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ScienceDirect
Journal of Sound and Vibration 317 (2008) 440–448
JOURNAL OF SOUND AND VIBRATION
www.elsevier.com/locate/jsvi

Rapid Communication

Higher-order approximate solutions for nonlinear vibration of a constant-tension string

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Received 11 April 2008; accepted 6 May 2008
Handling Editor: L.G. Tham
Available online 20 June 2008

Abstract

A new harmonic balance approach for solving the large amplitude nonlinear vibration of a constant-tension string is introduced. The coupling of Newton's method with harmonic balancing takes the advantage of reducing the deficiency and complexity of the classical harmonic balance method in dealing with the nonlinear systems. The solutions are directly induced from a set of linear algebraic equations instead of a set of complicated, coupled nonlinear algebraic equations. Illustrative examples are selected and compared to some published data to verify the accuracy of the higher-order solutions.

Consider a partial differential wave equation which models the pure, geometrical nonlinear effect of the curvature for large amplitude transverse vibrations of a flexible string under constant tension [1–5]

$$c^2\frac{\partial^2 u}{\partial x^2}=\left[1+\left(\frac{\partial u}{\partial x}\right)^2\right]^2\frac{\partial^2 u}{\partial t^2} \tag{1}$$

where $u(x,t)$ is the transverse amplitude in relation to the spatial x and temporal t coordinates, $c=\sqrt{\tau_0/\rho_0}$ is the velocity of transverse wave with τ_0 and ρ_0 being the tension and the mass per unit length, respectively. The string vibrates between the fixed end-points of length L governed by the following boundary conditions:

$$u(0,t)=u(L,t)=0 \tag{2}$$

By virtue of solving Eq. (1) readily, Gottlieb [3] proposed the reduction procedure to turn the wave equation into an ordinary differential equation by means of the averaging technique. The partial differential equation can be transformed into spatial or temporal ordinary differential equations relying upon the amplitude function deemed as the harmonic form (i.e. $u(x,t)=U(x)\cos(\omega t)$) or the fundamental modal shape for linear

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0022-460X/$ - see front matter
doi:10.1016/j.jsv.2008.05.003

wave equation (i.e. $u(x,t) = V(t)\sin(\pi x/L)$), respectively. In this communication, it mainly investigates the solution approach for the temporal nonlinear ordinary nonlinear differential equation [3,4]. Taking

$$u(x,t) = V(t)\sin\left(\frac{\pi}{L}x\right) \tag{3}$$

Substituting Eq. (3) into Eq. (1) and averaging over the string length L (i.e. a Galerkin procedure is performed to take the multiplication of Eq. (3) by Eq. (1) and integrate such equation to x from 0 to L) result in an ordinary second-order differential equation as

$$\frac{\mathrm{d}^2 S}{\mathrm{d}t^2} = -\frac{\beta S}{1 + S^2/2 + S^4/8} \tag{4}$$

where

$$S(t) = \frac{\pi}{L}V(t) \tag{5}$$

$$\beta = \left(\frac{\pi c}{L}\right)^2 \tag{6}$$

with the initial conditions

$$S(0) = a, \quad \frac{\mathrm{d}S(0)}{\mathrm{d}t} = 0 \tag{7}$$

The ordinary differential equation (4) has been investigated by the classical harmonic balance (HB) method [3] but not the perturbation method because of the absence of a small parameter [6]. Although the HB method [6] is a powerful technique for solving nonlinear systems with large parameters, higher-order analytical approximations are extremely hard to be developed owing to the appearance and coupling of nonlinear terms in a set of nonlinear algebraic equations. Hence, the dilemma encountered in solving nonlinear algebraic equations is that some nonlinear terms must be either neglected in order to give way for an asymptotic expression or implicated by numerical analysis to obtain numerical results. As a result, the main focus of this communication is to introduce an effective and accurate approximate analytical approach by coupling Newton's method with the harmonic balance method (NHB) [7] to increase the flexibility and overcome the deficiency of the classical HB method. The higher-order solutions of the NHB method have been developed in multiple applications such as nonlinear Jerk equation involving the third-temporal derivative of displacement [8], the Duffing-harmonic oscillator having a rational form for the restoring force [9], and the post-buckling deformation of a ring under uniform hydrostatic pressure [10]. Thus, the NHB method is adopted herein for solving the temporal ordinary differential equation derived from the nonlinear vibration of a constant-tension string.

