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Vibration amplitude control for a van der Pol–Duffing oscillator with time delay

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Abstract

Periodic solutions for the fundamental resonance response of a van der Pol–Duffing system under time-delayed position and velocity feedbacks are investigated. Using the asymptotic perturbation method, two slow-flow equations for the amplitude and phase of the fundamental resonance response are derived. Their fixed points correspond to limit cycles (phase-locked periodic solutions) for the original system. In the uncontrolled system, periodic solutions exist only for fixed values of amplitude and phase and depend on the system parameters and excitation amplitude. In many cases, the amplitudes of periodic solutions do not correspond to the technical requirements. It is demonstrated that, if the vibration control terms are added, stable periodic solutions with arbitrarily chosen amplitude can be accomplished. Therefore, the results obtained show that an effective vibration amplitude control is possible if appropriate time delay and feedback gains are chosen.

1. Introduction

It is well known that a self-excited system can exhibit remarkable complex dynamical behavior. The earliest mathematical model has been proposed by van der Pol to describe vacuum tube circuit [1] and subsequently many researchers have studied the complex phenomena observed from experiments and computer simulations (mechanical, electric, ecological, chemical systems and so on). Undesirable bifurcations, high-amplitude vibrations, quasi-periodic motion and chaotic behavior may occur and cause degradation or catastrophic failure of the system.

The task of suppressing the dangerous vibrations is very important for engineering science and bifurcation control theory has received a great deal of attention in the last years and various papers have been dedicated to the control of resonantly forced systems in various applicative fields [2]. For example, Atay [3] studied the effect of delayed position feedback on the response of a van der Pol oscillator and Hu et al. [4] considered primary resonance and the 1/3 subharmonic resonance of a forced Duffing oscillator with time delay state feedback. Using the multiple scales method, they demonstrated that appropriate choices of the feedback gains and the time delay are possible for a successful vibration control. Zhu et al. [5] applied a new stochastic

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averaging method to predict the response of a van der Pol–Duffing oscillator under both external and parametric excitation of wide-band stationary random processes. Ji and Leung [6] demonstrated that in parametrically excited Duffing systems the stable region of the trivial solution can be broadened, a discontinuous bifurcation can be transformed into a continuous one and the jump phenomenon in the response can be removed, if an appropriate feedback control is used. The same authors [7] studied the primary, subharmonic and superharmonic resonances of a Duffing system with damping under linear feedback control with two time delays.

Xu and Chung [8] discussed a van der Pol–Duffing oscillator with time-delayed position feedback and found two routes to chaos (period-doubling bifurcation and torus breaking). Kakmeni et al. [9] examined the strange attractors and chaos control in a Duffing–van der Pol oscillator with two external periodic forces. Li et al. [10] considered the response of a van der Pol–Duffing oscillator under delayed feedback control and found that unwanted multiple solutions can be prevented. It is also shown that coupled nonlinear state feedback control can be replaced by uncoupled nonlinear state feedback control.

In recent papers, a new time-delayed feedback control method for nonlinear oscillators has been proposed. The method has been used to suppress high-amplitude response and two-period quasi-periodic motion of a parametrically or externally excited van der Pol oscillator [11,12]. In particular, it has been shown that vibration control and quasi-periodic motion suppression are possible for appropriate choices of time delay and feedback gains. The method has been also applied to the primary resonance of a cantilever beam [13] and to two nonlinearly coupled and parametrically excited van der Pol oscillators [14].

In this paper, we demonstrate the existence of stable periodic solutions with arbitrarily chosen amplitudes in the fundamental resonance of a van der Pol–Duffing system, if we use a vibration control technique based on time delay and feedback gains.

The relevant nonlinear difference-differential equation is

$$\ddot{X} + \omega^2 X - [a - bX^2]\dot{X} + cX^3 = 2F\cos(\Omega t) + AX_T + B\dot{X}_T, \tag{1}$$

where $X = X(t)$, dot denotes differentiation with respect to time, $X_T = X(t-T)$, $\dot{X}_T = \dot{X}(t-T)$, T is the time delay, c is the nonlinear stiffness parameter, a is the coefficient of the steady source of energy ($a>0$), b the nonlinear damping coefficient ($b>0$), ω the natural frequency, F the fundamental excitation amplitude, A and B the feedback gains and the excitation frequency is $\Omega \approx \omega$. The control force is then represented by the two last terms of the RHS of Eq. (1) that can be considered a model of an active feedback control system. Various sensors detect the vibrations of the structure and generate the feedback signal. By means of a digital computer or an analog device, the feedback signal is examined and transformed into an appropriate output signal. Finally, the output signal arrives at the actuator, which act on the structure, thereby suppressing its unwanted vibrations. We note that time-delayed systems similar to Eq. (1) have been studied for $F = 0$ in biology [15], physics [16,17] and other fields [18–20].

