Available online at www.sciencedirect.com
ScienceDirect
Journal of Sound and Vibration 317 (2008) 1035–1041
JOURNAL OF SOUND AND VIBRATION
www.elsevier.com/locate/jsvi

Axisymmetric transverse vibrations of circular cylindrical shells with variable thickness

W.H. Duan, C.G. Koh*

Department of Civil Engineering, National University of Singapore, Kent Ridge, Singapore 117576, Singapore

Received 12 January 2007; received in revised form 15 January 2008; accepted 30 March 2008
Handling Editor: L.G. Tham
Available online 21 May 2008

Abstract

In this study, analytical solutions for axisymmetric transverse vibration of cylindrical shells with thickness varying monotonically in arbitrary power form due to forces acting in the transverse direction are derived for the first time, in terms of generalized hypergeometric function. To illustrate the use of the closed form solutions presented, free vibration analyses of a cylindrical shell with thickness variation under simply supported and clamped ends conditions are performed. The benchmark solutions may be used to check against numerical solutions for analysis of shells under dynamic load.

1. Introduction

Dynamics of cylindrical shells has been studied both numerically and experimentally by many researchers over a long period of time owing to its wide range of applications in engineering. Pellicano [1] proposed an analytical method based on the Sanders–Koiter theory to study linear and nonlinear vibrations of circular cylindrical shells with different boundary conditions, and the analytical results were verified by experiments. Other related references may be found in the well-known work of Leissa [2] and Love [3]. However, relatively few *analytical* solutions available in the published literature address the effect of the thickness variations on the vibration behavior of shells. Zhang and Xiang [4] presented the exact solutions for the vibration of circular cylindrical shells with step-wise thickness variations in the axial direction. The torsional vibration of circular cylindrical shells has been studied by Soni et al. [5] for varying thickness case. Radhamohan and Maith obtained the natural frequencies and buckling loads for cylindrical shells having linearly varying thickness by using a segmentation technique [6]. Fisher [7] obtained the analytical solutions of the transverse vibrations of circular cylindrical shells of uniform and linear varying thickness.

In the present note the axisymmetrical free vibration of thin circular cylindrical shells with thickness varying monotonically in arbitrary power form due to the transverse inertial force which produces transverse vibrations is studied. Transformation of variable is introduced such that the governing equation for the free vibration of varying thickness in power form is converted into a fourth-order generalized hypergeometric

*Corresponding author. Tel.: +65 6516 2163.
E-mail address: cgkoh@nus.edu.sg (C.G. Koh).

0022-460X/$ - see front matter
doi:10.1016/j.jsv.2008.03.069

Nomenclature

x, θ, and r coordinates in the longitudinal, circumferential and normal to the surface directions of a cylindrical shell, respectively
L, L_0 distances measured from the origin of the shell to the shell ends
a radius of a cylindrical shell
$\tau_1 = a/L$ dimensionless radius
h_0 shell wall thickness which occurs at $x = L$
$\tau_2 = h_0/L$ dimensionless shell wall thickness
m thickness variation parameter
$\bar{N}_x$, $\bar{N}_\theta$ and $\bar{Q}_x$, $\bar{Q}_\theta$ normal and transverse shearing forces in the x and θ directions, respectively
$\bar{N}_{x\theta} = (\bar{N}_{\theta x})$ in-plane shearing force
$\bar{M}_x$, $\bar{M}_\theta$ and $\bar{M}_{x\theta}$ bending moment and twisting moment, respectively
$\bar{u}$, $\bar{v}$ and $\bar{w}$ displacements in the x, θ and r directions, respectively
Overbar " ¯ " the normal and shearing forces, bending and twisting moments, and displacements without the overbar denote as dimensionless quantities. For example, N_x is the dimensionless form of $\bar{N}_x$
E modulus of elasticity
μ Poisson's ratio
p circular frequency
ρ mass density
n_m frequency parameter

equation. The method of Frobenius is then employed for solving the governing differential equations to arrive at the analytical solution in terms of generalized hypergeometric function. This method has been successfully applied by Duan et al. [8] in the investigation of the transverse vibration of thin plate with varying thickness.

