedom in full domain and in each substructure ($E = 4\,\mathrm{MPa}$, $\rho = 2800\,\mathrm{kg/m^3}$, $\nu = 0.3$). (a) Full domain and (b) two-substructures.

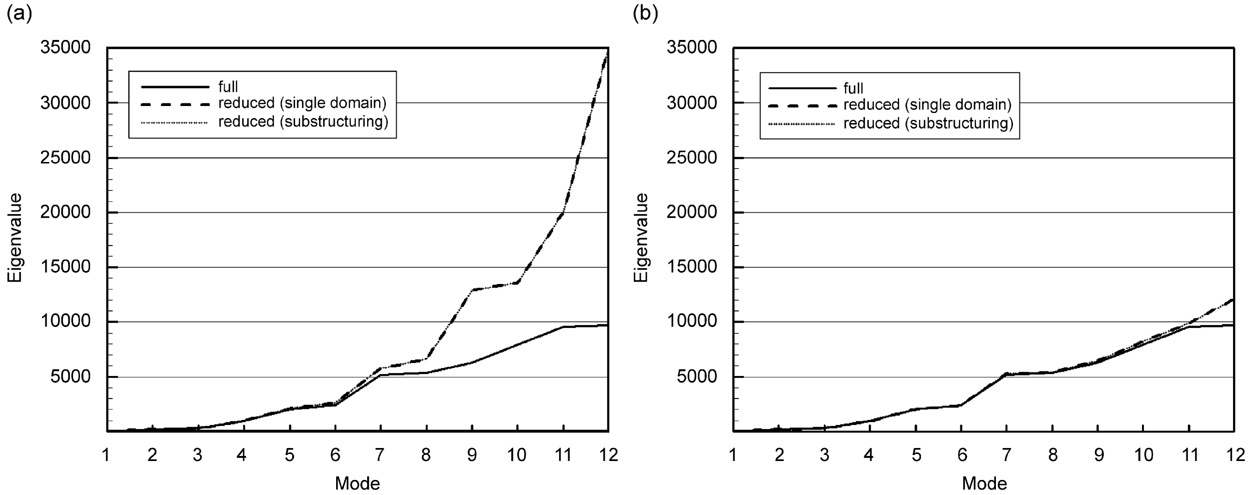


Fig. 3. Comparison of eigenvalues from the single reduced system and substructuring. (a) 0th iteration and (b) 10th iteration.

same selection of master degrees of freedom in full domain and in two substructures. The results of Fig. 3 show that the eigenvalues calculated from the two methods are exactly same. The present substructuring reduction scheme is also applicable to nonclassically damped system in the same manner. A proof that the reduced model reproduces the lower eigenproperties of the full system is given in the Appendix.

## 6. Numerical examples

To illustrate the convergence and effectiveness of the proposed method, numerical examples for both systems are considered. In the examples, the "converge" implies that the eigenproperties calculated from the reduced system are approaching to those obtained from the global system. Therefore, the absolute relative error of modal frequency both in undamped system and in nonclassically damped system is defined as

$$\text{relative error} : \varepsilon_\omega = \frac{|\omega_{\text{reduced}} - \omega_{\text{full}}|}{\omega_{\text{full}}} \times 100 \tag{54}$$

In Eq. (54), $\omega_{\text{full}}$, and $\omega_{\text{reduced}}$ are the modal frequencies calculated from global system and reduced system, respectively.

Especially for nonclassically damped systems, there are a few things to be considered. First, it needs to assume that the damping matrices in different parts are proportional to their corresponding stiffness matrices using different proportionality constants. At this time, the proportionality constants are selected properly at each part. If the proportionality constants are chosen improperly, the modes become overdamped, i.e., the corresponding eigenvalues have zero imaginary parts. In this case the present method cannot be applied because the eigenvalues calculated from reduced matrix are not converged to those of the global system. Thus, it is necessary to check the damping ratio of the system before. The damping matrices are constructed as follows:

$$\mathbf{C}_i = \gamma_i * \mathbf{K}_i, \quad i = 1, 2, 3, \ldots, N \text{ (no sum on } i) \tag{55}$$

In Eq. (55), $\mathbf{C}_i$, $\mathbf{K}_i$ and $\gamma_i$ are the damping, stiffness matrix, and proportionality constant of $i$th substructure, respectively. Because all eigenvalues are in complex conjugate pairs, only one value of each pair is considered. The eigenvalue corresponding to the $i$th mode is denoted as

$$\tilde{\Omega}_{ii} = -\xi_i \omega_i \pm \mathrm{i}\omega_{Di} \tag{56}$$

And the equivalent natural frequency ($\omega_i$), damping natural frequency ($\omega_{Di}$), and damping ratio ($\xi_i$) can be obtained as follows:

$$\omega_i = |\tilde{\Omega}_i|, \quad \omega_{Di} = \omega_i\sqrt{(1-\xi_i^2)}, \quad \xi_i = \frac{-\text{real}(\tilde{\Omega}_i)}{|\tilde{\Omega}_i|} \tag{57}$$