By introducing an independent variable, $\tau = \sqrt{\hat{\Omega}}t$, Eqs. (4) and (7) are rewritten as

$$\Omega S''\left(1 + \frac{1}{2}S^2 + \frac{1}{8}S^4\right) + S = 0 \tag{8}$$

and

$$S(0) = a, \quad \frac{\mathrm{d}S(0)}{\mathrm{d}\tau} = 0 \tag{9}$$

where $\Omega = \hat{\Omega}/\beta$ and a prime denotes differentiation with respect to τ. The independent variable is chosen in such a way that the solution of Eq. (8) is a periodic solution of τ of period 2π. In Eq. (4), the restoring force function $-f(S) = -\beta S/(1+S^2/2+S^4/8)$ is an odd function (i.e. $-f(S) = f(-S)$), so that the system vibrates around the equilibrium position between the symmetric limits $[-a, a]$ and the periodic solution is written as $S(\tau) = \sum_{j=0}^{\infty} k_{2j+1}\cos[(2j+1)\tau]$. By means of Newton's method [7], the squared angular frequency parameter $\Omega = \hat{\Omega}/\beta$ and periodic solution S are set as

$$\Omega = \Omega_1 + \Delta\Omega_1 \tag{10}$$

$$S = S_1 + \Delta S_1 \tag{11}$$

Substituting both Eqs. (10) and (11) into Eq. (8) and linearizing the governing equation obtain

$$\begin{aligned}&\Omega_1\left(S_1'' + \Delta S_1'' + \tfrac{1}{2}S_1''S_1^2 + \tfrac{1}{2}S_1^2\Delta S_1'' + \tfrac{1}{8}S_1''S_1^4 + \tfrac{1}{8}S_1^4\Delta S_1'' + S_1''S_1\Delta S_1 + \tfrac{1}{2}S_1''S_1^3\Delta S_1\right)\\ &\quad + \Delta\Omega_1\left(S_1'' + \tfrac{1}{2}S_1''S_1^2 + \tfrac{1}{8}S_1''S_1^4\right) + S_1 + \Delta S_1 = 0\end{aligned} \tag{12}$$

To fulfill the initial conditions of Eq. (9), it implies that

$$\Delta S_1(0) = 0, \quad \Delta S_1'(0) = 0 \tag{13}$$

where $\Delta S_1(\tau)$ is a periodic function of τ with a period of 2π to be determined later.

For the lowest-order or first-order analytical approximation, we initially set

$$S_1(\tau) = a\cos\tau \tag{14}$$

$$\Delta\Omega_1 = 0, \quad \Delta S_1 = 0 \tag{15}$$

Substituting Eqs. (14) and (15) into Eq. (12) and setting the coefficient of $\cos\tau$ to zero yields

$$\Omega_1(a) = \frac{\hat{\Omega}_1(a)}{\beta} = \frac{64}{64 + 24a^2 + 5a^4} \tag{16}$$

Therefore, the first-order analytical approximation of period and periodic solution for the temporal equation (4) is

$$T_1(a) = \frac{2\pi}{\sqrt{\beta}}\left(\frac{64}{64 + 24a^2 + 5a^4}\right)^{-(1/2)} \tag{17}$$

and

$$S_1(t) = a\cos\sqrt{\beta\Omega_1}\,t \tag{18}$$

Eqs. (17) and (18) are equivalent to the first approximate solution derived by Gottlieb [3].

For the second-order analytical approximation, we substitute the following equation to Eq. (12):

$$\Delta S_1(\tau) = c_1(\cos\tau - \cos 3\tau) \tag{19}$$

Expanding the resulting expression into a trigonometric series and setting the coefficients of $\cos\tau$ and $\cos 3\tau$ to zero yield two linear algebraic equations as

$$\begin{aligned}&a + c_1 - a\,\Delta\Omega_1 - \tfrac{3}{8}a^3\,\Delta\Omega_1 - \tfrac{5}{64}a^5\,\Delta\Omega_1 - a\Omega_1 - \tfrac{3}{8}a^3\Omega_1 - \tfrac{5}{64}a^5\Omega_1\\ &\quad - c_1\Omega_1 + \tfrac{1}{4}a^2c_1\Omega_1 + \tfrac{15}{128}a^4c_1\Omega_1 = 0\end{aligned} \tag{20}$$

$$-c_1 - \tfrac{1}{8}a^3\,\Delta\Omega_1 - \tfrac{5}{128}a^5\,\Delta\Omega_1 - \tfrac{1}{8}a^3\Omega_1 - \tfrac{5}{128}a^5\Omega_1 + 9c_1\Omega_1 + \tfrac{19}{8}a^2c_1\Omega_1 + \tfrac{53}{128}a^4c_1\Omega_1 = 0 \tag{21}$$