The remainder of this paper is arranged as follows. In Section 2, using the asymptotic perturbation (AP) method a lowest order approximate solution of the nonlinear system (Eq. (1)) is constructed. The AP method is based on large temporal rescalings and balancing of harmonic terms with a simple iteration, and then can be considered as an attempt to link the most useful characteristics of harmonic balance and multiple scale methods [11–14]. Only a slow time scale is used and harmonics are introduced for the fast time scale. However, for the first-order approximate solution, results are identical to those obtainable with the other perturbation methods. Obviously, there may be other solutions, for example large-amplitude quasi-periodic motion or chaotic behavior, which the slow-flow equations do not describe.

We obtain two slow-flow modulation equations governing the amplitude and phases of the fundamental resonance responses. Their fixed points correspond to limit cycles (phase-locked periodic solutions) for the van der Pol–Duffing oscillator (Eq. (1)).

In Section 3, we find that in the nonlinear system (Eq. (1)) without time delay control periodic solutions exist only for certain values of the amplitude and phase (phase-locked solutions). We derive the conditions for the stability of a solution and demonstrate that the amplitudes of the stable periodic solutions can be arbitrarily chosen by appropriate choices for the feedback gains and the time delay. The best choices of the feedback gains and the time delay, from the viewpoint of vibration control, are found by analyzing the modulation equations of the amplitude and the phase. It is then demonstrated that a certain type of time-delayed feedback technique can be effective in obtaining periodic solutions with arbitrarily chosen amplitude and phase.

The paper closes with a discussion, along with some conclusions, in Section 4.

2. Perturbation analysis and approximate solution

In this section, in order to take into account the fundamental resonance, we set

$$\omega = \Omega + \varepsilon\sigma, \tag{2}$$

where ε is a bookkeeping device, which will be set equal to unity in the final analysis, and σ is the detuning parameter. In order to balance the effect of the nonlinearity and damping with the fundamental excitation, we scale the excitation coefficient F and the dissipative, nonlinear and time delay coefficients a, b, c, A, B as $\varepsilon(a, b, c, A, B, F)$. It is assumed that the requested solution $X(t)$ is of order 1.

Using Eq. (2), we rewrite Eq. (1) in the following form:

$$\ddot{X} + (\Omega^2 + 2\varepsilon\sigma\Omega + \varepsilon^2\sigma^2)X - \varepsilon((a - bX^2)\dot{X} - cX^3 + 2F\cos(\Omega t) + AX_T + B\dot{X}_T) = 0. \tag{3}$$

Modifications induced by nonlinearities and fundamental resonance are best described by the slow temporal scale,

$$\tau = \varepsilon t, \tag{4}$$

and the solution $X(t)$ of Eq. (3) can be expressed by means of a power series in the expansion parameter ε,

$$X(t) = \sum_{n=-\infty[\text{odd}]}^{+\infty} \varepsilon^{\gamma_n}\psi_n(\tau, \varepsilon)\exp(-\mathrm{i}n\Omega t), \tag{5}$$

with $\gamma_n = (|n|-1)/2$. Note that, being the assumed solution $X(t)$ real,

$$\psi_n(\tau, \varepsilon) = \psi^*_{-n}(\tau, \varepsilon). \tag{6}$$

We suppose that the functions $\psi_n(\tau, \varepsilon)$ can be expanded in power series of ε, i.e.

$$\psi_n(\tau, \varepsilon) = \sum_{i=0}^{\infty} \varepsilon^i\psi_n^{(i)}(\tau). \tag{7}$$

We have assumed in Eq. (7) that the limit of the $\psi_n(\tau;\varepsilon)$ for $\varepsilon \to 0$ exists and is finite. In the following for simplicity we use the abbreviation $\psi_1^{(0)} = \psi$ for $n = 1$. The solution is then a Fourier expansion in which the coefficients vary slowly in time and the lowest order terms correspond to the harmonic solution of the linear problem.

Evolution equations for the amplitudes of the harmonic terms are then derived by substituting the expression of the solution into the original equations and projecting onto each Fourier mode. Substituting Eqs. (5)–(7) into Eq. (3), considering the coefficients of the most important Fourier modes and collecting terms of the same power of ε yield, to order ε, for $n = 1$,

$$2\mathrm{i}\Omega\frac{\mathrm{d}\psi}{\mathrm{d}\tau} - 2\sigma\Omega\psi + \mathrm{i}\Omega b|\psi|^2\psi - \mathrm{i}\Omega a\psi - 3c|\psi|^2\psi + F + (A - \mathrm{i}B\Omega)\exp(\mathrm{i}\Omega T)\psi = 0. \tag{8}$$