Thus, the absolute relative error of damping ratio for each mode can be defined as

$$\text{relative error} : \varepsilon_\xi = \frac{|\xi_{\text{reduced}} - \xi_{\text{full}}|}{\xi_{\text{full}}} \times 100 \tag{58}$$

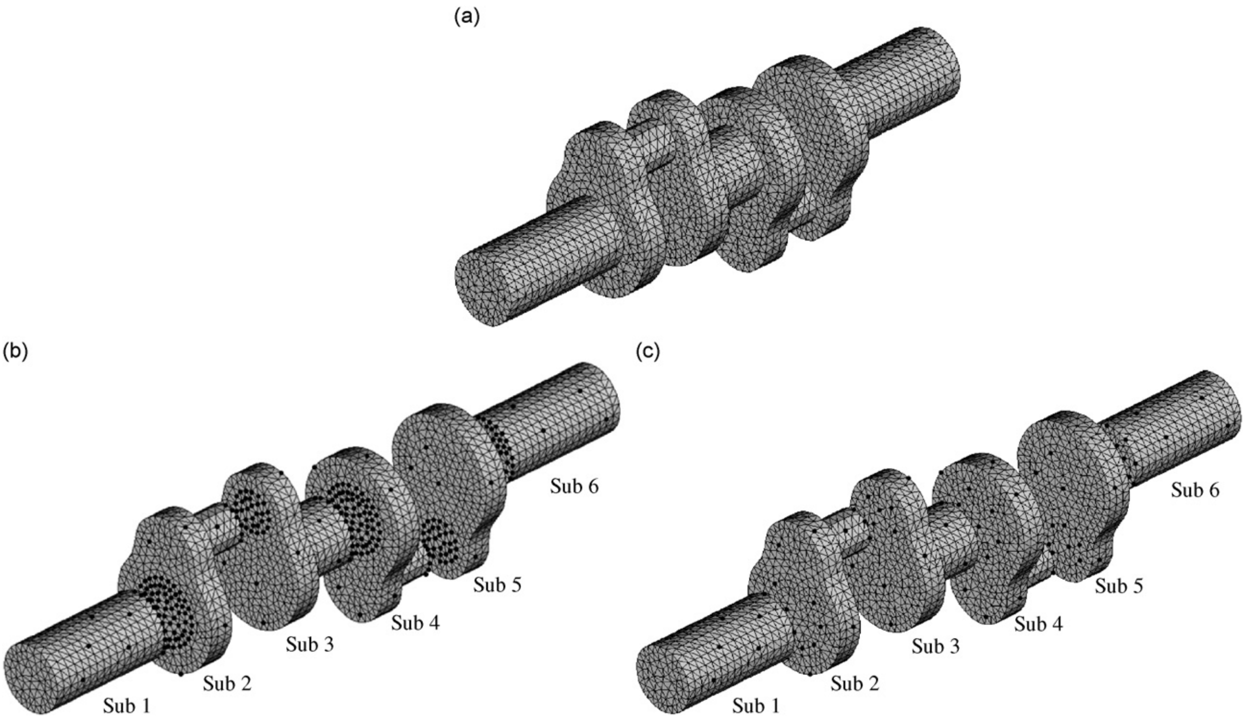


Fig. 4. Finite element model of the camshaft and the selection of master dofs in each substructure ($E = 83$ MPa, $\rho = 7000$ kg/m$^3$, $\nu = 0.3$). (a) Finite element model of the camshaft, (b) master dofs in the first reduced system and (c) master dofs in the second reduced system.

Table 1
Comparison of the number of dof in the full system and in the subsystem and the size of transformation matrix of the camshaft model

| | Total dof | Master dof | Slave dof | Interface dof | Transformation matrix |
|---|---|---|---|---|---|
| Full system | 22,323 | 1053 | 21,270 | 0 | [21,270 × 1053] |
| Subsystem | | | | | |
| Sub-1 | 3318 | 30 | 3063 | 225 | [3318 × 1053] |
| Sub-2 | 3747 | 30 | 3402 | 315 | [3747 × 1053] |
| Sub-3 | 5130 | 45 | 4767 | 318 | [5130 × 1053] |
| Sub-4 | 4077 | 30 | 3729 | 318 | [4077 × 1053] |
| Sub-5 | 3495 | 30 | 3150 | 315 | [3495 × 1053] |
| Sub-6 | 3414 | 30 | 3159 | 225 | [3414 × 1053] |