Therefore, the second-order analytical approximation of period and periodic solution for the temporal equation (4) is

$$T_2(a) = \frac{2\pi}{\sqrt{\beta\Omega_2}}, \quad \Omega_2 = \Omega_1 + \Delta\Omega_1 \tag{22}$$

and

$$S_2(t) = (a + c_1)\cos\sqrt{\beta\Omega_2}\,t - c_1\cos 3\sqrt{\beta\Omega_2}\,t \tag{23}$$

where

$$\begin{aligned}\Delta\Omega_1(a) = & -(16,384 - 163,840\Omega_1 - 47,104a^2\Omega_1 - 8704a^4\Omega_1 + 147,456\Omega_1^2 \\ & + 96,256a^2\Omega_1^2 + 33,024a^4\Omega_1^2 + 5184a^6\Omega_1^2 + 455a^8\Omega_1^2)/(-16,384 \\ & - 8192a^2 - 1920a^4 + 147,456\Omega_1 + 96,256a^2\Omega_1 + 33,024a^4\Omega_1 \\ & + 5184a^6\Omega_1 + 455a^8\Omega_1)\end{aligned} \tag{24}$$

$$\begin{aligned}c_1(a) = & (2048a^3 + 640a^5)/(-16,384 - 8192a^2 - 1920a^4 + 147,456\Omega_1 \\ & + 96,256a^2\Omega_1 + 33,024a^4\Omega_1 + 5184a^6\Omega_1 + 455a^8\Omega_1)\end{aligned} \tag{25}$$

and Ω_1 in Eqs. (22), (24) and (25) can be found from Eq. (16).

The derivation of third-order analytical approximation is based on the second-order analytical approximation, thus Eq. (12) is expressed as

$$\begin{aligned}&\Omega_2\left(S_2'' + \Delta S_2'' + \tfrac{1}{2}S_2''S_2^2 + \tfrac{1}{2}S_2^2\,\Delta S_2'' + \tfrac{1}{8}S_2''S_2^4 + \tfrac{1}{8}S_2^4\,\Delta S_2'' + S_2''S_2\,\Delta S_2 + \tfrac{1}{2}S_2''S_2^3\,\Delta S_2\right) \\ &+ \Delta\Omega_2\left(S_2'' + \tfrac{1}{2}S_2''S_2^2 + \tfrac{1}{8}S_2''S_2^4\right) + S_2 + \Delta S_2 = 0\end{aligned} \tag{26}$$

The function $\Delta S_2(\tau)$ is set as

$$\Delta S_2(\tau) = c_2(\cos\tau - \cos 3\tau) + c_3(\cos 3\tau - \cos 5\tau) \tag{27}$$

Hence, by substituting of Eqs. (23) and (27) into Eq. (26) and expanding of the resulting expression, then setting the coefficients of $\cos\tau$, $\cos 3\tau$ and $\cos 5\tau$ to zero result in three linear algebraic equations as

$$\kappa_1\,\Delta\Omega_2 + \kappa_2 c_2 + \kappa_3 c_3 + \kappa_4 = 0 \tag{28}$$

$$\kappa_5\,\Delta\Omega_2 + \kappa_6 c_2 + \kappa_7 c_3 + \kappa_8 = 0 \tag{29}$$

$$\kappa_9\,\Delta\Omega_2 + \kappa_{10} c_2 + \kappa_{11} c_3 + \kappa_{12} = 0 \tag{30}$$