2. Basic equation

Consider a circular cylindrical shell generated by rotating the shell wall thickness function $h = h_0(x/L)^m$ about the x-axis as shown in Fig. 1, where m is the thickness variation parameter. Shell wall thickness will be varied from h_0 occurring at $x = L$ to $h_0(L_0/L)^m$ occurring at $x = L_0$, where L and L_0 are the distances measured from the origin of the shell to the two shell ends. Let x, θ, and r be the coordinates in the longitudinal, circumferential and normal to the surface directions of a cylindrical shell, respectively. The five equilibrium equations for a shell surface element [2,9] are

$$\begin{aligned}
&a\bar{N}_{x,x} + \bar{N}_{\theta x,\theta} + p_x a = 0,\\
&\bar{N}_{\theta,\theta} + a\bar{N}_{x\theta,x} - \bar{Q}_\theta + p_\theta a = 0,\\
&\bar{Q}_{\theta,\theta} + a\bar{Q}_{x,x} + \bar{N}_\theta - p_r a = 0,\\
&\bar{M}_{\theta,\theta} + a\bar{M}_{x\theta,x} - a\bar{Q}_\theta = 0,\\
&a\bar{M}_{x,x} + \bar{M}_{\theta x,\theta} - a\bar{Q}_x = 0.
\end{aligned} \tag{1}$$

Eq. (1) may be considered as the equations of motion if the inertial forces are included in the external forces, p_x, p_θ and p_r, acting in the directions indicated by the subscripts. $\bar{N}_x$, $\bar{N}_\theta$ and $\bar{Q}_x$, $\bar{Q}_\theta$ are the normal and transverse shearing forces in the x and θ directions, respectively; $\bar{N}_{x\theta} = (\bar{N}_{\theta x})$ is the in-plane shearing force; $\bar{M}_x$, $\bar{M}_\theta$ and $\bar{M}_{x\theta} = (\bar{M}_{\theta x})$ are the bending moment and the twisting moment, respectively. A prime followed by a subscript denotes a derivative with respect to the subscript. The radius of the shell, a, is measured from the central axis to the mid surface of the shell.

For the axisymmetric case, only the inertia force p_r, $\bar{N}_\theta$, $\bar{M}_x$, $\bar{M}_\theta$, and $\bar{Q}_x$ remain in Eq. (1), i.e. $p_x = 0$, $p_\theta = 0$, $\bar{N}_x = 0$, $\bar{N}_{x\theta} = 0$, $\bar{Q}_\theta = 0$, and $\bar{M}_{x\theta} = 0$. By dropping all derivatives with respect to θ, Eq. (1) can be reduced to

$$\begin{aligned}
&a\bar{Q}_{x,x} + \bar{N}_\theta - p_r a = 0,\\
&\bar{M}_{x,x} - \bar{Q}_x = 0.
\end{aligned} \tag{2}$$

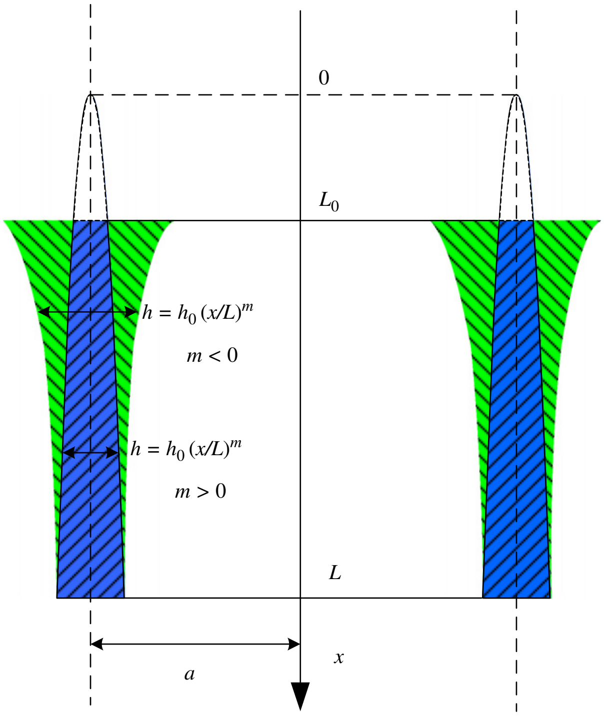

Fig. 1. Cylindrical shell with variable wall thickness.

