Available online at www.sciencedirect.com
ScienceDirect
Journal of Sound and Vibration 317 (2008) 366–377
JOURNAL OF SOUND AND VIBRATION
www.elsevier.com/locate/jsvi

A hybrid WKB–Galerkin method applied to a piezoelectric sandwich plate vibration problem considering shear force effects

Victor Z. Gristchak, Olga A. Ganilova*

Applied Mathematics Department, Faculty of Mathematics, Zaporizhzhya National University, Ukraine

Received 6 December 2007; received in revised form 26 March 2008; accepted 27 March 2008
Handling Editor: M.P. Cartmell
Available online 9 May 2008

Abstract

This paper deals with the problem of dynamic loading of a piezoelectric sandwich plate. The objective of the analysis is to obtain a closed form approximate analytical solution for the equilibrium equations of the loaded plate considering time variant damping, and to discuss the shear force effect for the problem. The solution to the problem is obtained using a hybrid WKB–Galerkin method.

1. Introduction

The development of contemporary engineering has promoted a generally wide application of anisotropic materials in aerospace applications, civil structural design, biomedical engineering and other fields [1–7]. One of the most attractive smart materials for many practical installations is based on piezoelectric technology, in which an actuator or sensor element usually has to be made up from several layers in order to be practically effective, thereby exhibiting anisotropy. The mechanical application of smart structures using these elements is therefore closely concerned with some appropriate application of the piezoelectric effect. The unique actuation characteristics of piezoelectric materials permit them to be exploited for deformation-control, shape control, and vibration control, as well as in the inverse role of sensing. The piezoelectric effect is a phenomenon in which certain crystals become polarised when they are mechanically stressed, or they exhibit mechanical strain when they experience an electric field, this being defined as the converse piezoelectric effect [5]. These basic properties of piezoelectric elements form the basis of their application as sensors and activators [7]. Piezoelectric actuators and sensors are compact and do not necessarily need supporting mechanisms to bear the reacting force in contrast with installations using shape memory alloy actuators [7]. Moreover, piezoelectric structures can be highly reliable and predictable in use since actuators will transfer force to the structural member according to the magnitude of the excitation voltage. They are also particularly attractive due to their fast activation response time, normally of the order of milliseconds.

*Corresponding author.
E-mail address: lionly@rambler.ru (O.A. Ganilova).

0022-460X/$ - see front matter © 2008 Elsevier Ltd. All rights reserved.
doi:10.1016/j.jsv.2008.03.043

Research results for various anisotropic plates applications, as given in Refs. [8–11], demonstrate that the results of iterative multilayer plate theory, which considers a correction associated with the shear force effect, are more accurate and correct than results obtained by taking the classical multilayer plate theory. Therefore, the analysis presented in this work has been undertaken with due attention paid to iterative multilayer plate theory.

According to Refs. [4,12], the finite element method, and other numerical methods, are invariably used in this field of mechanics, but it is also important to investigate anisotropic structural problems by applying appropriate analytical methods. This is one of the motivations behind this particular work. In this paper a piezoelectric orthotropic sandwich plate is considered. The structure is subjected to dynamic loading, and additionally a time variant damping coefficient is taken into consideration, to make the problem more general and relevant to possible applications in civil and aerospace engineering. In order to solve the problem, a hybrid Wentzel–Kramer–Brillouin–Galerkin (or WKB–Galerkin) method has been applied, principally because hybrid methods [1,13] have already shown considerable advantages in various branches of mechanics. Notably, they provide an opportunity to obtain the approximate solution in an asymptotic form. Since piezoelectric structures comprising several layers exhibit anisotropy some effects due to shear force action cannot be investigated in terms of classical multilayer plate theory. Thus the solution is obtained by taking the piezoelectric constant responsible for shear force into account, and in terms of iterative multilayer plate theory, according to Ref. [8], so as to observe the influence of these two approaches on the solution. Both the piezoelectric constant and the use of iterative theory provide a good opportunity to take shear force into account. The piezoelectric constant is widely used in modelling the behaviour of piezoelectric multilayer plates [6,14], and iterative theory can readily be applied to anisotropic laminated plates [8–10]. Therefore, an analysis is presented here of the influence of shear force on a symmetrical orthotropic piezoelectric sandwich plate in order to compare these two techniques.

2. Fundamentals of the problem

A rectangular, simply supported, symmetrical orthotropic sandwich plate $(a \times b)$ is considered. The structure has thickness h and is subjected to some appropriate dynamic electrical loading $V^k(x,y,t)$ on the kth lamina, in this case conveniently chosen to be defined as in [6],

$$V^k(x, y, t) = \varphi^k(t) \sin\frac{\pi x}{a} \sin\frac{\pi y}{b}, \tag{1}$$

where $V^k(x,y,t)$ [6], is assumed to consist of two separate functions in the form of a spatial part, $\Lambda^k(x, y) = \sin(\pi x/a)\sin(\pi y/b)$, and a temporal part, $\varphi^k(t) = \varphi_0^k \tilde{\varphi}^k(t)$, where φ_0^k is the maximum voltage amplitude applied to the plate; and $\Lambda^k(x,y)$ and $\tilde{\varphi}^k(t)$ are spatial and time variant input functions. In practice $\Lambda^k(x,y)$ can be obtained by varying the surface electrode location or the layer thickness [6].

A dynamic mechanical loading, $Z(x,y,t)$, is applied normal to the plate. In this work two cases of mechanical loading are considered,

$$Z = q(t) \sin\frac{\pi x}{a} \sin\frac{\pi y}{b} - 2\rho\varepsilon(t)h\frac{\partial w}{\partial t} - \rho h\frac{\partial^2 w}{\partial t^2} \tag{2}$$

and

$$Z = q(t) \cos\frac{\pi x}{a} \cos\frac{\pi y}{b} - 2\rho\varepsilon(t)h\frac{\partial w}{\partial t} - \rho h\frac{\partial^2 w}{\partial t^2} \tag{3}$$

where the first term represents the transverse load, for which $q(t) = Q_0 \cos\theta t$ with loading amplitude Q_0, and the second term is responsible for oscillation damping taking into consideration a dynamic damping coefficient $\varepsilon(t)$. The quantity $\rho = \sum_{k=1}^{3}(\rho_k h_k/h)$ invokes ρ_k and h_k which are the density and thickness of the kth layer. It is used in this form mainly for analytical convenience.