Solving the simultaneous equations (28)–(30) obtains

$$\Delta\Omega_2(a) = -\frac{-\kappa_4\kappa_7\kappa_{10} + \kappa_3\kappa_8\kappa_{10} + \kappa_4\kappa_6\kappa_{11} - \kappa_2\kappa_8\kappa_{11} - \kappa_3\kappa_6\kappa_{12} + \kappa_2\kappa_7\kappa_{12}}{-\kappa_3\kappa_6\kappa_9 + \kappa_2\kappa_7\kappa_9 + \kappa_3\kappa_5\kappa_{10} - \kappa_1\kappa_7\kappa_{10} - \kappa_2\kappa_5\kappa_{11} + \kappa_1\kappa_6\kappa_{11}} \tag{31}$$

$$c_2(a) = -\frac{-\kappa_4\kappa_7\kappa_9 + \kappa_3\kappa_8\kappa_9 + \kappa_4\kappa_5\kappa_{11} - \kappa_1\kappa_8\kappa_{11} - \kappa_3\kappa_5\kappa_{12} + \kappa_1\kappa_7\kappa_{12}}{\kappa_3\kappa_6\kappa_9 - \kappa_2\kappa_7\kappa_9 - \kappa_3\kappa_5\kappa_{10} + \kappa_1\kappa_7\kappa_{10} + \kappa_2\kappa_5\kappa_{11} - \kappa_1\kappa_6\kappa_{11}} \tag{32}$$

$$c_3(a) = -\frac{\kappa_4\kappa_6\kappa_9 - \kappa_2\kappa_8\kappa_9 - \kappa_4\kappa_5\kappa_{10} + \kappa_1\kappa_8\kappa_{10} + \kappa_2\kappa_5\kappa_{12} - \kappa_1\kappa_6\kappa_{12}}{\kappa_3\kappa_6\kappa_9 - \kappa_2\kappa_7\kappa_9 - \kappa_3\kappa_5\kappa_{10} + \kappa_1\kappa_7\kappa_{10} + \kappa_2\kappa_5\kappa_{11} - \kappa_1\kappa_6\kappa_{11}} \tag{33}$$

Therefore, the third-order analytical approximation of period and periodic solution for the temporal equation (4) is

$$T_3(a) = \frac{2\pi}{\sqrt{\beta\Omega_3}}, \quad \Omega_3 = \Omega_2 + \Delta\Omega_2 \tag{34}$$

and

$$S_3(t) = (a + c_1 + c_2)\cos\sqrt{\beta\Omega_3}t + (c_3 - c_2 - c_1)\cos 3\sqrt{\beta\Omega_3}t - c_3\cos 5\sqrt{\beta\Omega_3}t \tag{35}$$

where c_1 in Eq. (35) is derived in Eq. (25) and κ_i ($i = 1$–12) in Eqs. (28)–(33) are presented in Appendix A. Further analytical approximations can be developed based on the procedure as above for the first three approximate solutions.

For reference, the exact period $T_e(a)$ and exact frequency $\hat{\omega}_e(a) = \sqrt{\beta}\omega_e(a)$ are derived here via direct integration of Eq. (4) with the help of Eq. (7) as

$$T_e(a) = \frac{2\pi}{\sqrt{\beta}\omega_e(a)} = \frac{2}{\sqrt{\beta}} \int_0^{\pi/2} \frac{a \cos\theta}{\sqrt{\tan^{-1}(1 + (a^2/2)) - \tan^{-1}(1 + (a^2 \sin^2\theta/2))}} \, d\theta \tag{36}$$

The subscripts "HB1", "HB2", "1", "2", "3" and "e" of period T and periodic solution S denote, respectively, the first-order HB solution [3], the second-order HB solution [3], the first-order NHB solution, the second-order NHB solution, the third-order NHB solution and the exact solution. Because the exact period and frequency are written in terms of an implicit function in Eq. (36), they cannot provide an overall view of the nature of the systems in response to changes in parameters that affect nonlinearity. From a computational point of view, the exact periodic solutions cannot be explicitly obtained by substituting the initial conditions as explained above.

As observed in Tables 1 and 2, it is obvious that the third-order analytical approximation $\sqrt{\beta}T_3$ in these cases shows the best agreement as compared to the exact solution $\sqrt{\beta}T_e$ among the other solutions. The accuracy of the first-order $\sqrt{\beta}T_{\mathrm{HB1}}$ and second-order $\sqrt{\beta}T_{\mathrm{HB2}}$ analytical approximations constructed by the HB method [3] is, respectively, similar to the first-order $\sqrt{\beta}T_1$ and second-order $\sqrt{\beta}T_2$ analytical approximations developed herein by the NHB method. Although the third-order solutions of Eq. (4) can be alternatively constructed using the classical HB method, the most salient advantage of the NHB method over the classical ones is its solutions obtained from a set of linear algebraic equations instead of nonlinear algebraic equations.