Table 2
First 20 modal frequencies of the camshaft model in the first condensation step

| Iteration/mode | Frequency (rad/s) | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
| 0 | 408.7680 | 413.7540 | 653.8955 | 896.6588 | 1167.9451 | 1271.5427 | 1393.3177 | 1518.5835 | 1520.0756 | 1829.9331 |
| 1 | | 413.7540 | 653.8955 | 896.6588 | 1167.9451 | 1271.5427 | | 1518.5835 | 1520.0756 | 1829.9330 |
| 2 | | | | | | 1271.5427 | | | | |
| Exact | 408.7684 | 413.7536 | 653.8955 | 896.6588 | 1167.9452 | 1271.5421 | 1393.3177 | 1518.6029 | 1520.0560 | 1829.9330 |
| | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 |
| 0 | 2107.4253 | 2710.7123 | 3141.5663 | 3532.1674 | 3973.4039 | 4308.7223 | 4344.7276 | 4377.8047 | 4462.2191 | 5365.4190 |
| 1 | 2107.4252 | 2710.7117 | 3141.5647 | 3532.1633 | 3973.4011 | 4308.7090 | 4344.7178 | 4377.7835 | 4462.2043 | 5365.2495 |
| 2 | | | 3141.5646 | 3532.1626 | 3973.4010 | 4308.7076 | 4344.7177 | 4377.7808 | 4462.2040 | 5365.2474 |
| Exact | 2107.4251 | 2710.7117 | 3141.5643 | 3532.1622 | 3973.4005 | 4308.7119 | 4344.7156 | 4377.7740 | 4462.1998 | 5365.2131 |

Table 3
Percent errors in modal frequencies in the first condensation step

| Iteration/mode | Percent error | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
| 0 | 0.0001 | 0.0001 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0013 | 0.0013 | 0.0000 |
| 1 | | | | | | | | 0.0013 | 0.0013 | |
| 2 | | | | | | | | 0.0013 | 0.0013 | |
| | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 |
| 0 | 0.0000 | 0.0000 | 0.0001 | 0.0001 | 0.0001 | 0.0002 | 0.0003 | 0.0007 | 0.0005 | 0.0038 |
| 1 | | | 0.0000 | 0.0000 | 0.0000 | 0.0001 | 0.0001 | 0.0002 | 0.0001 | 0.0007 |
| 2 | | | | | | 0.0001 | 0.0001 | 0.0002 | 0.0001 | 0.0006 |

Table 4
First 20 modal frequencies of the camshaft model in the second condensation step

| Iteration/mode | Frequency (rad/s) | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
| 0 | 408.7680 | 413.7540 | 653.8955 | 896.6588 | 1167.9452 | 1271.5427 | 1393.3177 | 1518.5836 | 1520.0757 | 1829.9331 |
| 1 | | 413.7540 | 653.8955 | 896.6588 | 1167.9451 | 1271.5427 | | 1518.5835 | 1520.0756 | 1829.9331 |
| 2 | | | | | 1167.9451 | 1271.5427 | | 1518.5835 | 1520.0756 | 1829.9331 |
| Exact | 408.7684 | 413.7536 | 653.8955 | 896.6588 | 1167.9452 | 1271.5421 | 1393.3177 | 1518.6029 | 1520.0560 | 1829.9330 |
| | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 |
| 0 | 2107.4259 | 2710.7132 | 3141.5716 | 3532.2235 | 3973.4351 | 4308.7358 | 4344.7636 | 4377.8208 | 4462.2348 | 5365.4726 |
| 1 | 2107.4254 | 2710.7122 | 3141.5663 | 3532.1850 | 3973.4116 | 4308.7175 | 4344.7313 | 4377.7953 | 4462.2108 | 5365.3076 |
| 2 | 2107.4253 | 2710.7120 | 3141.5657 | 3532.1787 | 3973.4073 | 4308.7139 | 4344.7262 | 4377.7904 | 4462.2078 | 5365.2811 |
| Exact | 2107.4251 | 2710.7117 | 3141.5643 | 3532.1622 | 3973.4005 | 4308.7119 | 4344.7156 | 4377.7740 | 4462.1988 | 5365.2131 |

### *6.1. Undamped system—camshaft*

A camshaft model using tetrahedron element is shown in Fig. 4(a). The camshaft is constrained at both end sides and it contains a total of 7441 nodes, 34,095 elements, and 22,323 degrees of freedom. To apply the