For the structure of Fig. 1, it is arbitrarily assumed that

$$\delta_1 = \delta_2 = \delta, \quad B'_{mn} = B''_{mn} = B_{mn}, \tag{4}$$

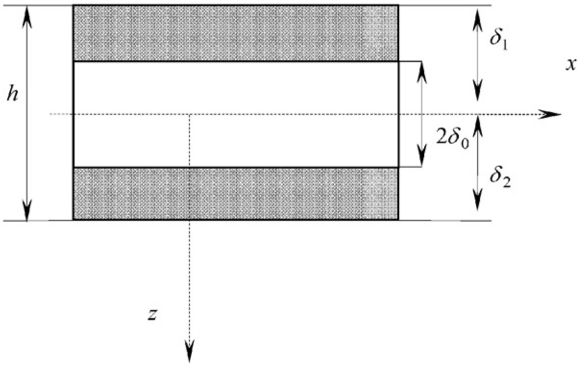

Fig. 1. A symmetrical orthotropic sandwich plate.

without significant loss of generality. According to Ref. [8], for a symmetrical orthotropic plate the following conditions can be applied:

$$u_0 = \upsilon_0 = 0, \quad u = \upsilon = 0, \quad K_{mn} = 0, \quad C_{mn} = 2\lfloor \delta_0 B^0_{mn}(\delta - \delta_0)B_{mn} \rfloor, \tag{5}$$

where $2\delta_0$, $\delta_1{-}\delta_0$, $\delta_2{-}\delta_0$ are the thicknesses of the middle, top and bottom layers, respectively; u_0, υ_0; u, υ are displacements of the system mid-plane in the x and y direction for the classical and iterative theories respectively; B^i_{mn} are predetermined layer elasticity coefficients; i is the index of the ply such that $i = 0$ for the middle layer, and for the top and bottom plies the indices are defined by $'$ and $''$. Also, we note that C_{mn} is the stretching—compression stiffness; and the subscripts m, $n = 1, 2, 6$.

Since the structure is subjected to electrical and mechanical dynamic loading, the equations of motion, with reference to [2,3,6,14,15], can be expressed in the following form:

$$L_{11}(C_{mn})u_0 + L_{12}(C_{mn})\upsilon_0 - L_{13}(K_{mn})w_0 = \rho h \frac{\partial^2 u_0}{\partial t^2} + \sum_{k=1}^{N} e^{0^k}_{31}\frac{\partial V^k}{\partial x} + \sum_{k=1}^{N} e^{0^k}_{36}\frac{\partial V^k}{\partial y},$$

$$L_{22}(C_{mn})\upsilon_0 + L_{12}(C_{mn})u_0 - L_{23}(K_{mn})w_0 = \rho h \frac{\partial^2 \upsilon_0}{\partial t^2} + \sum_{k=1}^{N} e^{0^k}_{32}\frac{\partial V^k}{\partial y} + \sum_{k=1}^{N} e^{0^k}_{36}\frac{\partial V^k}{\partial x},$$

$$L_{33}(D_{mn})w_0 - L_{13}(K_{mn})u_0 - L_{23}(K_{mn})\upsilon_0 = Z - \sum_{k=1}^{N} e^{0^k}_{31}z^k_0\frac{\partial^2 V^k}{\partial x^2} - 2\sum_{k=1}^{N} e^{0^k}_{36}z^k_0\frac{\partial^2 V^k}{\partial x\,\partial y} - \sum_{k=1}^{N} e^{0^k}_{32}z^k_0\frac{\partial^2 V^k}{\partial y^2}, \tag{6}$$

where the piezoelectric stress-charge constant e^0_{36} infers the induction and detection of shear force, e^0_{31} and e^0_{32} denote the presence of transverse bending moments and angular deflection [6], D_{mn} is the bending stiffness defined in Ref. [3], the number of layers $N = 3$, and the linear operators are defined as

$$L_{11}(C_{mn}) = C_{11}\frac{\partial^2}{\partial x^2} + C_{66}\frac{\partial^2}{\partial y^2}, \quad L_{22}(C_{mn}) = C_{22}\frac{\partial^2}{\partial y^2} + C_{66}\frac{\partial^2}{\partial x^2},$$

$$L_{12}(C_{mn}) = (C_{12} + C_{66})\frac{\partial^2}{\partial x\,\partial y}, \quad L_{13}(K_{mn}) = K_{11}\frac{\partial^3}{\partial x^3} + (K_{12} + 2K_{66})\frac{\partial^3}{\partial x\,\partial y^2},$$

$$L_{23}(K_{mn}) = K_{22}\frac{\partial^3}{\partial y^3} + (K_{12} + 2K_{66})\frac{\partial^3}{\partial y\,\partial x^2}, \quad L_{33}(D_{mn}) = D_{11}\frac{\partial^4}{\partial x^4} + D_{22}\frac{\partial^4}{\partial y^4} + 2(D_{12} + 2D_{66})\frac{\partial^4}{\partial x^2\,\partial y^2}.$$

In order to investigate the influence of the piezoelectric constant e^0_{36}, which introduces shear force in Eqs. (6), and iterative theory correction for the classical multilayer plate theory, required to represent the shear force effect in elastic structures, we consider two cases.

Initially the plate is considered to be subjected to an external mechanical force (2) and an electrical loading (1), and the term responsible for shear force, e^0_{36}, is neglected. For this problem the iterative multilayer plate theory is used. To apply the iterative technique according to Ref. [8], a solution based on classical multilayer plate theory has to be obtained as a zeroth-order approximation, and then the result of the iterative theory will be the first-order approximation.

Therefore, we get the following system of differential equations:

$$L_{11}(C_{mn})u_0 + L_{12}(C_{mn})v_0 - L_{13}(K_{mn})w_0 = \rho h \frac{\partial^2 u_0}{\partial t^2} + \sum_{k=1}^{N} e_{31}^{0^k} \frac{\partial V^k}{\partial x},$$
$$L_{22}(C_{mn})v_0 + L_{12}(C_{mn})u_0 - L_{23}(K_{mn})w_0 = \rho h \frac{\partial^2 v_0}{\partial t^2} + \sum_{k=1}^{N} e_{32}^{0^k} \frac{\partial V^k}{\partial y},$$
$$L_{33}(D_{mn})w_0 - L_{13}(K_{mn})u_0 - L_{23}(K_{mn})v_0 = Z - \sum_{k=1}^{N} e_{31}^{0^k} z_0^k \frac{\partial^2 V^k}{\partial x^2} - \sum_{k=1}^{N} e_{32}^{0^k} z_0^k \frac{\partial^2 V^k}{\partial y^2}. \quad (7)$$

The coordinate of the middle plane of the kth layer, according to Ref. [6] is given by $z_0^k = (1/2)(z_k + z_{k-1})$.