To determine the exact period as $a \to 0$, we can define $S \ll 1$ in Eq. (4). The temporal nonlinear differential equation is reduced to a linear harmonic oscillating system $d^2S/dt^2 + \beta S = 0$. Such reduction to a linear harmonic system is similar to the designation of a rational conservative Duffing-harmonic oscillator [9,11]. The exact period is then obviously derived as $T_e = 2\pi \big/ \sqrt{\beta}$ as $a \to 0$. From Eqs. (17), (22) and (34), we can also derive the following relations:

$$\underset{\lim\, a\to 0}{\sqrt{\beta}T_1} = \underset{\lim\, a\to 0}{\sqrt{\beta}T_2} = \underset{\lim\, a\to 0}{\sqrt{\beta}T_3} = 2\pi \tag{37}$$

Table 1
Comparison of the exact and approximate solutions for the period parameters $\sqrt{\beta}T$

$S(0) = a$	$\sqrt{\beta}T_{\mathrm{HB1}}$, Eq. (4.2) in Ref. [3]	$\sqrt{\beta}T_e$, Eq. (36) or Ref. [3]	$\sqrt{\beta}T_1$, Eq. (17)	$\sqrt{\beta}T_2$, Eq. (22)	$\sqrt{\beta}T_3$, Eq. (34)
0.1	6.294980	6.294977	6.294980	6.294977	6.294977
0.2	6.33052	6.33047	6.33052	6.33047	6.33047
0.5	6.58576	6.58379	6.58576	6.58377	6.58379
1.0	7.57411	7.54147	7.57411	7.54010	7.54140
2.0	12.1673	11.7148	12.1673	11.6772	11.7066
5.0	48.345	41.918	48.345	42.173	41.875
10.0	179.9	148.8	179.9	152.9	149.4

Table 2
Comparison of the exact and approximate solutions for the period parameters $\sqrt{\beta}T$

$S(0) = a$	$\sqrt{\beta}T_{\mathrm{HB2}}$, Eq. (4.3) in Ref. [3]	$\sqrt{\beta}T_e$, Eq. (36) or Ref. [3]	$\sqrt{\beta}T_1$, Eq. (17)	$\sqrt{\beta}T_2$, Eq. (22)	$\sqrt{\beta}T_3$, Eq. (34)
0.099984373	6.294973	6.294973	6.294976	6.294973	6.294973
0.19987494	6.330415	6.330415	6.330464	6.330415	6.330415
0.49804732	6.581367	6.581404	6.583346	6.581380	6.581404
0.98495702	7.499928	7.502130	7.532834	7.500862	7.502064
1.9134615	11.1700	11.2321	11.6201	11.2000	11.2255
4.6552500	37.145	37.144	42.530	37.288	37.099
9.2746941	132.0	128.9	155.4	132.2	129.3

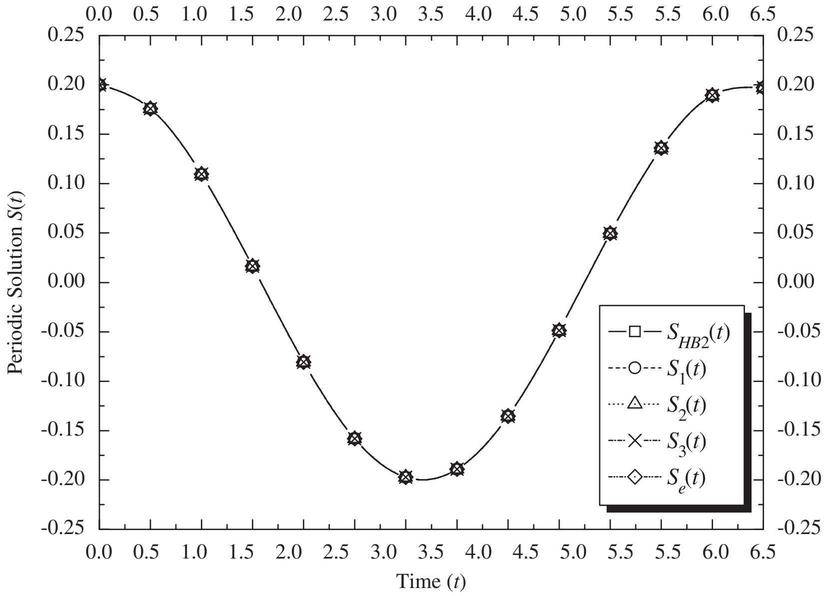

Fig. 1. Comparison of the approximate solutions with the exact solution for $a = 0.2$ and $\beta = 1$.