As we can see from Eq. (8), we can derive a differential equation for the evolution of the complex amplitude ψ,

$$\frac{\mathrm{d}\psi}{\mathrm{d}\tau} = (\alpha_1 + \mathrm{i}\alpha_2)\psi + (\beta_1 + \mathrm{i}\beta_2)\psi + (\gamma_1 + \mathrm{i}\gamma_2)|\psi|^2\psi + \mathrm{i}\delta, \tag{9}$$

with

$$\alpha_1 = \frac{a}{2}, \quad \beta_1 = \frac{1}{2\Omega}(\Omega B \cos(\Omega T) - A \sin(\Omega T)),$$
$$\beta_2 = \frac{1}{2\Omega}(A \cos(\Omega T) + \Omega B \sin(\Omega T)), \tag{10}$$

$$\alpha_2 = -\sigma, \quad \gamma_1 = -\frac{b}{2}, \quad \gamma_2 = -\frac{3c}{2\Omega}, \quad \delta = \frac{F}{2\Omega}. \tag{11}$$

To analyze the combined effects of the nonlinearity and fundamental resonance, we substitute the polar form,

$$\psi = \rho \exp(\mathrm{i}\vartheta), \tag{12}$$

into Eq. (9), use the definitions

$$K = \sqrt{\left(\frac{B}{2}\right)^2 + \left(\frac{A}{2\Omega}\right)^2}, \quad \cos\varphi = \frac{B}{2K}, \quad \sin\varphi = \frac{A}{2\Omega K}, \tag{13}$$

separate real and imaginary parts and obtain (if $\rho \neq 0$)

$$\frac{\mathrm{d}\rho}{\mathrm{d}\tau} = \rho(\alpha_1 + \beta_1 + \gamma_1\rho^2) + \delta \sin\vartheta, \tag{14}$$

$$\frac{\mathrm{d}\vartheta}{\mathrm{d}\tau} = \alpha_2 + \beta_2 + \gamma_2\rho^2 + \frac{\delta}{\rho}\cos\vartheta, \tag{15}$$

where now

$$\beta_1 = K\cos(\Omega T + \varphi), \quad \beta_2 = K\sin(\Omega T + \varphi). \tag{16}$$

From Eqs. (14) and (15) we can see that the fundamental excitations of the uncontrolled and controlled system are essentially the same, when the coefficients α_1 (with the energy source coefficient a) and α_2 (with the detuning parameter σ) are properly substituted $\alpha_1 \to \alpha_1 + \beta_1$, $\alpha_2 \to \alpha_2 + \beta_2$.

Taking into account Eqs. (5), (6) and (12), the approximate solution of Eq. (3) can be written as a sum of a contribution of order 1 and a contribution of order ε,

$$X(t) = 2\rho(t)\cos(\Omega t - \vartheta(t)) + O(\varepsilon), \tag{17}$$

where the amplitude ρ and the phase ϑ are given by Eqs. (14) and (15).

The validity of the approximate solution should be expected to be restricted on bounded intervals of the τ-variable and then on time scale $t = O(1/\varepsilon)$. If one wishes to construct approximate solutions on larger intervals such that $\tau = O(1/\varepsilon)$, then the higher terms will in general affect the solution and must be included. Moreover, the approximate solution (Eq. (17)) will be within $O(\varepsilon)$ of the true solution on bounded intervals of the τ-variable and, for example if the solution is periodic, for all t.

3. The vibration control method

If all the feedback gains are equal to zero ($A = B = \beta_1 = \beta_2 = 0$), Eqs. (14) and (15) correspond to the modulation equations for the uncontrolled system. First of all, we recall that phase-locked periodic solutions of the nonlinear system (Eq. (3)) correspond to fixed points of the model system (Eqs. (14) and (15)), i.e. to the solutions of the equations $\mathrm{d}\rho/\mathrm{d}\tau = \mathrm{d}\vartheta/\mathrm{d}\tau = 0$. A steady-state finite-amplitude response (ρ_0, ϑ_0) exists and is given by

$$(\alpha_1 + \gamma_1\rho_0^2)^2\rho_0^2 + (\alpha_2 + \gamma_2\rho_0^2)^2\rho_0^2 = \delta^2,$$
$$\tan(\vartheta_0) = \frac{\alpha_1 + \gamma_1\rho_0^2}{\alpha_2 + \gamma_2\rho_0^2}. \tag{18}$$

We conclude that only well determined values of the amplitude are acceptable solutions and in many cases these amplitude cannot match the technical requirements. The amplitude of the phase-locked solutions depends on the system parameters and the excitation force.

In this section, we now demonstrate that arbitrary stable responses are possible if we include the vibration control terms, because the response is strongly influenced by the time delay terms and an appropriate choice of the time delay and feedback gains can modify the nonlinear dynamic characteristics and the response amplitude, avoid some kinds of dynamic behavior and perform an efficient vibration control.