Taking into account Eqs. (1), (4) and (5), the first two equations of the system (7) form an independent plane problem, and the third one forms a lateral bending problem which is considered in this paper,

$$D_{11}\frac{\partial^4 w_0}{\partial x^4} + 2(D_{12} + 2D_{66})\frac{\partial^4 w_0}{\partial x^2 \partial y^2} + D_{22}\frac{\partial^4 w_0}{\partial y^4} = q(t)\sin\frac{\pi x}{a}\sin\frac{\pi y}{b} - 2\rho\varepsilon(t)h\frac{\partial w_0}{\partial t} - \rho h \frac{\partial^2 w_0}{\partial t^2}$$
$$+ \sum_{k=1}^{N} e_{31}^{0^k} z_0^k \varphi^k(t)\left(\frac{\pi}{a}\right)^2 \sin\frac{\pi x}{a}\sin\frac{\pi y}{b} + \sum_{k=1}^{N} e_{32}^{0^k} z_0^k \varphi^k(t)\left(\frac{\pi}{b}\right)^2 \sin\frac{\pi x}{a}\sin\frac{\pi y}{b}. \quad (8)$$

Using the substitution,

$$w_0(t) = f(t)\sin\frac{\pi x}{a}\sin\frac{\pi y}{b} \quad (9)$$

Eq. (8) can be rewritten as

$$\pi^4\left(\frac{D_{11}}{a^4}f(t) + 2\frac{D_{12}+2D_{66}}{a^2b^2}f(t) + \frac{D_{22}}{b^4}f(t)\right) = \hat{q}(t) - 2\rho h\varepsilon(t)f'(t) - \rho h f''(t), \quad (10)$$

where

$$\hat{q}(t) = q(t) + \left[\sum_{k=1}^{N} e_{31}^{0^k} z_0^k \left(\frac{\pi}{a}\right)^2 + \sum_{k=1}^{N} e_{32}^{0^k} z_0^k \left(\frac{\pi}{b}\right)^2\right]\varphi^k(t).$$

By assuming that

$$\tilde{D} = \pi^4\left(\frac{D_{11}}{a^4} + 2\frac{D_{12}+2D_{66}}{a^2b^2} + \frac{D_{22}}{b^4}\right), \quad \tilde{\tilde{D}} = \frac{\tilde{D}}{\rho a}, \quad \tilde{q}(t) = \frac{\hat{q}(t)}{\rho a}, \quad \bar{\varepsilon}(t) = \frac{h}{a}\varepsilon(t), \quad \lambda^2 = \frac{h}{a} \quad (11)$$

Eq. (10) can be rearranged into the form

$$\lambda^2 f''(t) + 2\bar{\varepsilon}(t)f'(t) + \tilde{\tilde{D}}f(t) = \tilde{q}(t) \quad (12)$$

in which λ^2 is a dimensionless parameter multiplying the highest order derivative.

The solution of this equation can be expressed as

$$f(t) = f_c(t) + f_p(t) \quad (13)$$

where $f_c(t)$ is a complementary function and $f_p(t)$ is a particular solution.

The homogeneous differential equation with the variable damping coefficient is solved in order to find the complementary function

$$\lambda^2 f''(t) + 2\bar{\varepsilon}(t)f'(t) + \tilde{\tilde{D}}f(t) = 0. \quad (14)$$

Then, according to the hybrid method described of Refs. [1,13], the general WKB-solution has the following form:

$$f(t,\lambda) = \exp\left[\int_a^t \left(\frac{1}{\lambda} f_0(t) + f_1(t)\right) \mathrm{d}t\right] \tag{15}$$

Substituting (15) into Eq. (14) leads to

$$\lambda^2\left[\frac{1}{\lambda}f_0' + f_1' + \frac{1}{\lambda^2}f_0^2 + f_1^2 + 2\frac{1}{\lambda}f_0 f_1\right] + 2\bar{\varepsilon}(t)\left[\frac{1}{\lambda}f_0 + f_1\right] + \tilde{\bar{D}} = 0. \tag{16}$$

From there, equating the coefficients of like powers generates this system of equations,

$$\begin{cases} f_0^2 + 2\bar{\varepsilon}(t) f_1 + \tilde{\bar{D}} = 0 \\ f_0' + 2 f_0 f_1 = 0 \end{cases} \quad \text{or} \quad \begin{cases} f_1 = -\frac{1}{2}\dfrac{\mathrm{d}}{\mathrm{d}t}\ln f_0, \\ \bar{\varepsilon}(t) f_0' - f_0^3 - \tilde{\bar{D}} f_0 = 0. \end{cases} \tag{17}$$

The second equation within the alternative system (17) can be solved using the standard substitution

$$f_0 = U(t)V(t) \tag{18}$$

This leads to

$$\bar{\varepsilon}(t)(U'V + UV') - U^3V^3 - \tilde{\bar{D}} UV = 0 \tag{19}$$

from which we obtain

$$V = \mathrm{e}^{\tilde{\bar{D}}\int \mathrm{d}t/\bar{\varepsilon}(t)} \quad \text{and} \quad U = \pm\mathrm{i}\left[2\int \frac{V^2}{\bar{\varepsilon}(t)}\mathrm{d}t\right]^{-1/2}. \tag{20}$$

Therefore, considering Eqs. (18) and (20) the solution of the system of Eqs. (17) can be expressed as

$$\begin{cases} f_1 = -\frac{1}{2}\dfrac{\mathrm{d}}{\mathrm{d}t}\ln f_0, \\ f_0 = \pm\mathrm{i}\left[2\int \frac{\mathrm{e}^{2\tilde{\bar{D}}\int \mathrm{d}t/\bar{\varepsilon}(t)}}{\bar{\varepsilon}(t)}\mathrm{d}t\right]^{-1/2} \mathrm{e}^{\tilde{\bar{D}}\int \mathrm{d}t/\bar{\varepsilon}(t)}. \end{cases} \tag{21}$$

Thus, the solution based on the application of the WKB method can be defined by (15), taking into consideration Eq. (21).

For the second step of the solution procedure, and according to the hybrid WKB–Galerkin method, the Bubnov–Galerkin technique is applied using just the first term of the WKB-solution (f_0). Thus we consider the solution in this form:

$$\tilde{f}(t,\lambda) = \exp\left[\int_a^t [\delta_{01}(\lambda) + \mathrm{i}\delta_{02}(\lambda)] f_0(t)\,\mathrm{d}t\right]. \tag{22}$$

According to the hybrid approach described in Refs. [1,13], expression (22) is then substituted into Eq. (14), leading to

$$\lambda^2\left[(\delta_{01} + \mathrm{i}\delta_{02}) f_0' + (\delta_{01}^2 + 2\mathrm{i}\delta_{01}\delta_{02} - \delta_{02}^2) f_0^2\right] + 2\bar{\varepsilon}(t)[\delta_{01} + \mathrm{i}\delta_{02}] f_0 + \tilde{\bar{D}} = 0. \tag{23}$$