Table 5
Percent errors in modal frequencies in the second condensation step

| Iteration/mode | Percent error | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
| 0 | 0.0001 | 0.0001 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0013 | 0.0013 | 0.0000 |
| 1 | | | | | | | | 0.0013 | 0.0013 | |
| 2 | | | | | | | | 0.0013 | 0.0013 | |
| | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 |
| 0 | 0.0000 | 0.0001 | 0.0002 | 0.0017 | 0.0009 | 0.0006 | 0.0011 | 0.0011 | 0.0008 | 0.0048 |
| 1 | | 0.0000 | 0.0001 | 0.0006 | 0.0003 | 0.0001 | 0.0004 | 0.0005 | 0.0003 | 0.0018 |
| 2 | | | 0.0000 | 0.0005 | 0.0002 | 0.0000 | 0.0002 | 0.0004 | 0.0002 | 0.0013 |

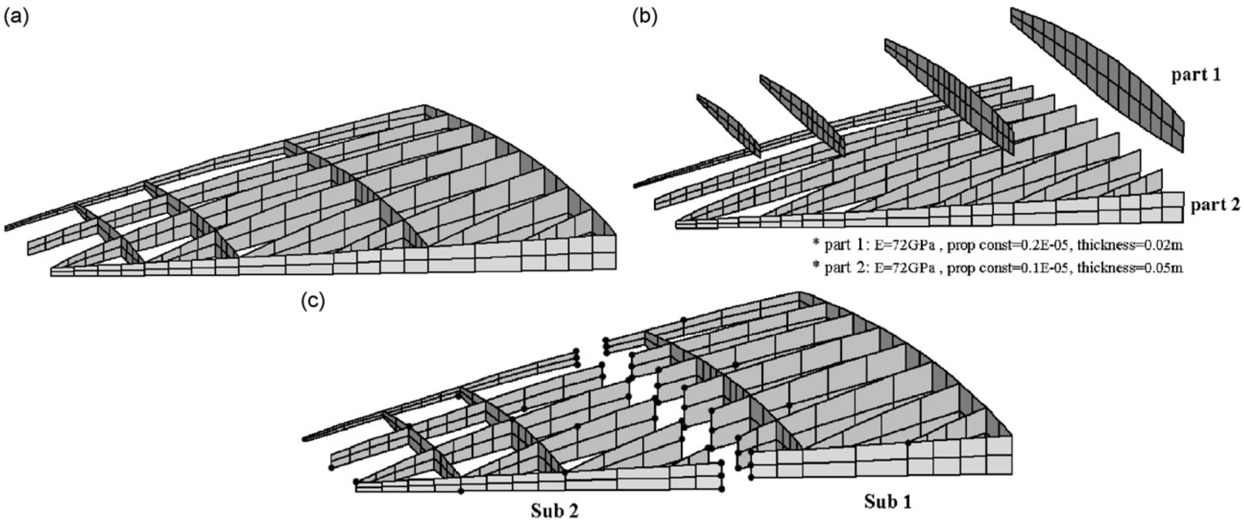


Fig. 5. Finite element model of the aircraft wing and the selection of master dofs in each substructure ($E = 72\,\text{GPa}$, $\rho = 2800\,\text{kg/m}^3$, $v = 0.3$). (a) Finite element model of the aircraft wing, (b) material properties in each part (rib, spar) and (c) master degrees of freedom in each substructure.

substructuring technique to the camshaft model, the whole system is divided into six substructures. As shown in Fig. 4(b) the total amount of 1053 arbitrary degrees of freedom is selected as the master degrees of freedom including the interface degrees of freedom. The final reduced system is just 4.7% of the global system. Table 1 shows the number of degrees of freedom and the size of transformation matrices in the full system and in each subsystem. It can clearly be seen that the size of the full system has reduced to the size of each subsystem. Table 2 shows the first 20 modal frequencies calculated from the reduced system in the first condensation step. The modal frequencies converge to the global ones as the iteration continues. Table 3 shows the percent error of modal frequencies. The highly accurate modal frequencies with the relative errors within 0.0013% are obtained.

Fig. 2(c) shows the results of selection of the master degrees of freedom in further condensation step. The total amount of 526 arbitrary degrees of freedom is selected as the master degrees of freedom. Thus, the reducing ratio is 2.4% of the full system. Tables 4 and 5 represent the first 20 modal frequencies of the second reduced matrices and percent errors, respectively. Reliable modal frequencies within the required error bound are also obtained. A little lower values in errors show in comparison with the ones in the first step.