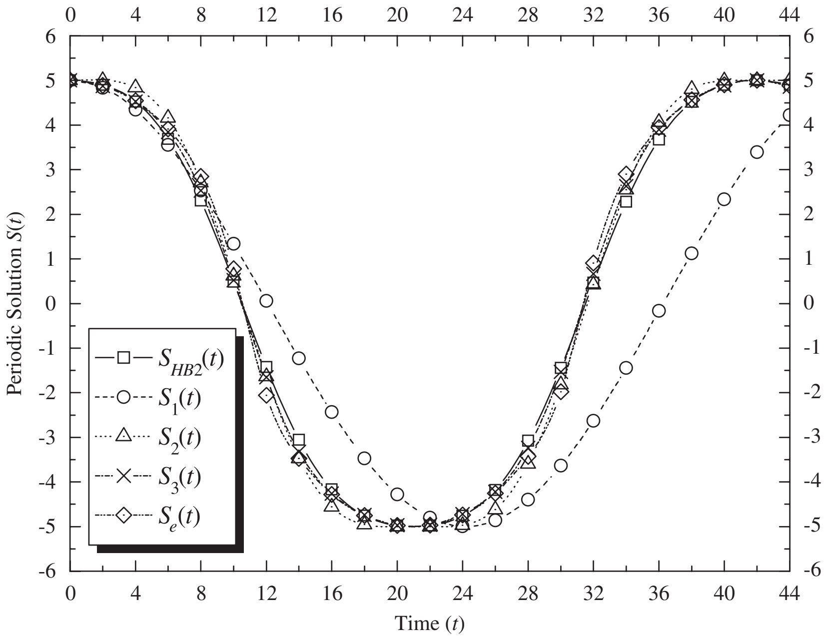

Fig. 2. Comparison of the approximate solutions with the exact solution for $a = 5.0$ and $\beta = 1$.

To further illustrate and verify the accuracy for this new approximate analytical approach, the time history periodic response of Eq. (4) derived from various approximations is presented and compared in Figs. 1–3. As the parameter $\beta = (\pi c/L)^2$ in Eq. (4) does not affect the accuracy of the analytical approximations, $T_i = 2\pi\Big/\sqrt{\beta\Omega_i}\quad (i = 1, 2, 3)$, comparing to the exact solution, $T_e = 2\pi\Big/\left(\sqrt{\beta}\omega_e\right)$, the parameter β can be normalized to a unit value (i.e. $\beta = 1$) in order to compare the periodic solutions of various sources readily. In Fig. 1, all solutions show excellent agreement with respect to the exact solution for a small initial condition