Following the orthogonality condition of the Bubnov–Galerkin method of the F and $N+1$ coordinate functions on the interval $[a,b]$, i.e. $\int_a^b F(\delta_0,\ldots,\delta_N, f_0,\ldots,f_N, f_0',\ldots,f_N^{(n-1)}, t,\lambda) f_i(t)\,\mathrm{d}t = 0$, or in this case, $\int_a^b F(\delta_{01},\delta_{02}, f_0, f_0', t,\lambda) f_0(t)\mathrm{d}t = 0$, where $F(\delta_{01},\delta_{02}, f_0, f_0', t,\lambda)$ is the left-hand side of Eq. (23), the following is obtained:

$$\lambda^2\left[(-\delta_{01} - \mathrm{i}\delta_{02})\bar{f}_0'\bar{f}_0 + (-\delta_{01}^2\mathrm{i} + 2\delta_{01}\delta_{02} + \mathrm{i}\delta_{02}^2)\left(\pm\bar{f}_0^3\right)\right] + 2\bar{\varepsilon}(t)[-\delta_{01} - \mathrm{i}\delta_{02}]\bar{f}_0^2 + \mathrm{i}\tilde{\bar{D}}\left(\pm\bar{f}_0\right) = 0, \tag{24}$$

where $f_0 = \pm\mathrm{i}\bar{f}_0$ and f_0 is given by Eq. (21).

Equating the coefficients of the *Real* and *Imaginary* terms leads to the following system of equations:

$$\begin{cases} A\delta_{02} - B\delta_{01}^2 + B\delta_{02}^2 + W = 0, \\ A\delta_{01} + 2B\delta_{01}\delta_{02} = 0, \end{cases} \tag{25}$$

where

$$A = \int_a^b [-\lambda^2 \bar{f}_0' \bar{f}_0 - 2\bar{\varepsilon}(t)\bar{f}_0^2]\mathrm{d}t, \quad B = \pm \int_a^b \lambda^2 \bar{f}_0^3\,\mathrm{d}t, \quad W = \pm \int_a^b \tilde{\bar{D}}\bar{f}_0\,\mathrm{d}t. \tag{26}$$

Solving the system of Eqs. (25) gives

$$\begin{cases} \delta_{01} = \dfrac{\sqrt{4BW - A^2}}{2B}, \\ \delta_{02} = \mp\dfrac{A}{2B}. \end{cases} \tag{27}$$

Therefore, the solution based on the hybrid WKB–Galerkin method can be expressed in the form of Eq. (22), by applying Eqs. (21) and (27). The hybrid solution of Eq. (22) is a complementary function $f_c(t)$ of Eq. (12), so in order to get the particular solution by applying the variation of parameters method, we state that

$$\varepsilon(t) = Ct, \tag{28}$$

where C is an arbitrary constant.

Using Eqs. (21), (27) and (28) the hybrid solution (22) can be rewritten as

$$\tilde{f} = \mathrm{e}^{-\delta_{02}\sqrt{\tilde{\bar{D}}}t}\left(c_1 \sin \delta_{01}\sqrt{\tilde{\bar{D}}}t + c_2 \cos \delta_{01}\sqrt{\tilde{\bar{D}}}t\right), \tag{29}$$

where c_1 and c_2 are arbitrary constants. It should be noted that in case $\varepsilon(t) = M\mathrm{e}^{Kt}$, where M, K are arbitrary constants, it is possible to obtain the hybrid solution in the same form (29).

The particular solution can be obtained in the form of

$$f_p = \mathrm{e}^{-\delta_{02}\sqrt{\tilde{\bar{D}}}t}\left(\bar{c}_1 \sin \delta_{01}\sqrt{\tilde{\bar{D}}}t + \bar{c}_2 \cos \delta_{01}\sqrt{\tilde{\bar{D}}}t\right), \tag{30}$$

where

$$\bar{c}_1 = \int \frac{\tilde{q}(t)\mathrm{e}^{\delta_{02}\sqrt{\tilde{\bar{D}}}t} \cos \delta_{01}\sqrt{\tilde{\bar{D}}}t}{\lambda^2\delta_{01}\sqrt{\tilde{\bar{D}}}}\mathrm{d}t \quad \text{and} \quad \bar{c}_2 = -\int \frac{\tilde{q}(t)\mathrm{e}^{\delta_{02}\sqrt{\tilde{\bar{D}}}t} \sin \delta_{01}\sqrt{\tilde{\bar{D}}}t}{\lambda^2\delta_{01}\sqrt{\tilde{\bar{D}}}}\mathrm{d}t.$$

Then, from Eq. (13), the general solution of Eq. (12) can be expressed as

$$f(t) = \mathrm{e}^{-\delta_{02}\sqrt{\tilde{\bar{D}}}t}\left((\bar{c}_1 + c_1)\sin \delta_{01}\sqrt{\tilde{\bar{D}}}t + (\bar{c}_2 + c_2)\cos \delta_{01}\sqrt{\tilde{\bar{D}}}t\right). \tag{31}$$

As a result the solution (9) of Eq. (8) can be written in terms of Eqs. (27) and (31), according to the classical multilayer plate theory.

Since the solution $w_0(t)$ has already been obtained above in Eqs. (9) and (31), we can then apply the iterative multilayer plate theory. This theory, summarised in Ref. [8], is based on the idea of correcting terms, which describe the in-plane shear effect. These terms are dependent on the plate thickness, the mechanical characteristics of the plate material, and the loading. If the solution based on the classical theory of a multilayer plate is regarded as a zeroth-order approximation then the result of the iterative theory will necessarily be a first-order approximation. The shear force effect is taken into account due to additive 'correction' terms M_x^*, M_y^*, H^* which appear due to internal forces and moments induced by the shear force effect. These correction terms are based on the classical multilayer plate solution since the basic hypothesis of the iterative theory is stated as [8], 'it is supposed that tangential stresses τ_{xz}^i, τ_{yz}^i are defined by certain expressions based on τ_{xz}^i, τ_{yz}^i determined for the classical multilayer plate theory, i.e. by taking into consideration the hypothesis of the unstrained normal for the entire plate'. Therefore, applying the $w_0(t)$ solution means that it is possible to find the correction terms M_x^*, M_y^*, H^* and then finally to get the problem solution according to iterative multilayer plate theory.