Table 6
Comparison of the number of dof in the full system and in the subsystem and the size of transformation matrix of the aircraft wing model (in state space)

| | Total dof | Master dof | Slave dof | Interface dof | Transformation matrix |
|---|---|---|---|---|---|
| Full system | 5040 | 420 | 4620 | 0 | [4620 × 420] |
| Subsystem | | | | | |
| Sub-1 | 2664 | 48 | 2364 | 252 | [2664 × 420] |
| Sub-2 | 2628 | 120 | 2256 | 252 | [2628 × 420] |

Table 7
First 20 modal frequencies and damping ratios of the aircraft wing model

| Iteration/mode | Frequency (rad/s) | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
| 0 | 11.9917 | 25.3007 | 26.7524 | 27.8205 | 33.6815 | 38.3294 | 39.0463 | 40.0256 | 40.6790 | 41.9520 |
| 1 | 11.9833 | 25.2715 | 26.7088 | 27.5503 | 33.3124 | 37.4382 | 38.1365 | 39.1713 | 39.7569 | 40.8115 |
| 2 | 11.9834 | 25.2766 | 26.7083 | 27.5435 | 33.3058 | 37.3605 | 38.0422 | 39.1306 | 39.6898 | 40.6950 |
| 3 | 11.9833 | 25.2712 | 26.7082 | 27.5433 | 33.3056 | 37.3577 | 38.0383 | 39.1283 | 39.6858 | 40.6916 |
| 5 | 11.9832 | 25.2721 | 26.7093 | 27.5433 | 33.3056 | 37.3562 | 38.0361 | 39.1268 | 39.6837 | 40.6887 |
| 10 | 11.9832 | 25.2712 | 26.7081 | 27.5432 | 33.3054 | 37.3546 | 38.0337 | 39.1257 | 39.6820 | 40.6860 |
| Exact | 11.9832 | 25.2712 | 26.7081 | 27.5429 | 33.3052 | 37.3498 | 38.0276 | 39.1230 | 39.6778 | 40.6764 |
| | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 |
| 0 | 44.9422 | 48.2925 | 55.9675 | 57.3443 | 59.3721 | 59.6982 | 60.7474 | 61.0709 | 61.3717 | 63.4285 |
| 1 | 43.0959 | 44.3769 | 51.2293 | 52.5587 | 53.9274 | 54.7265 | 54.9337 | 55.2709 | 56.3579 | 57.2858 |
| 2 | 42.5990 | 44.3442 | 51.0749 | 51.4761 | 52.3348 | 53.6590 | 54.2972 | 54.5334 | 54.8832 | 55.6032 |
| 3 | 42.6746 | 44.3413 | 51.0649 | 51.2056 | 52.3191 | 53.6383 | 53.8604 | 54.5010 | 54.8483 | 55.5078 |
| 5 | 42.8580 | 44.3430 | 51.0603 | 51.1585 | 52.3142 | 53.6217 | 53.6671 | 54.4802 | 54.8320 | 55.4762 |
| 10 | 42.6488 | 44.3396 | 51.0555 | 51.1199 | 52.3083 | 53.4850 | 53.6264 | 54.4585 | 54.8141 | 55.4425 |
| Exact | 42.6279 | 44.3377 | 50.9847 | 51.0471 | 52.2925 | 53.1754 | 53.6073 | 54.4202 | 54.7762 | 55.3381 |

| | Damping ratio ($\times 10^{-5}$) | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
| 0 | 0.6234 | 1.3900 | 1.5102 | 1.6038 | 1.9888 | 2.0846 | 2.1804 | 2.2995 | 2.3897 | 2.5307 |
| 1 | 0.6532 | 1.2794 | 1.5829 | 1.6509 | 1.9776 | 2.0715 | 2.1631 | 2.1865 | 2.3033 | 2.4126 |
| 2 | 0.1683 | 2.7050 | 4.3951 | 1.6129 | 1.9648 | 2.0360 | 2.0796 | 2.2187 | 2.2747 | 2.3886 |
| 3 | 0.7351 | 1.5369 | 1.4723 | 1.5908 | 1.9468 | 2.0122 | 2.1511 | 2.2566 | 2.2768 | 2.4598 |
| 5 | 0.6270 | 2.0031 | 3.8570 | 1.5945 | 1.9484 | 1.9891 | 2.0835 | 2.2559 | 2.3415 | 2.3741 |
| 10 | 0.5845 | 1.3824 | 1.5011 | 1.5968 | 1.9592 | 2.0269 | 2.0887 | 2.2165 | 2.2820 | 2.4237 |
| Exact | 0.6229 | 1.3889 | 1.5092 | 1.5939 | 1.9601 | 2.0324 | 2.1163 | 2.2488 | 2.2930 | 2.4472 |
| | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 |
| 0 | 2.4418 | 2.7501 | 3.2236 | 3.1540 | 3.0792 | 3.1201 | 3.1581 | 3.1493 | 3.2995 | 3.5237 |
| 1 | 2.4910 | 2.3739 | 2.9944 | 2.8540 | 2.8577 | 2.8903 | 2.9817 | 2.8495 | 2.9753 | 2.1029 |
| 2 | 2.8632 | 2.6109 | 2.9509 | 1.9461 | 2.8425 | 2.8061 | 0.7996 | 2.7475 | 2.6590 | 2.4417 |
| 3 | 2.2776 | 2.3944 | 2.9105 | 1.8919 | 2.9019 | 2.7964 | 0.8579 | 2.7311 | 2.6728 | 2.4260 |
| 5 | 2.2745 | 2.2883 | 2.9616 | 1.9467 | 2.7903 | 2.4640 | 1.4078 | 2.7507 | 2.7028 | 2.4462 |
| 10 | 2.4242 | 2.3984 | 2.9419 | 2.0581 | 2.8419 | 1.2937 | 2.8278 | 2.7520 | 2.7286 | 2.5096 |
| Exact | 2.4548 | 2.4019 | 2.7409 | 2.9710 | 2.8607 | 2.7039 | 2.8517 | 2.8228 | 2.8199 | 2.8912 |