We choose an arbitrary amplitude ρ_A and determine the feedback gains and time delay in such a way that it is a stable solution of the system (Eqs. (14) and (15)). We can obtain the appropriate values for the feedback gains, because for hypothesis (ρ_A) is a solution. Then, Eqs. (14) and (15) and the conditions $d\rho/d\tau = d\vartheta/d\tau = 0$ yield

$$\beta_1 = K\cos(\Omega T + \varphi) = -\alpha_1 - \gamma_1 \rho_A^2 - \frac{\delta}{\rho_A}\sin\vartheta_A, \tag{19}$$

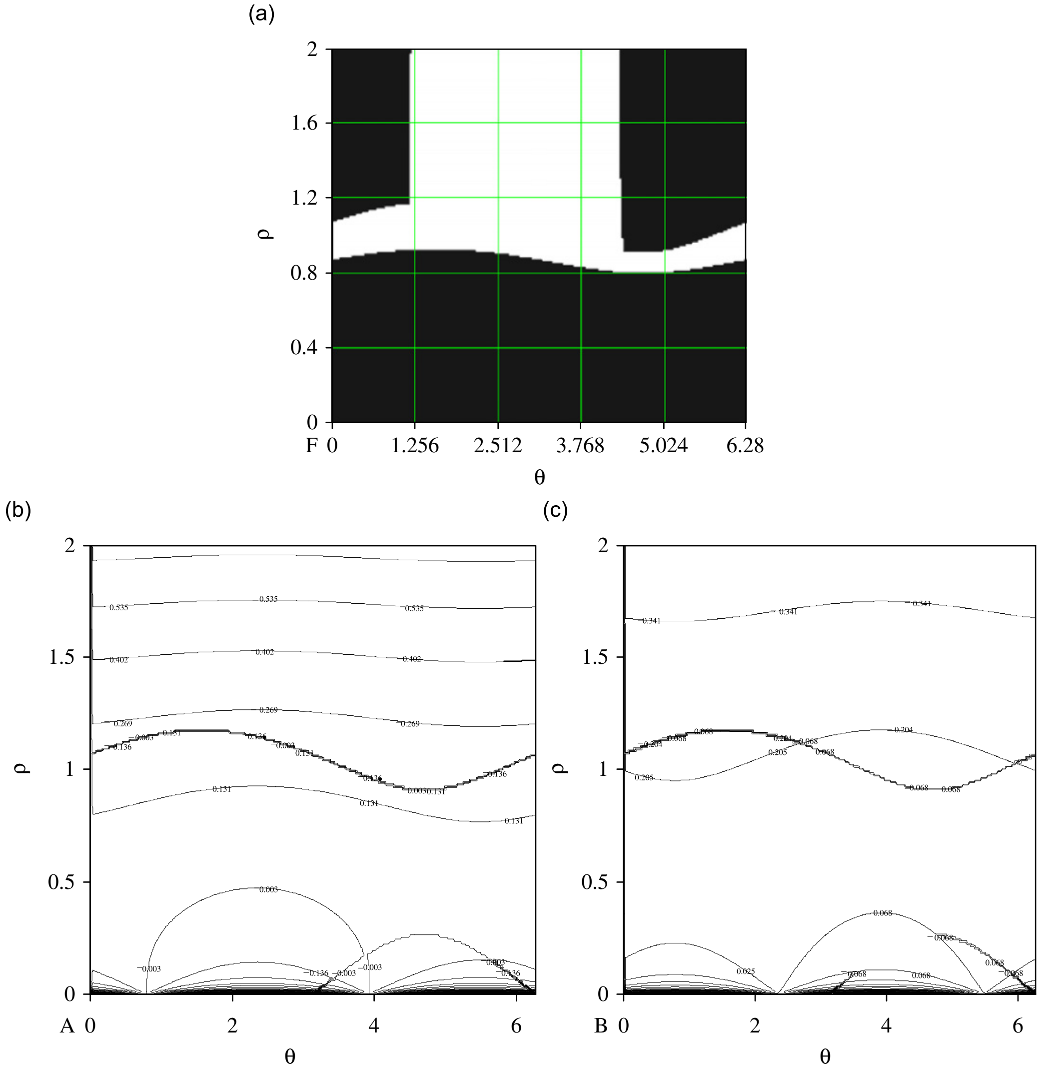

Fig. 1. Stability chart (a) in the plane (ϑ, ρ) for the nonlinear system (Eq. (3)) with the parameter values $a = 0.08$, $b = 0.07$, $c = -0.06$, $F = 0.02$, $\sigma = -0.04$, $\Omega = 1$. The phase varies from 0 to 2π and the response from 0 to 1. White (black) regions stand for stable (unstable) solutions. Values of the vibration control terms A and B (b and c) in the plane (ϑ, ρ).