Thus, we now consider the iterative theory of multilayer plates. According to this theory [8] Eq. (8) can be rewritten as follows:

$$D_{11}\frac{\partial^4 w}{\partial x^4}+2(D_{12}+2D_{66})\frac{\partial^4 w}{\partial x^2\,\partial y^2}+D_{22}\frac{\partial^4 w}{\partial y^4}=q(t)\sin\frac{\pi x}{a}\sin\frac{\pi y}{b}-2\rho\varepsilon(t)h\frac{\partial w}{\partial t}-\rho h\frac{\partial^2 w}{\partial t^2}$$
$$+\sum_{k=1}^{N}e_{31}^{0k}z_0^k\varphi^k(t)\left(\frac{\pi}{a}\right)^2\sin\frac{\pi x}{a}\sin\frac{\pi y}{b}+\sum_{k=1}^{N}e_{32}^{0k}z_0^k\varphi^k(t)\left(\frac{\pi}{b}\right)^2\sin\frac{\pi x}{a}\sin\frac{\pi y}{b}+\left(\frac{\partial^2 M_x^*}{\partial x^2}+2\frac{\partial^2 H^*}{\partial x\,\partial y}+\frac{\partial^2 M_y^*}{\partial y^2}\right). \quad (32)$$

When considering the correction terms M_x^*, M_y^*, H^*, it can be noted that the derivatives $\partial^2 M_x^*/\partial x^2$, $\partial^2 H^*/\partial x\,\partial y$, $\partial^2 M_y^*/\partial y^2$ in the right-hand side of Eq. (32) are all given in Ref. [8].

The form of these correction terms can be simplified by applying assumption (9), and they can therefore be written in the following form:

$$\frac{\partial^2 M_x^*}{\partial x^2}=\tilde{M}_x f(t)\sin\frac{\pi x}{a}\sin\frac{\pi y}{b},$$
$$\frac{\partial^2 M_y^*}{\partial y^2}=\tilde{M}_y f(t)\sin\frac{\pi x}{a}\sin\frac{\pi y}{b},$$
$$\frac{\partial^2 H^*}{\partial x\,\partial y}=\tilde{H} f(t)\sin\frac{\pi x}{a}\sin\frac{\pi y}{b}, \quad (33)$$

where $\tilde{M}_x, \tilde{M}_y, \tilde{H}$ are stated in the Appendix.

Thus, the expression in the right-hand side of Eq. (32) can be rearranged into

$$\left(\frac{\partial^2 M_x^*}{\partial x^2}+2\frac{\partial^2 H^*}{\partial x\,\partial y}+\frac{\partial^2 M_y^*}{\partial y^2}\right)=Mf(t)\sin\frac{\pi x}{a}\sin\frac{\pi y}{b}, \quad (34)$$

where $f(t)$ is given by Eq. (31) and $M=\tilde{M}_x+2\tilde{H}+\tilde{M}_y$.

From this, Eq. (32) can be written as

$$D_{11}\frac{\partial^4 w}{\partial x^4}+2(D_{12}+2D_{66})\frac{\partial^4 w}{\partial x^2\,\partial y^2}+D_{22}\frac{\partial^4 w}{\partial y^4}=\hat{q}(t)\sin\frac{\pi x}{a}\sin\frac{\pi y}{b}-2\rho\varepsilon(t)h\frac{\partial w}{\partial t}-\rho h\frac{\partial^2 w}{\partial t^2}$$
$$+Mf(t)\sin\frac{\pi x}{a}\sin\frac{\pi y}{b}. \quad (35)$$

For the case of simply supported boundary conditions the solution to Eq. (35) is expressed as

$$w(t)=\bar{u}(t)\sin\frac{\pi x}{a}\sin\frac{\pi y}{b}. \quad (36)$$

After simplifications similar to those defined for Eqs. (11), Eq. (35) can be rewritten as

$$\frac{h}{a}\bar{u}''(t)+2\frac{h}{a}\varepsilon(t)\bar{u}'(t)+\frac{\tilde{D}}{\rho a}\bar{u}(t)=\frac{\hat{q}(t)+Mf(t)}{\rho a}, \quad (37)$$

where $\tilde{D}$ is given by Eq. (11).

It is important to point out that the solution obtained for Eq. (12), and given by Eq. (31), is based on the classical theory and is independent of the form of the function $\tilde{q}(t)$, which appears in the right-hand side of Eq. (12). Therefore, solution (31), obtained from using the hybrid WKB–Galerkin method, can be applied to Eq. (37) based on the iterative theory, by changing the function $\tilde{q}(t)$ in solution (31) on the right-hand side of Eq. (37).

For the second step, we consider that the plate as described is subjected to the external mechanical loading of Eq. (3) and the electrical loading of Eq. (1), using the term responsible for the shear force e_{36}^0. As a result, from the system of Eq. (6), a lateral bending problem is obtained in the form,

$$D_{11}\frac{\partial^4 \tilde{w}_0}{\partial x^4}+2(D_{12}+2D_{66})\frac{\partial^4 \tilde{w}_0}{\partial x^2\,\partial y^2}+D_{22}\frac{\partial^4 \tilde{w}_0}{\partial y^4}=q(t)\cos\frac{\pi x}{a}\cos\frac{\pi y}{b}-2\rho\varepsilon(t)h\frac{\partial \tilde{w}_0}{\partial t}-\rho h\frac{\partial^2 \tilde{w}_0}{\partial t^2}$$
$$+\sum_{k=1}^{N} e_{31}^{0^k} z_0^k \varphi(t)\left(\frac{\pi}{a}\right)^2 \sin\frac{\pi x}{a}\sin\frac{\pi y}{b}-2\left(\frac{\pi^2}{ab}\right)\sum_{k=1}^{N} e_{36}^{0^k} z_0^k \varphi(t)\cos\frac{\pi x}{a}\cos\frac{\pi y}{b}$$
$$+\sum_{k=1}^{N} e_{32}^{0^k} z_0^k \varphi(t)\left(\frac{\pi}{b}\right)^2 \sin\frac{\pi x}{a}\sin\frac{\pi y}{b}, \tag{38}$$

where it is reasonable to assume that the electrical loading is equal for each layer $\varphi^k(t)=\varphi(t)$. To consider a specific example we assume that $\varphi(t)=\varphi_0\tilde{\varphi}(t)=q_0\cos\theta t$ and then introduce a constant $Z_0=2(\pi^2/ab)\sum_{k=1}^{N} e_{36}^{0^k} z_0^k$. Since Q_0 can be arbitrarily chosen for our example then mathematically we assume that $Q_0=Z_0 q_0$ for $q(t)=Q_0\cos\theta t$. It should be noted that in this case the first and the fifth terms cancel out, due to the choice of Z_0. However in practice, these terms may not cancel out, since the value of Z_0 may be greater or smaller than the one chosen here as an arbitrary example to observe the influence of the fifth term, and the fifth one may only reduce the value of the first term, or vice versa.

In this case Eq. (38) can be rewritten as

$$D_{11}\frac{\partial^4 \tilde{w}_0}{\partial x^4}+2(D_{12}+2D_{66})\frac{\partial^4 \tilde{w}_0}{\partial x^2\,\partial y^2}+D_{22}\frac{\partial^4 \tilde{w}_0}{\partial y^4}=-2\rho\varepsilon(t)h\frac{\partial \tilde{w}_0}{\partial t}-\rho h\frac{\partial^2 \tilde{w}_0}{\partial t^2}$$
$$+\left[\sum_{k=1}^{N} e_{31}^{0^k} z_0^k\left(\frac{\pi}{a}\right)^2+\sum_{k=1}^{N} e_{32}^{0^k} z_0^k\left(\frac{\pi}{b}\right)^2\right]\varphi(t)\sin\frac{\pi x}{a}\sin\frac{\pi y}{b}. \tag{39}$$

This equation can be solved by applying a closed form analytical solution as described above in Eqs. (9) and (31).