Table 8
Percent errors in modal frequencies of the aircraft wing model

| Iteration/mode | Percent error | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
| 0 | 0.0709 | 0.1166 | 0.1656 | 0.9979 | 1.1171 | 2.5558 | 2.6090 | 2.2550 | 2.4612 | 3.0406 |
| 1 | 0.0001 | 0.0010 | 0.0029 | 0.0267 | 0.0216 | 0.2361 | 0.2857 | 0.1232 | 0.1989 | 0.3310 |
| 2 | 0.0012 | 0.0214 | 0.0009 | 0.0023 | 0.0018 | 0.0285 | 0.0386 | 0.0194 | 0.0300 | 0.0457 |
| 3 | 0.0002 | 0.0001 | 0.0004 | 0.0016 | 0.0012 | 0.0210 | 0.0282 | 0.0134 | 0.0201 | 0.0373 |
| 5 | 0.0000 | 0.0033 | 0.0047 | 0.0013 | 0.0011 | 0.0172 | 0.0225 | 0.0096 | 0.0149 | 0.0302 |
| 10 | 0.0001 | 0.0002 | 0.0002 | 0.0009 | 0.0007 | 0.0129 | 0.0160 | 0.0069 | 0.0104 | 0.0235 |
| | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 |
| 0 | 5.1494 | 8.1893 | 8.9030 | 10.9814 | 11.9241 | 10.9263 | 11.7538 | 10.8902 | 10.7469 | 12.7551 |
| 1 | 1.0860 | 0.0884 | 0.4774 | 2.8760 | 3.0318 | 2.8341 | 2.4145 | 1.5392 | 2.8065 | 3.3998 |
| 2 | 0.0679 | 0.0148 | 0.1767 | 0.8335 | 0.0809 | 0.9012 | 1.2707 | 0.2077 | 0.1950 | 0.4767 |
| 3 | 0.1100 | 0.0082 | 0.1570 | 0.3096 | 0.0509 | 0.8629 | 0.4699 | 0.1484 | 0.1315 | 0.3056 |
| 5 | 0.5369 | 0.0120 | 0.1482 | 0.2178 | 0.0415 | 0.8321 | 0.0115 | 0.1102 | 0.1018 | 0.2490 |
| 10 | 0.0491 | 0.0043 | 0.1386 | 0.1424 | 0.0303 | 0.5787 | 0.0357 | 0.0704 | 0.0692 | 0.1883 |

### *6.2. Nonclassically damped system—aircraft wing*

A simple aircraft wing clamped along wing root section, shown in Fig. 5(a), is considered. The Aminpour's shell element with 6 degrees of freedom per node is used. The model contains a total of 420 nodes, 316 elements, and 2520 degrees of freedom. Thus, the size of system matrices **A** and **B** in state space is [$5040 \times 5040$], respectively. In this example, as shown in Fig. 5(b), the model is divided into two different parts, which are wing rib and spar. They have different thickness and proportionality constant. The global structure is divided into two substructures. Fig. 5(c) shows the result of selection of master degrees of freedom including interface degrees of freedom. A total of 210 randomly distributed degrees of freedom are selected as master degrees of freedom out of each substructure. This nonclassically damped eigenvalue problem is condensed to a ratio of 8.3% from the full system. Table 6 represents the number of degrees of freedom and the size of transformation matrices in full system and in each subsystem.

The results in Table 7 clearly indicate that all modal frequencies and damping ratios are converged. The accuracy of the solutions is increased by making more iteration. After the ten iteration steps the first 20 modal frequencies have relative errors less than 0.5787% as shown in Table 8.