$$\beta_2 = K \sin(\Omega T + \varphi) = -\alpha_2 - \gamma_2 \rho_A^2 - \frac{\delta}{\rho_A} \cos \vartheta_A, \tag{20}$$

where (ϑ_A) is the associated phase (its appropriate value will be chosen later) and then

$$K^2 = \left(\alpha_1 + \gamma_1 \rho_A^2 + \frac{\delta}{\rho_A} \sin \vartheta_A\right)^2 + \left(\alpha_2 + \gamma_2 \rho_A^2 + \frac{\delta}{\rho_A} \cos \vartheta_A\right)^2, \tag{21}$$

$$\tan \Phi = \tan(\Omega T + \varphi) = \frac{(\alpha_2 \rho_A + \gamma_2 \rho_A^3 + \delta \cos \vartheta_A)}{(\alpha_1 \rho_A + \gamma_1 \rho_A^3 + \delta \sin \vartheta_A)}. \tag{22}$$

From Eq. (22) we obtain the phase φ, if we have chosen an arbitrary value for the time delay,

$$\varphi = \Phi - \Omega T. \tag{23}$$

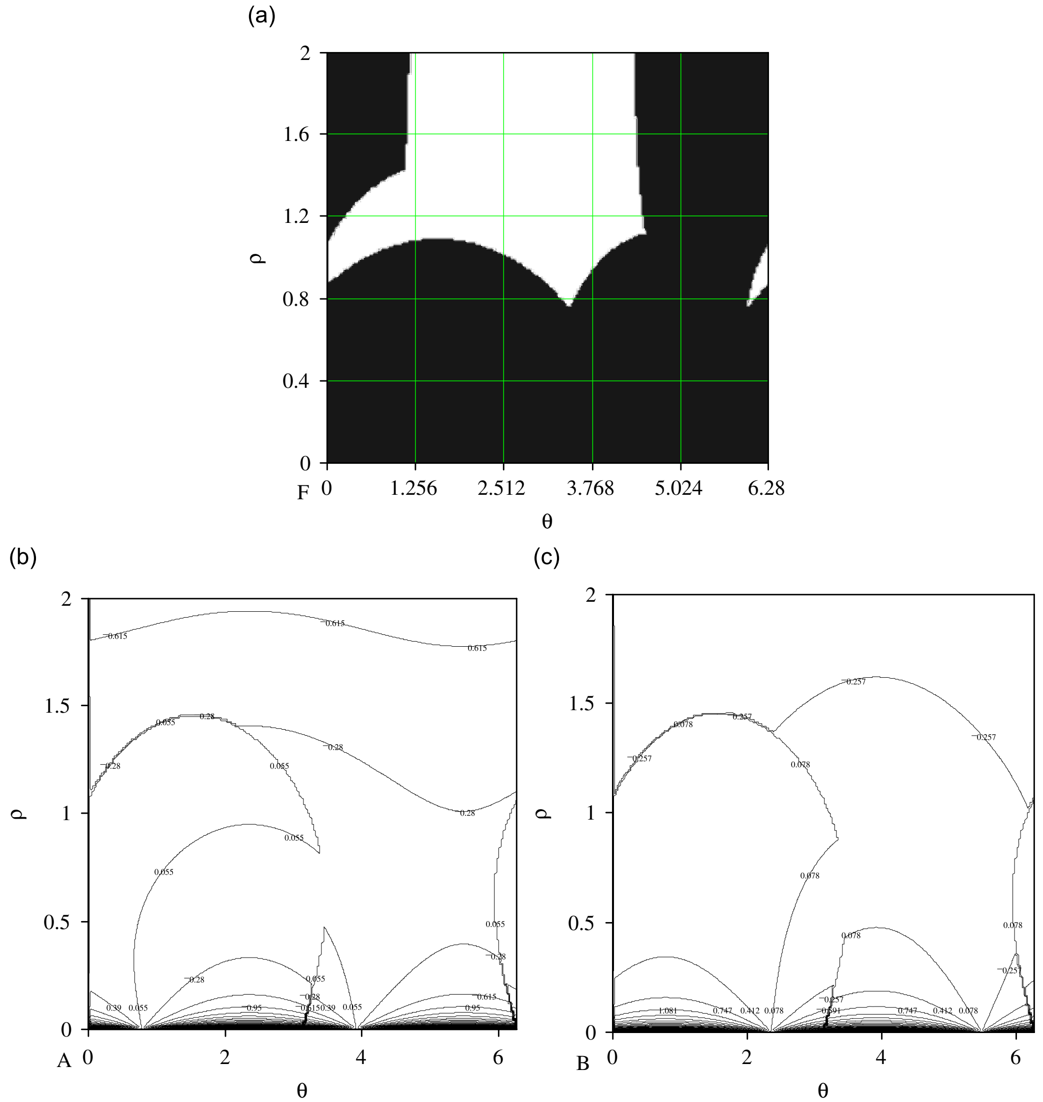

Fig. 2. Stability chart in the plane (ϑ, ρ) for the nonlinear system (Eq. (3)) with the parameter values $a = 0.08$, $b = 0.07$, $c = -0.06$, $F = 0.1$, $\sigma = -0.04$, $\Omega = 1$. The phase varies from 0 to 2π and the response from 0 to 1. White (black) regions stand for stable (unstable) solutions. Values of the vibration control terms A and B (b and c) in the plane (ϑ, ρ).