3. Numerical results

To validate the solutions obtained above, graphical results for the problem for some predetermined parameters are presented. It is supposed that a plate ($a=b=2m$) is made of transversely isotropic layers, and the planes of isotropy are parallel to the $z=0$ plane, i.e. xOy. The piezoelectric material of the top and bottom layers is $BaTiO_3$ and the middle layer is made of a passive elastic material. Therefore, the parameters for the top and bottom piezoelectric layers are as given in [3],

$$Q_{11}=Q_{22}=120\times 10^9\,\mathrm{N\,m^{-2}},\quad Q_{12}=36.2\times 10^9\,\mathrm{N\,m^{-2}},\quad Q_{66}=42\times 10^9\,\mathrm{N\,m^{-2}},\quad D_{mn}=\int_{-h/2}^{h/2} Q_{mn}z^2\,\mathrm{d}z$$
$$e_{31}^0=e_{32}^0=e_{36}^0=-12.3\,\mathrm{C\,m^{-2}}, \tag{40}$$

and for the middle, non-piezoelectric, layer the coefficients of elasticity are expressed as in [8]

$$B_{11}^0=B_{22}^0=\frac{E_0}{1-\nu_0^2},\quad B_{12}^0=\frac{\nu_0 E_0}{1-\nu_0^2},\quad B_{66}^0=\frac{E_0}{2(1+\nu_0)},\quad a_{44}^0=a_{55}^0=\frac{1}{G_0'}, \tag{41}$$

where E_0, G_0', ν_0 are the modulus of elasticity, the modulus of in-plane shear, and Poisson's ratio. It is also assumed that,

$$\nu=\nu_0=0.3,\quad \frac{E}{E_0}=n,\quad \frac{E_0}{G_0'}=s_0. \tag{42}$$

For this particular example the following data is arbitrarily assumed, $\rho=5.7\times 10^3\,\mathrm{kg\,m^{-3}}$, $\varepsilon(t)=t$, and functions of electrical and mechanical loadings are assumed to be $\varphi(t)=10^8\cos t$ and $q(t)=-9.1\times 10^8\cos t$; and parameters $h/a=0.1$, $h=0.2\,\mathrm{m}$, $\delta=2\delta_0=h/2=0.1\,\mathrm{m}$, where h is thickness of the entire plate, are taken from Ambartsumian [8].

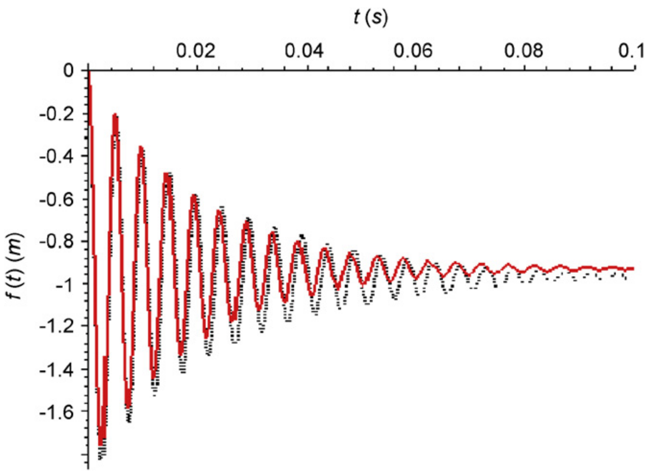

Fig. 2. Characteristic functions $f(t)$ (—) and $\bar{u}(t)$ (- - - -) of the solutions in terms of classical and iterative theories.

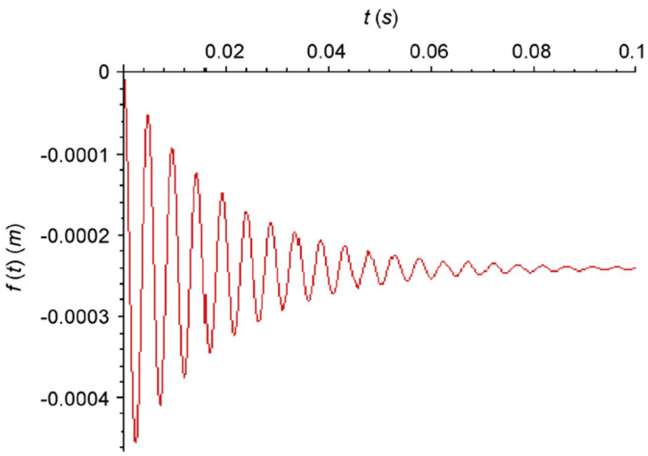

Fig. 3. Behaviour of $f(t)$ if the mechanical loading compensates for the electrical loading.

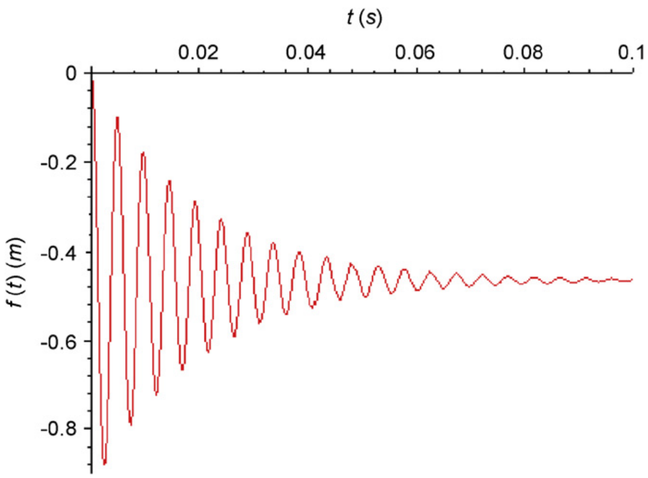

Fig. 4. Solution obtained for the case where $e_{36}^{0} \neq 0$.

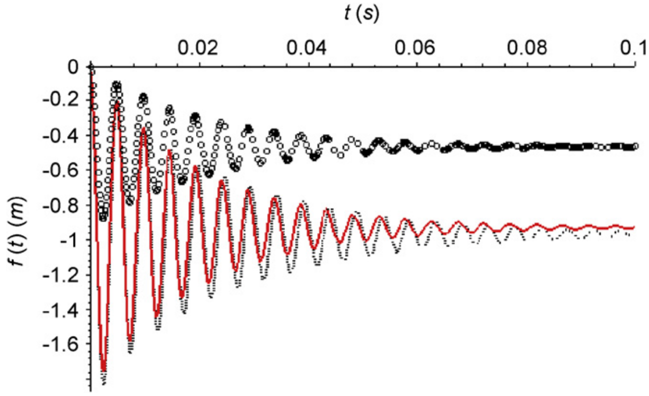

Fig. 5. Comparison of solutions for $e_{36}^0 = 0$ (— and - - - -) and $e_{36}^0 \neq 0$ (°°°).