## 7. Conclusion

An iterated IRS method combined with a substructuring scheme is presented for efficient eigenanalysis. The key point of the present method is on the iterative update of the transformation matrix from the global degrees of freedom to the selected master degrees of freedom in each substructure. In particular, the present method is effectively applicable to the dynamic analysis for large structures even under the environment of limited computer storage. Numerical examples for undamped and nonclassically damped structural problems demonstrated the convergence and accuracy of the present method. In the present method, the reduced system is expressed as the degrees of freedom of the finite element model including physical information. Thus it can be very useful in the repeated analysis of dynamic problems such as vibration analysis and control, system identification and structural optimization. The present algorithm is very efficient to the problem in two-dimensional configurations such as plate and shell structures since this kind problem has relatively small number of interface degrees of freedom at the interfaces between adjacent substructures. However, the present substructuring method is not so efficient for the problem with a large number of degrees of freedom at the interface between substructures such as three-dimensional solid problems. More research work is required to reduce the interface degrees of freedom without using large-sized memory storage in the present substructuring reduction method.

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## Appendix

From Eq. (30) it can be shown that the reduced model constructed by the substructuring scheme reproduces the lower eigenvalues and their associated eigenvectors of the original system. The transformation matrix for the substructuring is

$$\mathbf{T} = \begin{bmatrix} \mathbf{t}_{(1)} \\ \mathbf{I}_m \\ \mathbf{t}_{(2)} \end{bmatrix} \tag{A.1}$$

And the eigenvector estimated by using the converged transformation matrix $\mathbf{T}$ is

$$\mathbf{\Phi} = \begin{bmatrix} \mathbf{\Phi}_s^{(1)} \\ \mathbf{\Phi}_m \\ \mathbf{\Phi}_s^{(2)} \end{bmatrix} = \begin{bmatrix} \mathbf{t}_{(1)} \\ \mathbf{I}_m \\ \mathbf{t}_{(2)} \end{bmatrix} \mathbf{\Phi}_m = \mathbf{T}\mathbf{\Phi}_m \tag{A.2}$$

Since $\mathbf{\Phi}_m$ is an eigenvector of the reduced system

$$\mathbf{M}_R^{-1}\mathbf{K}_R\mathbf{\Phi}_m = \Lambda_m\mathbf{\Phi}_m \tag{A.3}$$

And the slave degrees of freedom field in each substructure can be expressed as

$$\begin{aligned} \mathbf{\Phi}_s^{(1)} &= \mathbf{t}_{(1)}\mathbf{\Phi}_m = \left[-\left(\mathbf{K}_{ss}^{(1)}\right)^{-1}\mathbf{K}_{sm}^{(1)} + \Lambda_m\left(\mathbf{K}_{ss}^{(1)}\right)^{-1}\left(\mathbf{M}_{sm}^{(1)} + \mathbf{M}_{ss}^{(1)}\mathbf{t}_{(1)}\right)\right]\mathbf{\Phi}_m \\ \mathbf{\Phi}_s^{(2)} &= \mathbf{t}_{(2)}\mathbf{\Phi}_m = \left[-\left(\mathbf{K}_{ss}^{(2)}\right)^{-1}\mathbf{K}_{sm}^{(2)} + \Lambda_m\left(\mathbf{K}_{ss}^{(2)}\right)^{-1}\left(\mathbf{M}_{sm}^{(2)} + \mathbf{M}_{ss}^{(2)}\mathbf{t}_{(2)}\right)\right]\mathbf{\Phi}_m \end{aligned} \tag{A.4}$$

In Eq. (A.4), premultiplying $\mathbf{K}_{ss}^{(1)}$ and $\mathbf{K}_{ss}^{(2)}$ in each equation

$$\begin{aligned} \mathbf{K}_{ss}^{(1)}\mathbf{\Phi}_s^{(1)} &= -\mathbf{K}_{sm}^{(1)}\mathbf{\Phi}_m + \Lambda_s\mathbf{M}_{sm}^{(1)}\mathbf{\Phi}_m + \Lambda_s\mathbf{M}_{ss}^{(1)}\mathbf{\Phi}_s^{(1)} \\ \mathbf{K}_{ss}^{(2)}\mathbf{\Phi}_s^{(2)} &= -\mathbf{K}_{sm}^{(2)}\mathbf{\Phi}_m + \Lambda_s\mathbf{M}_{sm}^{(2)}\mathbf{\Phi}_m + \Lambda_s\mathbf{M}_{ss}^{(2)}\mathbf{\Phi}_s^{(2)} \end{aligned} \tag{A.5}$$

