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Fundamental frequencies of mechanical systems with N-piecewise constant properties

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Abstract

This paper presents an analytical derivation of a set of transcendental equations for the normal modes of a one-dimensional mechanical system composed of an arbitrary number of components and provides experimental validation for the case of a two-component system consisting of a fiber partially submerged in liquid. The transcendental equations relate thickness of layers and their elastic moduli and allow possible experimental measurement of thin film thickness and elastic moduli. An effective model of the wave propagation in such systems gives an *a-priori* qualitative explanation of the natural frequencies change and allows one to predict them quantitatively.

1. Introduction

Natural frequency of a single degree of freedom oscillator or normal modes of a string are fundamental characteristics of mechanical systems. An inaccurate determination of natural frequencies can result in unwanted noise, lower effectiveness or even in the destructive damage of mechanical system [1,2]. Theoretical determination of normal modes is quite complicated problem; therefore in most classical textbooks (see, for example [3]), the vibrating behavior is considered mainly for systems with constant mechanical properties. However many real applications involve not a constant but piecewise constant properties. As examples, the longitudinal or torsional oscillations of the rod consisted of two or more different materials [4], or vibration of a string partially immersed in liquid [5] can be mentioned. Here we show that normal modes of a mechanical system with N-piecewise constant properties are the roots of a transcendental equation and an explicit form of the transcendental equations for any N are derived analytically.

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2. Mathematical background

2.1. Statement of the problem for the N-steps function

This analysis starts with a one-dimensional system (rod, shaft or string) of length l consisting of N homogeneous parts with different properties. An N-step function $f_N(x)$ can be represented as

$$f_N(x) = \sum_{i=1}^{N} f_i \, \mathrm{rect}(x, x_{i-1}, x_i), \tag{1}$$

where $\mathrm{rect}(x, a, b) = H(x-a)H(b-x)$ is a rectangular window on an interval (a, b), $H(x)$ is the unit step function, $x_0 = 0$, $x_N = l$ and f_i is the amplitude on interval (x_{i-1}, x_i).

Then a differential equation describing the system oscillations can be written as

$$\rho_N(x)u_{tt} = F_N(x)u_{xx}. \tag{2}$$

In case of longitudinal vibrations of a rod, $\rho_N(x)$ and $F_N(x) \equiv E_N(x)$ are the density and Young's modulus distributions over the rod; in case of torsional oscillations, $\rho_N(x)$ and $F_N(x) \equiv G_N(x)$ are the density and shear modulus distributions over the rod, and in case of string immersed in N-layers pie of immiscible liquids, $F_N(x) =$ constant is the string tension, $\rho_N(x)$ is the linear density of the string with due account of added mass.

Splitting Eq. (2) into N equations with constant coefficients yields

$$u_{itt} = c_i^2 u_{ixx}, h_{i-1} < x < h_i, \quad i = 1, 2, \ldots N, \quad h_0 = 0, \quad h_N = l, \tag{3}$$

with these corresponding matching conditions:

$$u_i(h_i, t) = u_{i+1}(h_i, t), \tag{4}$$

$$k_i u_{ix}(h_i, t) = k_{i+1} u_{(i+1)x}(h_i, t), \tag{5}$$

where $c_i = (F_i/\rho_i)^{1/2}$ is the speed of wave propagation, $k_i = E_i S_i$ (S_i is the cross sectional area) in case of longitudinal oscillations of a rod, $k_i = G_i I_{Pi}$ (I_{Pi} is the polar moment of inertia) in case of torsional oscillations of a rod and $k_i = 1$ in case of a string.

2.2. Exact solution for $N = 3$

We start with an analytical solution for a particular case with $N = 3$ to illustrate the main idea of the method, and afterwards will make a generalization for an arbitrary N.