Eliminating $\bar{Q}_x$ yields

$$a\bar{M}_{x,xx} + \bar{N}_\theta - p_r a = 0. \tag{3}$$

Let

$$\begin{aligned}
&\bar{u} = h_0 u(\eta)\exp(\mathrm{i}pt), \quad \bar{v} = h_0 v(\eta)\exp(\mathrm{i}pt), \quad \bar{w} = h_0 w(\eta)\exp(\mathrm{i}pt),\\
&\bar{N}_x = Eh_0 N_x(\eta)\exp(\mathrm{i}pt), \quad \bar{N}_\theta = Eh_0 N_\theta(\eta)\exp(\mathrm{i}pt),\\
&\bar{M}_x = Eh_0 a M_x(\eta)\exp(\mathrm{i}pt), \quad h = h_0\eta^m, \quad p_r = \rho h_0 \eta^m p^2 h_0 w(\eta),
\end{aligned} \tag{4}$$

where t is time, E is the modulus of elasticity, p is the circular frequency, ρ is the mass density, $\bar{u}$, $\bar{v}$ and $\bar{w}$ are the displacements in the x, θ and r directions, respectively, and $\eta = x/L$ is a dimensionless variable.

In the context of linearly elastic response, the following force–displacement relations may be used:

$$\begin{aligned}
N_x &= \frac{\tau_2\eta^m}{1-\mu^2}\left(\frac{\mu}{\tau_1}w + u_{,\eta}\right),\\
N_\theta &= \frac{\tau_2\eta^m}{1-\mu^2}\left(\frac{w}{\tau_1} + \mu u_{,\eta}\right),\\
M_x &= \frac{\eta^{3m}(\tau_2)^3 w_{,\eta\eta}}{12(1-\mu^2)\tau_1},
\end{aligned} \tag{5a − c}$$

in which

$$\tau_1 = \frac{a}{L}, \quad \tau_2 = \frac{h_0}{L}, \tag{6}$$

and μ is Poisson's ratio. Substituting Eqs. (5a)–(5c) into Eq. (3) yields

$$M_{x,\eta\eta} + \frac{1}{\tau_1^2}N_\theta - \frac{\tau_2\omega^2\eta^m}{\tau_1}w = 0, \tag{7}$$

where $\omega^2 = (\rho p^2 L^2/E)$.

3. Closed form solutions

Since only vibration in the transverse direction is considered, we have

$$N_x = 0. \tag{8}$$

Thus, Eqs. (5a) and (5b) yield

$$u_{,\eta} = -\frac{\mu}{\tau_1}w, \tag{9}$$

$$N_\theta = \frac{\tau_2}{\tau_1}\eta^m w. \tag{10}$$

Substituting Eqs. (10) and (5c) into Eq. (7) leads to a homogeneous linear ordinary differential equation with variable coefficients

$$\eta^2 w_{,\eta\eta\eta\eta} + 6m\eta w_{,\eta\eta\eta} + 3m(3m-1)w_{,\eta\eta} - n_m^4\pi^4\eta^{2-2m}w = 0, \tag{11}$$

where frequency parameter n_m is defined as $n_m^4\pi^4 = (12(1-\mu^2)/\tau_1^2\tau_2^2)(\omega^2\tau_1^2 - 1)$, therefore

$$\omega^2 = \frac{1}{\tau_1^2}\left(\frac{\tau_1^2\tau_2^2}{12(1-\mu^2)}n_m^4\pi^4 + 1\right). \tag{12}$$

A novel transformation to solve Eq. (11) is introduced, in which m can be any number except $m \neq 2$, given by

$$r = \frac{n_m^4\pi^4\eta^{4-2m}}{(4-2m)^4}. \tag{13}$$

Thus, Eq. (11) is transformed to a generalized hypergeometric equation:

$$\left\{1 - \frac{1}{r}\prod_{i=1}^{4}(\vartheta + \gamma_i - 1)\right\}w(r) = 0, \tag{14}$$

where

$$\vartheta = r\frac{\partial}{\partial r}, \quad \gamma_1 = 1, \quad \gamma_2 = 1 - \frac{1}{4-2m}, \quad \gamma_3 = 1 + \frac{3m-3}{4-2m}, \quad \gamma_4 = 1 + \frac{3m-2}{4-2m}. \tag{15}$$