Rearranging gives

$$\Lambda_s\left[\mathbf{M}_{sm}^{(1)} \quad \mathbf{M}_{ss}^{(1)} + \mathbf{M}_{ss}^{(2)} \quad \mathbf{M}_{sm}^{(2)}\right]\begin{Bmatrix} \mathbf{\Phi}_s^{(1)} \\ \mathbf{\Phi}_m \\ \mathbf{\Phi}_s^{(2)} \end{Bmatrix} = \left[\mathbf{K}_{sm}^{(1)} \quad \mathbf{K}_{ss}^{(1)} + \mathbf{K}_{ss}^{(2)} \quad \mathbf{K}_{sm}^{(2)}\right]\begin{Bmatrix} \mathbf{\Phi}_s^{(1)} \\ \mathbf{\Phi}_m \\ \mathbf{\Phi}_s^{(2)} \end{Bmatrix} \tag{A.6}$$

From Eq. (A.6), the slave eigenvalues and associated eigenvectors are obtained.
Since $\Lambda_m$ is an eigenvalue of the reduced system with eigenvector $\mathbf{\Phi}_m$

$$\Lambda\mathbf{T}^{\mathrm{T}}\mathbf{M}\mathbf{T}\mathbf{\Phi}_m = \Lambda_m[\mathbf{t}_{(1)} \quad \mathbf{I}_m \quad \mathbf{t}_{(2)}]\mathbf{M}\begin{Bmatrix} \mathbf{\Phi}_s^{(1)} \\ \mathbf{\Phi}_m \\ \mathbf{\Phi}_s^{(2)} \end{Bmatrix} = [\mathbf{t}_{(1)} \quad \mathbf{I}_m \quad \mathbf{t}_{(2)}]\mathbf{K}\begin{Bmatrix} \mathbf{\Phi}_s^{(1)} \\ \mathbf{\Phi}_m \\ \mathbf{\Phi}_s^{(2)} \end{Bmatrix} = \Lambda_m\mathbf{T}^{\mathrm{T}}\mathbf{K}\mathbf{T}\mathbf{\Phi}_m \tag{A.7}$$

Multiplying out the above equation

$$\Lambda_m\left[\mathbf{M}_{ms}^{(1)} \quad \mathbf{M}_{mm}^{(1)} + \mathbf{M}_{mm}^{(2)} \quad \mathbf{M}_{ms}^{(2)}\right]\begin{Bmatrix} \mathbf{\Phi}_s^{(1)} \\ \mathbf{\Phi}_m \\ \mathbf{\Phi}_s^{(2)} \end{Bmatrix} = \left[\mathbf{K}_{ms}^{(1)} \quad \mathbf{K}_{mm}^{(1)} + \mathbf{K}_{mm}^{(2)} \quad \mathbf{K}_{ms}^{(2)}\right]\begin{Bmatrix} \mathbf{\Phi}_s^{(1)} \\ \mathbf{\Phi}_m \\ \mathbf{\Phi}_s^{(2)} \end{Bmatrix} \tag{A.8}$$

Combining Eq. (A.6) with Eq. (A.8) gives

$$\Lambda_{s+m}\begin{bmatrix} \mathbf{M}_{ss}^{(1)} & \mathbf{M}_{sm}^{(1)} & \\ \mathbf{M}_{ms}^{(1)} & \mathbf{M}_{mm} & \mathbf{M}_{ms}^{(2)} \\ & \mathbf{M}_{sm}^{(2)} & \mathbf{M}_{ss}^{(2)} \end{bmatrix}\begin{Bmatrix} \mathbf{\Phi}_s^{(1)} \\ \mathbf{\Phi}_m \\ \mathbf{\Phi}_s^{(2)} \end{Bmatrix} = \begin{bmatrix} \mathbf{K}_{ss}^{(1)} & \mathbf{K}_{sm}^{(1)} & \\ \mathbf{K}_{ms}^{(1)} & \mathbf{K}_{mm} & \mathbf{K}_{ms}^{(2)} \\ & \mathbf{K}_{sm}^{(2)} & \mathbf{K}_{ss}^{(2)} \end{bmatrix}\begin{Bmatrix} \mathbf{\Phi}_s^{(1)} \\ \mathbf{\Phi}_m \\ \mathbf{\Phi}_s^{(2)} \end{Bmatrix}$$

$$= \Lambda\mathbf{M\Phi} = \mathbf{K\Phi} \quad \text{(A.9)}$$

From Eq. (A.9), we can easily demonstrate $\Lambda$ is an eigenvalue of the full system associated eigenvector $\mathbf{\Phi}$. In addition, the nonclassically damped system can also be derived in the same manner.

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