To close statement of this problem, the following initial and boundary conditions can be imposed

$$u(x, 0) = u_t(x, 0) = 0, \tag{6}$$

$$u_1(0, t) = 0, \quad u_3(l, t) = \sin(pt). \tag{7}$$

Applying the Laplace transform results in the following system:

$$U_{ixx} = \left(\frac{s}{c_i}\right)^2 U_i, h_{i-1} < x < h_i, \quad i = 1, 2, 3, \tag{8}$$

with the matching conditions

$$U_i(h_i, s) = U_{i+1}(h_i, s), \quad k_i U_{ix}(h_i, s) = k_{i+1} U_{(i+1)x}(h_i, s), \quad i = 1, 2, \tag{9}$$

and the boundary conditions

$$U_1(0, s) = 0, \quad U_3(l, s) = \chi, \quad \chi = \frac{p}{p^2 + s^2}. \tag{10}$$

The general solution of Eq. (8) is

$$U_i(x,s) = A_i \sinh\left(\frac{sx}{c_i}\right) + B_i \cosh\left(\frac{sx}{c_i}\right). \tag{11}$$

Accounting for matching and boundary conditions (9), (10) yields the solution in the imaginary space:

$$U_1(x,s) = \chi\kappa_2\kappa_3 \sinh\left(\frac{sx/c_1}{d(s)}\right), \tag{12}$$

$$U_2(x,s) = \chi\frac{\kappa_1\kappa_3\cosh(sh_1/c_1)\sinh[(s(x-h_1))/c_2] + \kappa_2\kappa_3\sinh((sh_1)/c_1)\cosh[s(x-h_1)/c2]}{d(s)}, \tag{13}$$

$$\begin{aligned} U_3(x,s) = \chi\{&\kappa_1\kappa_3\cosh(sh_1/c_1)\sinh[s(h_2-h_1)/c_2]\cosh[s(x-h_2)/c_3] \\ &+ \kappa_2\kappa_3\sinh(sh_1/c_1)\cosh[s(h_2-h_1)/c_2]\cosh[s(x-h_2)/c_3] \\ &+ \kappa_1\kappa_2\cosh(sh_1/c_1)\cosh[s(h_2-h_1)/c_2]\sinh[s(x-h_2)/c_3] \\ &+ \kappa_2^2\sinh(sh_1/c_1)\sinh[s(h_2-h_1)/c_2]\sinh[s(x-h_2)/c_3]\}/d(s), \end{aligned} \tag{14}$$

where

$$\begin{aligned} \kappa_i = k_i/c_i, \text{ and } d(s) &\\ = \kappa_1\kappa_3&\cosh(sh_1/c_1)\sinh[s(h_2-h_1)/c_2]\cosh[s(l-h_2)/c_3] \\ &+ \kappa_2\kappa_3\sinh(sh_1/c_1)\cosh[s(h_2-h_1)/c_2]\cosh[s(l-h_2)/c_3] \\ &+ \kappa_1\kappa_2\cosh(sh_1/c_1)\cosh[s(h_2-h_1)/c_2]\sinh[s(l-h_2)/c_3] \\ &+ \kappa_2^2\sinh(sh_1/c_1)\sinh[s(h_2-h_1)/c_2]\sinh[s(l-h_2)/c_3]. \end{aligned}$$

It can be seen from Eqs. (12)–(14) that all $U_i(x, s)$ have a common denominator $E(s)$ of the form

$$E(s) = \chi d(s). \tag{15}$$

The function $E(s)$ has two zeros $\pm$ ip and a countable set of zeros given by transcendental equation $d(s) = 0$. All zeros of $E(s)$ are simple poles of U_i, hence the inverse transform can be performed by integrating the function $(2\pi)^{-1}U_i e^{st}$ over the closed contour $C_R \cup [\sigma - iR, \sigma + iR]$ (C_R is a semicircle of radius R and the line $[\sigma - iR, \sigma + iR]$ is chosen so as to lie to the right of all the poles) at $R \to \infty$ and applying Cauchy's residue theorem [6]. In order to find the natural frequencies it is unnecessary to perform an inverse Laplace transform. Notice that poles $\pm$ ip correspond to the angular frequency of the external force. But if the frequency of external force coincides with a root s_n of transcendental equation $d(s) = 0$, then function U_i has pole of the second order at $s = s_n$. Corresponding to this pole, U_i contains a resonance term [7]. Hence, consecutive roots s_n of the transcendental equation $d(s)$ correspond to the natural frequencies. Introducing a new variable [8] $\lambda = is$ and using two trigonometric identities $\sin(ix) = i \cdot \sinh x$ and $\cos(ix) = \cosh x$, the equation $d(s) = 0$ can be brought into its final form