The appropriate values for A and B can be deduced from Eq. (13),

$$A = 2\Omega K \sin \varphi, \quad B = 2K \cos \varphi. \tag{24}$$

Moreover, the standard linearization method permits the computation of the eigenvalues of the Jacobian matrix relative to each equilibrium point. We superpose small perturbations in the steady-state solutions and the resulting equations are then linearized. Subsequently, we consider the eigenvalues of the corresponding system of first-order differential equations with constant coefficients (the Jacobian matrix). A positive real root indicates an unstable solution, whereas if the real parts of the eigenvalues are all negative then the steady-state solution is stable. When the real part of an eigenvalue is zero, bifurcations occur. A change from complex roots with negative real parts to complex roots with positive real parts usually indicates the presence of a Hopf bifurcation. The eigenvalues are

$$\lambda_1 = \alpha_1 + \beta_1 + 2\gamma_1\rho_A^2 + \sqrt{\Delta}, \quad \lambda_2 = \alpha_1 + \beta_1 + 2\gamma_1\rho_A^2 - \sqrt{\Delta}, \tag{25}$$

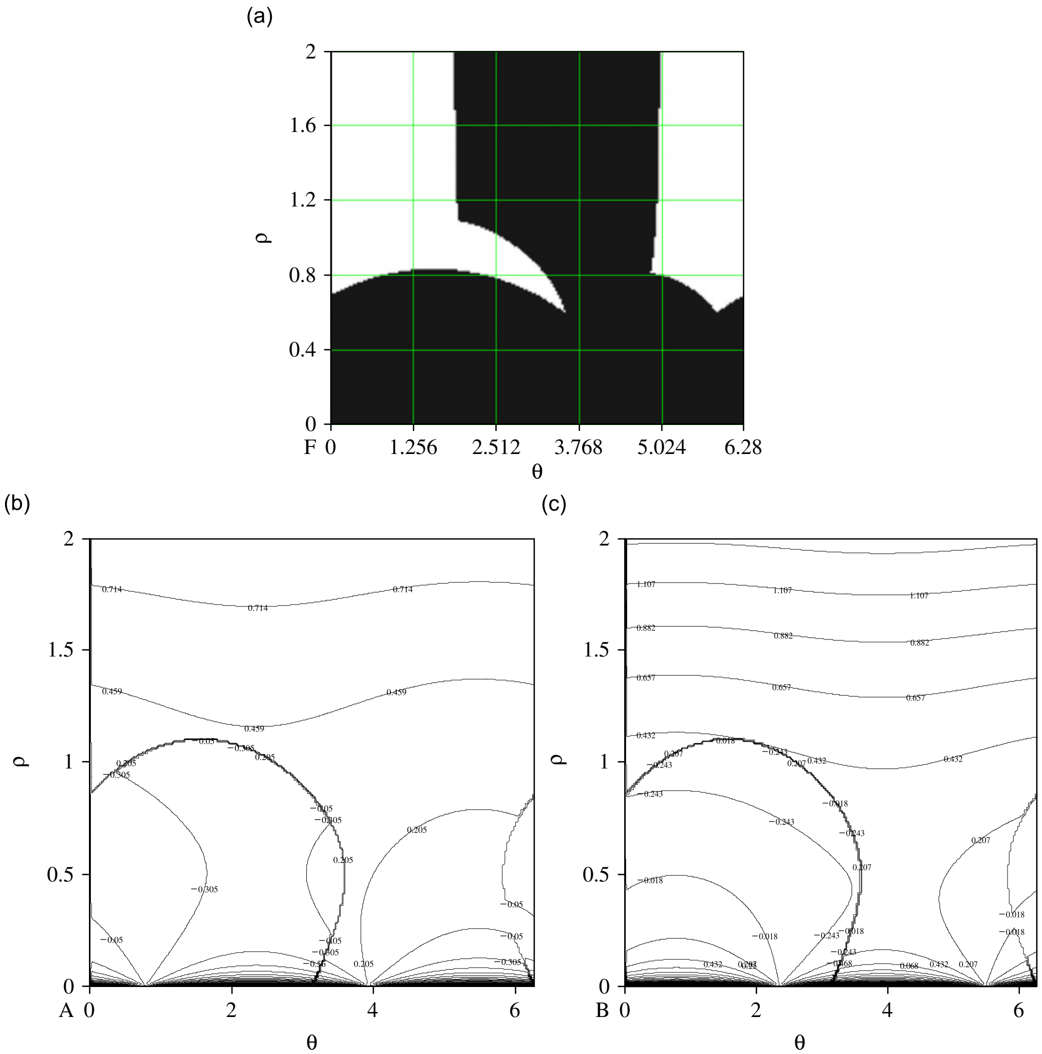

Fig. 3. Stability chart in the plane (ϑ, ρ) for the nonlinear system (Eq. (3)) with the parameter values $a = 0.08$, $b = 0.11$, $c = 0.12$, $F = 0.06$, $\sigma = 0.08$, $\Omega = 1$. The phase varies from 0 to 2π and the response from 0 to 1. White (black) regions stand for stable (unstable) solutions. Values of the vibration control terms A and B (b and c) in the plane (ϑ, ρ).

where

$$\Delta = (\alpha_1 + \beta_1 + 2\gamma_1\rho_A^2)^2 - (\alpha_1 + \beta_1 + \gamma_1\rho_A^2) \times (\alpha_1 + \beta_1 + 3\gamma_1\rho_A^2) - (\alpha_2 + \beta_2 + \gamma_2\rho_A^2)(\alpha_2 + \beta_2 + 3\gamma_2\rho_A^2). \tag{26}$$

The solution is asymptotically stable if and only if $\lambda_1 < 0$, $\lambda_2 < 0$, then from Eqs. (25) and (26) and the Routh–Hurwitz criterion we obtain that the following two inequalities hold simultaneously:

$$(\alpha_1 + \beta_1 + \gamma_1\rho_A^2)(\alpha_1 + \beta_1 + 3\gamma_1\rho_A^2) + (\alpha_2 + \beta_2 + \gamma_2\rho_A^2) \times (\alpha_2 + \beta_2 + 3\gamma_2\rho_A^2) > 0, \quad \alpha_1 + \beta_1 + 2\gamma_1\rho_A^2 < 0. \tag{27}$$

In Fig. 1, we show a stability chart for the solutions (ρ_A, ϑ_A) of the modulation system (Eqs. (14) and (15)), where white (black) regions correspond to stable (unstable) solutions. Other similar examples can be found in Figs. 2–4. In all the figures the time delay has been set to $T = \tau/4\Omega$. In particular, Figs. 1a and 2a show that if the external excitation parameter F increases from 0.02 to 0.1, the stability zone decreases, but amplitudes