To illustrate the results graphically appropriate code was written within the MAPLE™ software package and used for numerical visualisation. According to assumptions (9), (31) and (36), the characteristic functions $f(t)$ and $\bar{u}(t)$ of solutions $w_0(t)$ and $w(t)$, which were obtained by applying the classical and iterative multilayer plate theories for the first problem, i.e. where $e_{36}^0 = 0$, are given in Fig. 2.

If we consider the case when the mechanical loading does not promote vibration created by the electric field, and also opposes the electrical loading, i.e. $q(t) = 9.1 \times 10^8 \cos t$, then we obtain the function given in Fig. 3.

Considering the second problem described by Eq. (38), i.e. we take into account the shear force piezoelectric constant e_{36}^0, it is obvious that for the particular example described by Eq. (39) the shear force effect compensates for the mechanical loading, and as a result, the action of the electrical loading is not increased by the mechanical loading effect as shown in Fig. 4, in contrast to the case when $e_{36}^0 = 0$.

Therefore, the correlation of results obtained above for $e_{36}^0 = 0$ in terms of classical and iterative multilayer plate theories, and for $e_{36}^0 \neq 0$ can be illustrated in Fig. 5.

4. Concluding remarks

Considering initially the first problem defined by Eqs. (8) and (32) in this paper, and illustrated in Fig. 2, in which the mechanical loading is defined by Eq. (2), the electric excitation by Eq. (1), and the term responsible for the shear force effect in the piezoelectric layers e_{36}^0 is neglected. Then, by analysing Eq. (8) it has been shown that if both the mechanical and electric loadings take the form of periodic functions, then, in terms of the converse piezoelectric effect, vibration of the structure caused by the mechanical loading is increased by the mechanical strain due to the electric load action. So, the amplitude of oscillation for the model described in Eq. (8), i.e. for $e_{36}^0 = 0$, is larger than that for the case of $e_{36}^0 \neq 0$, where the action of the mechanical loading is reduced by the shear force effect (Fig. 5).

Fig. 4 illustrates the vibration of the plate subjected to the mechanical loading of Eq. (3) and the electric excitation of Eq. (1). By analysing Eqs. (38) and (39), it is obvious that the terms responsible for the mechanical loading and the shear force effect (e_{36}^0) cancel out, and we observe the result of the electrical loading action. As a result of this we obtain a system for which the mechanical loading compensates for the shear force action. It is highly significant that this effect would not have been observable if the shear force had not been taken into consideration.

We can conclude that for the cases investigated here the results of applying classical multilayer plate theory differ significantly from the results obtained when taking the shear force effect into account.

Also, since piezoelectric structures are often required to be made of several layers, and therefore exhibit anisotropy, some effects due to shear force action cannot be investigated by applying classical multilayer plate theory, and the results specifically obtained from this research for an orthotropic sandwich plate subjected to dynamic loading confirm this assumption. According to Fig. 5 it is obvious that the effect of the presence of the piezoelectric constant responsible for the shear force effect is greater than the influence of the iterative multilayer plate theory application. Therefore the application of the piezoelectric terms, which take the shear forces for the piezoelectric material into consideration, is particularly justified for applications which involve piezoelectric sandwich plate vibration.

Appendix

Simplified expressions for the correction terms $\tilde{M}_x, \tilde{M}_y, \tilde{H}$ in Eq. (33)

$$\begin{aligned}
\tilde{M}_x = \Bigg[& B_{11}\left(\frac{\pi}{a}\right)^6 \left(\frac{\delta_0^3}{3} B_{11}^0 a_{55}^0 (\delta^2-\delta_0^2) + \frac{\delta_0}{2}(\delta^2-\delta_0^2)^2 B_{11} a_{55}^0 + \frac{(\delta-\delta_0)^3}{30}(8\delta^2+9\delta\delta_0+3\delta_0^2) B_{11} a_{55}\right) \\
& + (B_{12}+2B_{66})\frac{\pi^6}{a^4 b^2}\left(\frac{\delta_0^3(\delta^2-\delta_0^2)}{3}(B_{12}^0 a_{44}^0 + B_{11}^0 a_{55}^0) + \frac{\delta_0}{2}(\delta^2-\delta_0^2)^2 (B_{12} a_{44}^0 + B_{11} a_{55}^0)\right. \\
& \left. + \frac{(\delta-\delta_0)^3}{30}(8\delta^2+9\delta\delta_0+3\delta_0^2)(B_{12} a_{44} + B_{11} a_{55})\right) \\
& + B_{11}^0 \left(\frac{\pi}{a}\right)^6 \left(\frac{4\delta_0^5}{15} B_{11}^0 a_{55}^0 + \frac{(\delta^2-\delta_0^2)\delta_0^3}{3} B_{11} a_{55}^0\right) \\
& + (B_{12}^0 + 2B_{66}^0)\frac{\pi^6}{a^4 b^2}\left(\frac{4\delta_0^5}{15}(B_{12}^0 a_{44}^0 + B_{11}^0 a_{55}^0) + \frac{(\delta^2-\delta_0^2)\delta_0^3}{3}(B_{12} a_{44}^0 + B_{11} a_{55}^0)\right) \\
& + B_{22}\frac{\pi^6}{b^4 a^2}\left(\frac{\delta_0^3(\delta^2-\delta_0^2)}{3} B_{12}^0 a_{44}^0 + \frac{\delta_0}{2}(\delta^2-\delta_0^2)^2 B_{12} a_{44}^0 + \frac{(\delta-\delta_0)^3}{30}(8\delta^2+9\delta\delta_0+3\delta_0^2) B_{12} a_{44}\right) \\
& + B_{22}^0 \frac{\pi^6}{b^4 a^2}\left(\frac{4\delta_0^5}{15} B_{12}^0 a_{44}^0 + \frac{(\delta^2-\delta_0^2)\delta_0^3}{3} B_{12} a_{44}^0\right)\Bigg],
\end{aligned}$$