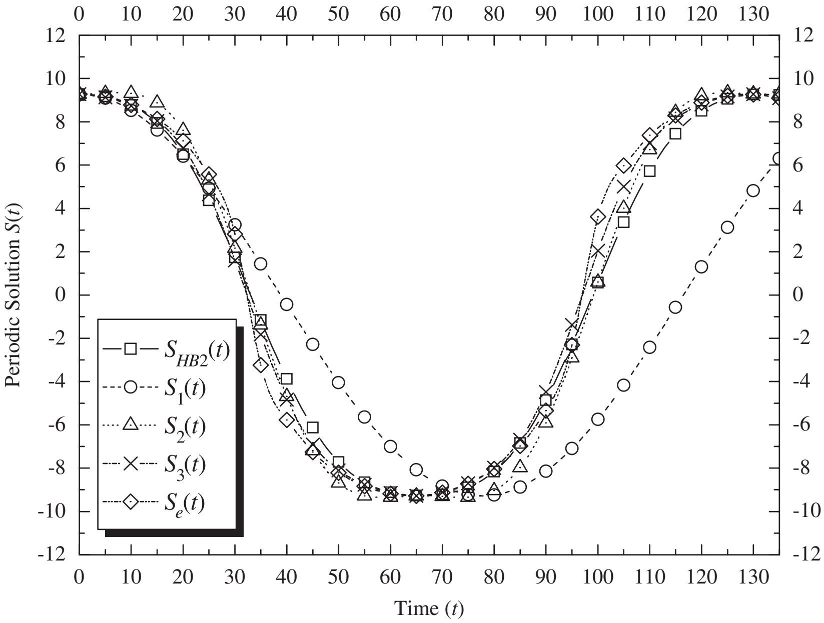

Fig. 3. Comparison of the approximate solutions with the exact solution for $a = 9.2746941$ and $\beta = 1$.

$a = 0.2$. In Figs. 2 and 3, the first-order analytical approximation $S_1(t)$ deviates from the exact solution $S_e(t)$ seriously except at the very beginning of time. The second-order solutions, $S_{\mathrm{HB2}}(t)$ and $S_2(t)$, are capable of achieving good results, but significantly disagreement is expected to increase as time progresses. This is mainly due to increased contribution of nonlinear effects, and hence the number of harmonic terms of the first- and second-order analytical approximations used is rather insufficient. However, the third-order solution $S_3(t)$ does not only able to provide very accurate result, but also expect to maintain very close agreement with respect to the exact solution as time progresses. After evaluating the periodic solution $S(t)$, the transverse amplitude $u(x, t)$ of a string under constant tension according to Eq. (3) can be expressed as follows:

$$u_i(x,t) = \frac{L}{\pi} S_i(t) \sin\left(\frac{\pi}{L} x\right) \tag{38}$$

in which the subscripts $i = 1, 2, 3$ and e of u correspond, respectively, to the first-, second- and third-order analytical approximations of the NHB method and the exact solution.

In summary, the NHB method has been successfully employed to develop accurate, higher-order solutions for the temporal ordinary nonlinear differential equation. At the same time, it removes the analytical difficulties encountered in applying the classical HB method. The NHB approximate solutions are established analytically and systematically. Furthermore, the higher-order solutions show an excellent convergence and accuracy as compared to the exact solution for small as well as large amplitude vibration of string because it is not restricted to the presence of a small parameter. In addition, the NHB method does not require a known initial condition a priori, which is a condition for numerous numerical methods.

Appendix A

The variables κ_i ($i = 1$–12) in Eqs. (28)–(33) are expressed as follows:

$$\begin{aligned}\kappa_1 = & -a - \frac{3a^3}{8} - \frac{5a^5}{64} - c_1 + \frac{a^2 c_1}{4} + \frac{15a^4 c_1}{128} - \frac{25ac_1^2}{8} - \frac{23a^3c_1^2}{32} - \frac{15c_1^3}{4} - \frac{73a^2c_1^3}{32} \\ & - \frac{105ac_1^4}{32} - \frac{245c_1^5}{128}\end{aligned} \tag{A.1}$$

$$\kappa_2 = 1 - \Omega_2 + \frac{a^2\Omega_2}{4} + \frac{15a^4\Omega_2}{128} - \frac{25ac_1\Omega_2}{4} - \frac{23a^3c_1\Omega_2}{16} - \frac{45c_1^2\Omega_2}{4} - \frac{219a^2c_1^2\Omega_2}{32} - \frac{105ac_1^3\Omega_2}{8} - \frac{1225c_1^4\Omega_2}{128} \tag{A.2}$$

$$\kappa_3 = -\frac{11a^2\Omega_2}{8} - \frac{9a^4\Omega_2}{32} - 2ac_1\Omega_2 - \frac{29a^3c_1\Omega_2}{16} + \frac{19c_1^2\Omega_2}{4} - \frac{3a^2c_1^2\Omega_2}{2} + \frac{105ac_1^3\Omega_2}{32} + \frac{165c_1^4\Omega_2}{32} \tag{A.3}$$

$$\kappa_4 = a + c_1 - a\Omega_2 - \frac{3a^3\Omega_2}{8} - \frac{5a^5\Omega_2}{64} - c_1\Omega_2 + \frac{a^2c_1\Omega_2}{4} + \frac{15a^4c_1\Omega_2}{128} - \frac{25ac_1^2\Omega_2}{8} - \frac{23a^3c_1^2\Omega_2}{32} - \frac{15c_1^3\Omega_2}{4} - \frac{73a^2c_1^3\Omega_2}{32} - \frac{105ac_1^4\Omega_2}{32} - \frac{245c_1^5\Omega_2}{128} \tag{A.4}$$