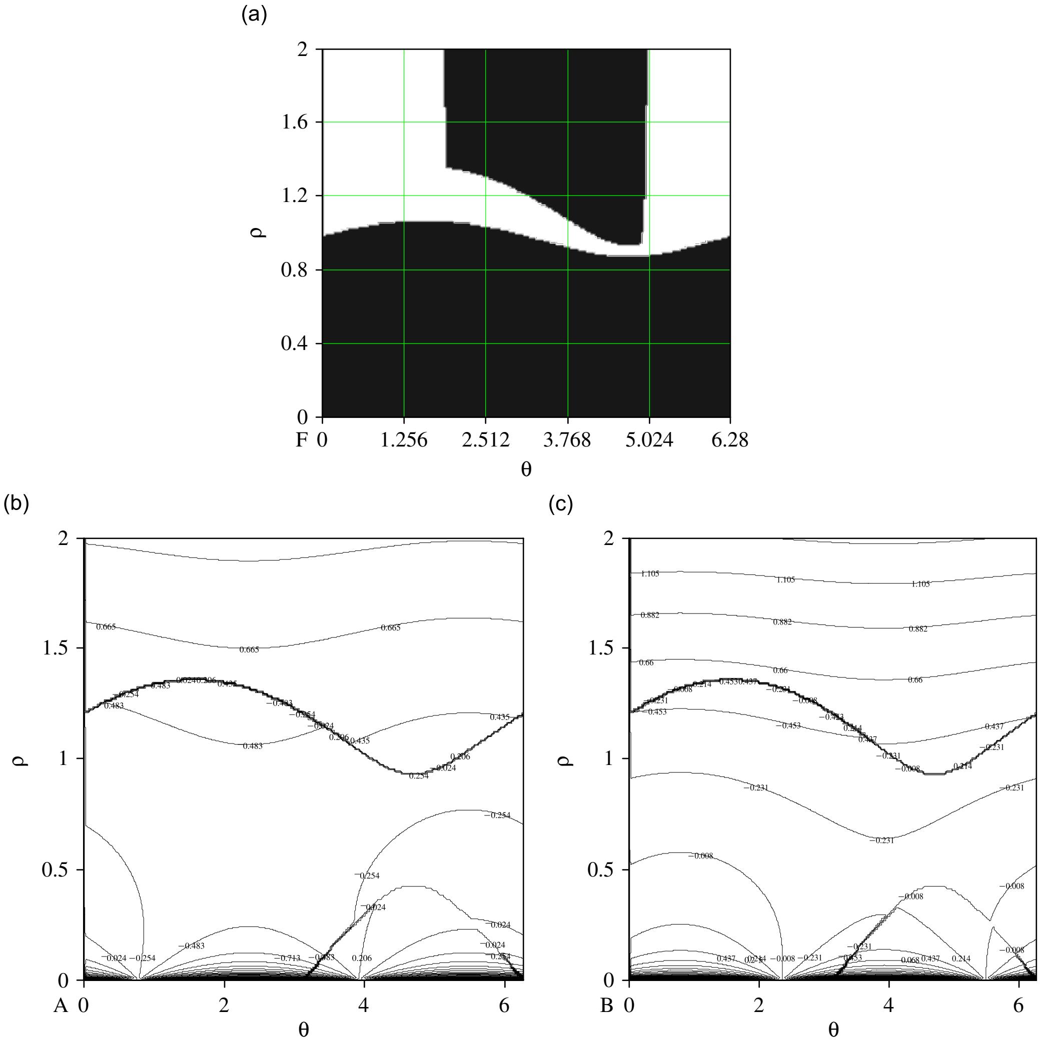

Fig. 4. Stability chart in the plane (ϑ, ρ) for the nonlinear system (Eq. (3)) with the parameter values $a = 0.16$, $b = 0.11$, $c = 0.12$, $F = 0.06$, $\sigma = 0.08$, $\Omega = 1$. The phase varies from 0 to 2π and the response from 0 to 1. White (black) regions stand for stable (unstable) solutions. Values of the vibration control terms A and B (b and c) in the plane (ϑ, ρ).

below $\rho_A \approx 0.8$ are always unstable for all values of the phase ϑ_A. Only in the first case for a generic value of the phase ϑ_A, there is always a stable amplitude ρ_A. Note however that, due to the presence of time-delay feedback control, it is possible to choose a phase (ϑ_A) in such a way to obtain an arbitrary value for the stable amplitude (ρ_A). For example in Fig. 1a, if we choose $\rho_A = 1.2$ and seek for a stable solution, Fig. 1a shows that an appropriate value for the phase is $\vartheta_A = 2.50$, while $A = -0.24$ (Fig. 1b) and $B = -0.22$ (Fig. 1c).

Figs. 3 and 4 show the dependence on the energy source coefficient (a, see Eq. (1)). An increase from 0.08 to 0.16 causes an extension of the stability region in such a way that for each value of the phase (ϑ_A) there is always a region of stable amplitude (ρ_A). Also in this case we must choose the amplitude (ρ_A) and then to find an associated phase (ϑ_A) in such a way that the selected amplitude is stable.

Note that in Figs. 1b, c, 2b, c, 3b, c and 4b, c the corresponding values of the feedback gains A and B are written on the contour lines. We observe the presence of regions characterized by very near contour lines that appear as close solid lines (for example in the lower part of Fig. 2a). In these regions small variations of the amplitude (ρ_A) and phase (ϑ_A) cause great changes of the feedback gains A and B.

In conclusion, the optimal choices for the time delay and the feedback gains are given by conditions (21)–(24), because in the nonlinear system (Eq. (3)) a stable periodic motion appears with the requested amplitude.