$$\begin{aligned}
\tilde{M}_y = \Bigg[& B_{22}\left(\frac{\pi}{b}\right)^6 \left(\frac{\delta_0^3}{3} B_{22}^0 a_{44}^0 (\delta^2-\delta_0^2) + \frac{\delta_0}{2}(\delta^2-\delta_0^2)^2 B_{22} a_{44}^0 + \frac{(\delta-\delta_0)^3}{30}(8\delta^2+9\delta\delta_0+3\delta_0^2) B_{22} a_{44}\right) \\
& + (B_{12}+2B_{66})\frac{\pi^6}{b^4 a^2}\left(\frac{\delta_0^3(\delta^2-\delta_0^2)}{3}(B_{12}^0 a_{55}^0 + B_{22}^0 a_{44}^0) + \frac{\delta_0}{2}(\delta^2-\delta_0^2)^2 (B_{12} a_{55}^0 + B_{22} a_{44}^0)\right. \\
& \left. + \frac{(\delta-\delta_0)^3}{30}(8\delta^2+9\delta\delta_0+3\delta_0^2)(B_{12} a_{55} + B_{22} a_{44})\right) \\
& + B_{22}^0 \left(\frac{\pi}{b}\right)^6 \left(\frac{4\delta_0^5}{15} B_{22}^0 a_{44}^0 + \frac{(\delta^2-\delta_0^2)\delta_0^3}{3} B_{22} a_{44}^0\right) \\
& + (B_{12}^0 + 2B_{66}^0)\frac{\pi^6}{b^4 a^2}\left(\frac{4\delta_0^5}{15}(B_{12}^0 a_{55}^0 + B_{22}^0 a_{44}^0) + \frac{(\delta^2-\delta_0^2)\delta_0^3}{3}(B_{12} a_{55}^0 + B_{22} a_{44}^0)\right) \\
& + B_{11}\frac{\pi^6}{a^4 b^2}\left(\frac{\delta_0^3(\delta^2-\delta_0^2)}{3} B_{12}^0 a_{55}^0 + \frac{\delta_0}{2}(\delta^2-\delta_0^2)^2 B_{12} a_{55}^0 + \frac{(\delta-\delta_0)^3}{30}(8\delta^2+9\delta\delta_0+3\delta_0^2) B_{12} a_{55}\right) \\
& + B_{11}^0 \frac{\pi^6}{a^4 b^2}\left(\frac{4\delta_0^5}{15} B_{12}^0 a_{55}^0 + \frac{(\delta^2-\delta_0^2)\delta_0^3}{3} B_{12} a_{55}^0\right)\Bigg],
\end{aligned}$$

$$\begin{aligned}
\tilde{H} = & \left[B_{11}\frac{\pi^6}{a^4 b^2} + (B_{12}+2B_{66})\frac{\pi^6}{a^2 b^4}\right]\left(\frac{\delta_0^3}{3} B_{66}^0 a_{55}^0 (\delta^2-\delta_0^2) + \frac{\delta_0}{2}(\delta^2-\delta_0^2)^2 B_{66} a_{55}^0\right. \\
& \left. + \frac{(\delta-\delta_0)^3}{30}(8\delta^2+9\delta\delta_0+3\delta_0^2) B_{66} a_{55}\right) \\
& + \left[B_{11}^0 \frac{\pi^6}{a^4 b^2} + (B_{12}^0 + 2B_{66}^0)\frac{\pi^6}{a^2 b^4}\right]\left(\frac{4\delta_0^5}{15} B_{66}^0 a_{55}^0 + \frac{(\delta^2-\delta_0^2)\delta_0^3}{3} B_{66} a_{55}^0\right)
\end{aligned}$$

$$+\left[B_{22}\frac{\pi^6}{b^4a^2}+(B_{12}+2B_{66})\frac{\pi^6}{b^2a^4}\right]\left(\frac{\delta_0^3}{3}B_{66}^0a_{44}^0(\delta^2-\delta_0^2)+\frac{\delta_0}{2}(\delta^2-\delta_0^2)^2B_{66}a_{44}^0\right.$$
$$\left.+\frac{(\delta-\delta_0)^3}{30}(8\delta^2+9\delta\delta_0+3\delta_0^2)B_{66}a_{44}\right)$$
$$+\left[B_{22}^0\frac{\pi^6}{b^4a^2}+(B_{12}^0+2B_{66}^0)\frac{\pi^6}{b^2a^4}\right]\left(\frac{4\delta_0^5}{15}B_{66}^0a_{44}^0+\frac{(\delta^2-\delta_0^2)\delta_0^3}{3}B_{66}a_{44}^0\right).$$

References

[1] V.Z. Gristchak, O.A. Ganilova, Application of a hybrid WKB–Galerkin method in control of the dynamic instability of a piezolaminated imperfect column, *Technische Mechanik* 26 (2) (2006) 106–116.

[2] D. Huang, B.H. Sun, Approximate solution on smart composite beams by using Matlab, *Composite Structures* 54 (2001) 197–205.

[3] M. Ishihara, N. Noda, Nonlinear dynamic behaviour of a piezothermoelastic laminated plate with anisotropic material properties, *Acta Mechanica* 166 (2003) 103–118.

[4] V.G. Karnauhov, A.V. Kozlov, E.V. Pyatetskaya, Damping of viscoelastic plate oscillation with a help of distributed piezoelectric enclosures, *Acoustic Bulletin* 5 (4) (2002) 15–32 (in Russian).

[5] S.O. Kasap, *Principles of Electrical Engineering Materials and Devices*, McGraw-Hill, Canada, 2000.

[6] S.E. Miller, H. Abramovich, Y. Oshman, Active distributed vibration control of anisotropic piezoelectric laminated plates, *Journal of Sound and Vibration* 183 (5) (1995) 797–817.

[7] S.P. Tompson, J. Loughlan, The active buckling control of some composite column strips using piezoceramic actuators, *Composite Structures* 32 (1995) 59–67.

[8] S.A. Ambartsumian, *Theory of Anisotropic Plates. Strength, Stability and Oscillation*, Nauka, Moscow, 1987 (in Russian).

[9] S.A. Ambartsumian, *General Theory of Anisotropic Shells*, Nauka, Moscow, 1974 (in Russian).

[10] S.A. Ambartsumian, *Theory of Anisotropic Shells*, Fizmatgiz, Moscow, 1961 (in Russian).

[11] S.G. Lehnitskiy, *Anisotropic Plates*, Fizmatgiz, Moscow, 1957 (in Russian).

[12] C.M. Mota Soares, C.A. Mota Soares, V.M. Franco Correia, Optimal design of piezolaminated structures, *Composite Structures* 47 (1999) 625–634.

[13] V.Z. Gristchak, Ye.A. Dmitrieva, Hybrid WKB–Galerkin method and its application, *Technische Mechanik* 15 (1995) 281–294.

[14] C.K. Lee, Theory of laminated piezoelectric plates for the design of distributed sensors and actuators. Part 1: governing equations and reciprocal relationships, *Journal of the Acoustical Society of America* 87 (3) (1990) 1144–1158.

[15] P. Lu, H.P. Lee, C. Lu, Exact solutions for simply supported functionally graded piezoelectric laminates by Stroh-like formalism, *Composite Structures* 72 (2006) 352–363.