$$\kappa_5 = -\frac{a^3}{8} - \frac{5a^5}{128} + 9c_1 + \frac{19a^2c_1}{8} + \frac{53a^4c_1}{128} + \frac{41ac_1^2}{8} + \frac{17a^3c_1^2}{16} + 6c_1^3 + \frac{97a^2c_1^3}{32} + \frac{605ac_1^4}{128} + \frac{385c_1^5}{128} \tag{A.5}$$

$$\kappa_6 = -1 + 9\Omega_2 + \frac{19a^2\Omega_2}{8} + \frac{53a^4\Omega_2}{128} + \frac{41ac_1\Omega_2}{4} + \frac{17a^3c_1\Omega_2}{8} + 18c_1^2\Omega_2 + \frac{291a^2c_1^2\Omega_2}{32} + \frac{605ac_1^3\Omega_2}{32} + \frac{1925c_1^4\Omega_2}{128} \tag{A.6}$$

$$\kappa_7 = 1 - 9\Omega_2 + \frac{5a^2\Omega_2}{8} + \frac{19a^4\Omega_2}{64} - \frac{15ac_1\Omega_2}{2} - \frac{5a^3c_1\Omega_2}{16} - \frac{73c_1^2\Omega_2}{4} - \frac{291a^2c_1^2\Omega_2}{64} - \frac{435ac_1^3\Omega_2}{32} - \frac{843c_1^4\Omega_2}{64} \tag{A.7}$$

$$\kappa_8 = -c_1 - \frac{a^3\Omega_2}{8} - \frac{5a^5\Omega_2}{128} + 9c_1\Omega_2 + \frac{19a^2c_1\Omega_2}{8} + \frac{53a^4c_1\Omega_2}{128} + \frac{41ac_1^2\Omega_2}{8} + \frac{17a^3c_1^2\Omega_2}{16} + 6c_1^3\Omega_2 + \frac{97a^2c_1^3\Omega_2}{32} + \frac{605ac_1^4\Omega_2}{128} + \frac{385c_1^5\Omega_2}{128} \tag{A.8}$$

$$\kappa_9 = -\frac{a^5}{128} + \frac{11a^2c_1}{8} + \frac{47a^4c_1}{128} + \frac{3ac_1^2}{8} + \frac{9a^3c_1^2}{16} - c_1^3 + \frac{49a^2c_1^3}{64} + \frac{25ac_1^4}{128} - \frac{49c_1^5}{128} \tag{A.9}$$

$$\kappa_{10} = \frac{11a^2\Omega_2}{8} + \frac{47a^4\Omega_2}{128} + \frac{3ac_1\Omega_2}{4} + \frac{9a^3c_1\Omega_2}{8} - 3c_1^2\Omega_2 + \frac{147a^2c_1^2\Omega_2}{64} + \frac{25ac_1^3\Omega_2}{32} - \frac{245c_1^4\Omega_2}{128} \tag{A.10}$$

$$\kappa_{11} = -1 + 25\Omega_2 + \frac{43a^2\Omega_2}{8} + \frac{61a^4\Omega_2}{64} + \frac{31ac_1\Omega_2}{2} + \frac{111a^3c_1\Omega_2}{32} + \frac{167c_1^2\Omega_2}{8} + \frac{579a^2c_1^2\Omega_2}{64} + \frac{463ac_1^3\Omega_2}{32} + \frac{691c_1^4\Omega_2}{64} \tag{A.11}$$

$$\kappa_{12} = -\frac{a^5\Omega_2}{128} + \frac{11a^2c_1\Omega_2}{8} + \frac{47a^4c_1\Omega_2}{128} + \frac{3ac_1^2\Omega_2}{8} + \frac{9a^3c_1^2\Omega_2}{16} - c_1^3\Omega_2 + \frac{49a^2c_1^3\Omega_2}{64} + \frac{25ac_1^4\Omega_2}{128} - \frac{49c_1^5\Omega_2}{128} \tag{A.12}$$

where Ω_2 and c_1 in Eqs. (A.1)–(A.12) are obtained from Eqs. (22) and (25), respectively.

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