The strategy control can be summarized in the following three steps:

(i) we choose the working amplitude ρ_A and by the stability chart find a suitable value for the phase ϑ_A in such a way that the resulting solution is stable;
(ii) the approximate solution is constructed from Eq. (17);
(iii) the appropriate values for the feedback gains and the time delay are given by Eqs. (21)–(24). Note that Eq. (22) furnishes only the sum $(\varphi + \Omega T) = \Phi$ and there are only two conditions to be satisfied for the three parameters A, B, T (or equivalently K, φ, T) and then φ or the time delay T can be arbitrarily chosen.

4. Conclusion

We have developed an analytical approach to study the response of the van der Pol–Duffing system (Eq. (1)), under external excitation and state feedback control with a time delay. In order to identify self-sustained oscillations, we have used the asymptotic perturbation method and derived two slow-flow equations, governing the amplitude and phase of approximate long time response. Although the structure of the slow-flow equations is similar to that of the uncontrolled system, the feedback gains and time delay coefficients can be used to implement an effective vibration control. Stability charts for the amplitude and phase of the periodic phase-locked solutions can be easily constructed. In this way, we can choose the appropriate amplitude of the stable periodic solutions and then calculate the necessary feedback gains and time delay.

Vibration control can be successfully performed, because we have demonstrated that the time delay and the feedback gains can enhance the control performance and obtain the desired vibration amplitude.

Finally, in some forthcoming papers, we will apply this vibration control method to two or more degrees-of-freedom systems with various excitation forces.

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