According to Frobenius theory, if no two values of γ_i are equal or differ by an integer value, the solutions of Eq. (14) are non-logarithmic and may be written in the form [8]

$$\begin{aligned}
w_1(r) &= r^{1-\gamma_1}{}_0F_3([\,],[1+\gamma_2-\gamma_1, 1+\gamma_3-\gamma_1, 1+\gamma_4-\gamma_1], r),\\
w_2(r) &= r^{1-\gamma_2}{}_0F_3([\,],[1+\gamma_1-\gamma_2, 1+\gamma_3-\gamma_2, 1+\gamma_4-\gamma_2], r),\\
w_3(r) &= r^{1-\gamma_3}{}_0F_3([\,],[1+\gamma_1-\gamma_3, 1+\gamma_2-\gamma_3, 1+\gamma_4-\gamma_3], r),\\
w_4(r) &= r^{1-\gamma_4}{}_0F_3([\,],[1+\gamma_1-\gamma_4, 1+\gamma_2-\gamma_4, 1+\gamma_3-\gamma_4], r),
\end{aligned} \tag{16}$$

where ${}_0F_3([\,],[1+\gamma_2-\gamma_1, 1+\gamma_3-\gamma_1, 1+\gamma_4-\gamma_1], r)$ is the generalized hypergeometric function. The series form of the function ${}_0F_q$ is given by

$${}_0F_q([\,],[b_1, b_2, \ldots, b_q], r) = 1 + \sum_{k=1}^{\infty}\frac{r^k}{\Pi_{j=1}^{q}(b_j)_k k!}, \tag{17}$$

where $(b_j)_k = (\Gamma(b_j + k)/\Gamma(b_j)) = b_j(b_j + 1)\cdots(b_j + k - 1)$. The complete solution of Eq. (14) can be expressed as

$$w(r) = \sum_{i=1}^{4} c_i w_i(r), \tag{18}$$

where c_i are non-zero constants.

For certain particular values of m such as

$$m = \frac{1}{3}, \frac{2}{3}, 2 + \frac{1}{2n}, 2 - \frac{4}{2n+3}, \text{ or } 2 - \frac{3}{2n+3}, \quad n \text{ is an integer}, \tag{19}$$

two of γ_i are equal or differ by an integer value. There is no loss of generality in taking these as γ_1 and γ_2, arranged with their real parts in ascending order. Under these conditions, the solution $w_2(r)$ is degenerated to logarithmic form. The logarithmic form solutions of Eq. (14) have been studied by Duan et al. [8]. Hence, $w_2(r)$ in logarithmic form is given as follows without derivations:

$$\begin{aligned} w_2(r) = {} & w_1(r)\ln r + r^{1-\gamma_1} \sum_{s=0}^{\infty} \Psi_{0s}^{10} r^s \prod_{i=1}^{4} \frac{\Gamma(1-\gamma_1+\gamma_i)}{\Gamma(1-\gamma_1+\gamma_i+s)} \\ & + {}_5F_0\left([1, 1, 1+\gamma_1-\gamma_2, 1+\gamma_1-\gamma_3, 1+\gamma_1-\gamma_4], [\,], \frac{1}{r}\right) \frac{1}{r^{\gamma_1}} \prod_{i=2}^{4} (\gamma_i - \gamma_1), \end{aligned} \tag{20}$$

where $\Psi_{0s}^{10} = \sum_{t=2}^{4} \varphi_0(1-\gamma_2+\gamma_t) + \varphi_0(\gamma_2-\gamma_1) - \sum_{t=1}^{4} \varphi_0(1-\gamma_1+\gamma_t+s)$ and φ_0 is a polygamma function.

4. Numerical results and discussion

The variation of the frequency parameter n_m with the taper (represented by the power of thickness function) of the thickness of the shell under both simply supported and clamped ends is considered. The simply supported condition does not permit the linear motion along the x-axis but does not hinder rotation of the shell cross-section at the supports. It follows that

$$w = 0, \quad M_x = 0 \quad \text{at } \eta = \beta, 1, \tag{21}$$

where $\beta = L_0/L$. As for the clamped condition, the linear motion and rotation along the x-axis become zero. That is

$$w = 0, \quad w_{,\eta} = 0 \quad \text{at } \eta = \beta, 1. \tag{22}$$

Substituting Eqs (18) and (5c) into Eqs. (21) and (22), the matrices involving the system frequencies can be formulated for simply supported and clamped shells, respectively. The matrices are too lengthy and not shown in this short communication. Setting the determinant of these matrices to zero yields the frequency parameter n_m.

In order to assess the validity of the results provided by the analytical approach, a finite element model (FEM) of the shell with clamped ends is prepared using ABAQUS 6.4. The parameters for the shell are as follows: $L = 1.0\,\text{m}$, $L_0 = 0.8\,\text{m}$, $a = 1\,\text{m}$, $h_0 = 0.01\,\text{m}$, $E = 70\,\text{GPa}$, $\mu = 0.3$, and $\rho = 2700\,\text{kg/m}^3$. The thickness variation parameter m is set to be 1 (linear varying thickness). The finite element model is represented by a mesh of 400 axisymmetric solid element CAX8R (8-node biquadratic, reduced integration). Lanczos iterative technique was adopted to compute the natural frequencies of the shell. The comparison between the analytical and numerical results is shown in Table 1. The good agreement of less than 5% maximum difference indicates that the correctness of proposed solutions in this paper.

The variation of the first four frequency parameters n_m with the power of thickness function m for the shell with clamped ends condition is presented in Table 2, where m is varied from -2 to 1.8 ($m \neq 2$ in view of Eq. (13)) and β is set to be 0.2. It is seen that the values of the frequency parameters n_m can be substantially altered due to the variation of thickness variation parameter m. For example, the first four frequency parameters n_m for the case of linear varying thickness ($m = 1$), i.e. 1.3692, 2.2674, 3.1708, and 4.0744, are rather smaller than those parameters for the case of uniform thickness ($m = 0$), i.e. 1.8820, 3.1246, 4.3749 and 5.6248.

Table 1
Comparison of frequency parameter n_m of the shell under clamped ends condition between FEM and proposed solutions for $m = 1$ (linear varying thickness)

m	1st mode	2nd mode	3rd mode	4th mode
Proposed solutions	7.1315	11.8397	16.5768	21.3127
FEM	7.1386	11.6574	16.1354	20.4568
Difference (%)	0.10	−1.56	−2.74	−4.18

Table 2
First four frequency parameter n_m under clamped ends condition

m	1st mode	2nd mode	3rd mode	4th mode
−2.0	3.8309	5.9022	7.9473	9.9861
−1.8	3.6027	5.5731	7.5242	9.4730
−1.6	3.3766	5.2495	7.1107	8.9732
−1.4	3.1541	4.9340	6.7096	8.4896
−1.2	2.9372	4.6293	6.3237	8.0239
−1.0	2.7283	4.3381	5.9548	7.5775
−0.8	2.5301	4.0624	5.6038	7.1505
−0.6	2.3451	3.8032	5.2710	6.7424
−0.4	2.1752	3.5609	4.9558	6.3527
−0.2	2.0209	3.3350	4.6574	5.9804
0.0	1.8820	3.1246	4.3749	5.6248
0.2	1.7571	2.9285	4.1073	5.2852
0.4	1.6448	2.7457	3.8538	4.9608
0.6	1.5434	2.5752	3.6137	4.6512
0.8	1.4518	2.4160	3.3863	4.3559
1.0	1.3692	2.2674	3.1708	4.0744
1.2	1.2949	2.1285	2.9668	3.8063
1.4	1.2288	1.9986	2.7737	3.5513
1.6	1.1702	1.8769	2.5909	3.3090
1.8	1.1183	1.7627	2.1681	3.0246

The percentage differences $(n_m|_{m=0} - n_m|_{m=1}/n_m|_{m=0}) \times 100\%$ in the frequency parameters are 27.25%, 27.43%, 27.52% and 27.56%, respectively. In addition, the first four frequency parameters n_m for the case of negative quadratic varying thickness ($m = -2$) are 3.8309, 5.9022, 7.9473 and 9.9861, respectively. The percentage differences $(n_m|_{m=-2} - n_m|_{m=0}/n_m|_{m=-2}) \times 100\%$ in the frequency parameters are 28.78%, 26.50%, 25.07% and 24.12%, respectively.

The first four frequency parameters n_m for the shell with simply supported ends condition versus the power of thickness function m are presented in Fig. 2, where m is varied from −1 to 1 and β is set to be 0.1. Firstly, when $m = 0$, n_m equal to 1, 2, 3, and 4 for the first four modes respectively, therefore Eq. (12) becomes

$$\omega^2 = \frac{1}{\tau_1^2}\left(\frac{\tau_1^2\tau_2^2}{12(1-\mu^2)}n^4\pi^4 + 1\right), \quad n \text{ is } 1, 2, 3 \text{ and } 4. \tag{23}$$

This is consistent with the results from the shell with uniform thickness [10]. This agreement verifies the proposed solution in the special case of uniform thickness. In addition, it can be seen in Fig. 1 that when the power m increase from −1 to 1, the thickness of the shell keeps the same as h_0 at the $\eta = 1$ while it varies from a larger value $h_0\eta^{-1}$ to a smaller value $h_0\eta$ at $\eta = \beta$. Hence, the frequency parameters n_m are decreased as reflected in Fig. 2. For example, the frequency parameters n_m for the first four modes are decreased from 1.4057, 3.0369, 4.6059 and 6.1664 to 0.70153, 1.5112, 2.2459 and 2.976, respectively, as the power m increase from −1 to 1.

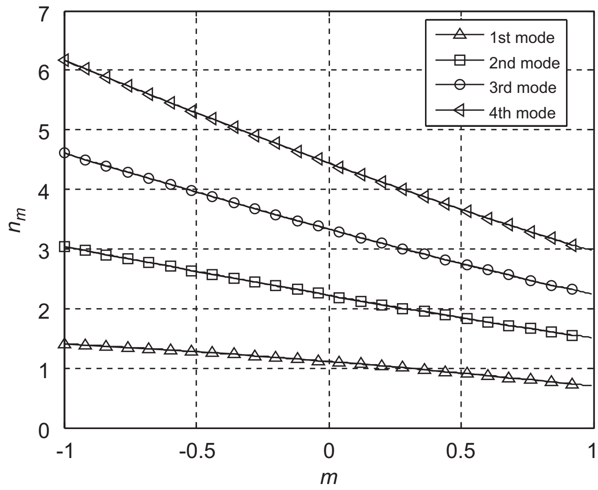

Fig. 2. Variation of first four frequency parameter n_m with m.

From Eq. (12), Fig. 2 and Table 1 it can be seen that, besides geometry parameters τ_1 and τ_2, the thickness function m allows significant adjustment for frequency of shells. This could be useful for the optimization for the design of shells under dynamics loads.

Acknowledgments

The authors are grateful to the financial support by Maritime and Port Authority of Singapore and the National University of Singapore.

References

[1] F. Pellicano, Vibrations of circular cylindrical shells: theory and experiments, *Journal of Sound and Vibration* 303 (2007) 154–170.

[2] A.W. Leissa, Vibration of Shells, Scientific and Technical Information Office, National Aeronautics and Space Administration, Washington, 1973.

[3] A.E.H. Love, *A Treatise on the Mathematical Theory of Elasticity*, fourth ed., Dover Publications, New York, 1944.

[4] L. Zhang, Y. Xiang, Exact solutions for vibration of stepped circular cylindrical shells, *Journal of Sound and Vibration* 299 (2006) 948–964.

[5] S.R. Soni, R.K. Jain, C. Prasad, Torsional vibrations of shells of revolution of variable thickness, *The Journal of the Acoustical Society of America* 53 (1973) 1445–1447.

[6] S.K. Radhamohan, M. Maiti, Vibrations of initially stressed cylinders of variable thickness, *Journal of Sound and Vibration* 53 (1977) 267–271.

[7] H.D. Fisher, The axisymmetric free vibration and transient response of a clamped cylindrical shell of linear thickness, *Nuclear Engineering and Design* 30 (1974) 278–285.

[8] W.H. Duan, S.T. Quek, Q. Wang, Generalized hypergeometric function solutions for transverse vibration of a class of non-uniform annular plates, *Journal of Sound and Vibration* 287 (2005) 785–807.

[9] W. Flügge, *Stresses in Shells*, second ed., Springer, Berlin/New York, 1973.

[10] T. Marku, *The Mechanics of Vibrations of Cylindrical Shells*, Elsevier, Amsterdam/New York, 1988.