$$\begin{aligned} &-\tan[\lambda(h_2-h_1)/c_2]/\kappa_2 - \tan(\lambda h_1/c_1)/\kappa_1 - \tan[\lambda(l-h_2)/c_3]/\kappa_3 \\ &+ \kappa_2/(\kappa_1\kappa_3)\tan(\lambda h_1/c_1)\tan[\lambda(h_2-h_1)/c_2]\tan[\lambda(l-h_2)/c_3] = 0. \end{aligned} \tag{16}$$

The cases $N = 1$ and 2 can be easily obtained from Eq. (16) by setting $h_1 = h_2 = 0$ and $h_1 = h_2 = h$, $c_1 = c_2 = c_s$, $c_3 = c$, respectively. For $N = 2$ and $k_i = 1$, Eq. (16) transforms to the following transcendental equation

$$c_s \tan\left(\frac{\lambda h}{c_s}\right) + c\tan\left[\frac{\lambda(l-h)}{c}\right] = 0, \tag{17}$$

which describes the natural frequencies of a string partially immersed in liquid.

Setting $h = 0$ or l in Eq. (17) gives the normal modes for a fully immersed string.

2.3. Generalization for an arbitrary N

Now the generalization for any N can be performed.

Similarly to the previous analysis, applying the Laplace transform to Eqs. (3)–(5), solving the system of N algebraic equations while accounting for the matching conditions, the denominator $d(s)$ of functions $U_i\,(x, s)$ can be found and, correspondingly, the transcendental equation:

$$\sum_{i=1}^{[N/2]} (-1)^i \Re_{2i-1}(N) = 0, \tag{18}$$

where $\Re_M\ (N)$ is the sum of $C_N^M = N!/((N-M)!M!)$ terms of the form $(\Pi_{p=1}^{m}\kappa_{i_{2p}}/\Pi_{p=1}^{m+1}\kappa_{i_{2p-1}})$ $(i_1 \times i_2 \times \ldots \times i_M)$, where $i_1 < i_2 < \ldots < i_M \leqslant N$, $M = 2m+1$ is an odd number; the integer number $i_j = n$ substitutes for tan $[\lambda(h_n - h_{n-1})/c_n]$; $[p]$ is the roundoff function, where p is a real number; $\Pi_{p=1}^{0}\kappa_{i_{2p}} = 1$.

For example, transcendental equations for $N = 5$ reads

$$\sum_{i=1}^{5} -(i/\kappa_i) + 123\frac{\kappa_2}{\kappa_1\kappa_3} + 124\frac{\kappa_2}{\kappa_1\kappa_4} + 125\frac{\kappa_2}{\kappa_1\kappa_5} + 134\frac{\kappa_3}{\kappa_1\kappa_4} + 135\frac{\kappa_3}{\kappa_1\kappa_5} + 145\frac{\kappa_4}{\kappa_1\kappa_5}$$
$$+ 234\frac{\kappa_3}{\kappa_2\kappa_4} + 235\frac{\kappa_3}{\kappa_2\kappa_5} + 245\frac{\kappa_4}{\kappa_2\kappa_5} + 345\frac{\kappa_4}{\kappa_3\kappa_5} - 12345\frac{\kappa_2\kappa_4}{\kappa_1\kappa_3\kappa_5} = 0.$$

From Eq. (18) it follows that in case of the N-steps function the transcendental equation comprises 2^{N-1} terms. The roots of Eq. (18) are the angular fundamental frequencies of the mechanical system with N-piecewise constant properties.

This makes finding the roots of Eq. (18) difficult for higher N. To circumvent this problem and find out the physical sense of the natural frequencies of such heterogeneous systems, it would be well to introduce an effective wave velocity model.

2.4. An effective model for the normal modes

It is known that the normal modes frequencies of a string fixed at both ends can be found from the condition $l = n\lambda_n/2$. Representing a wavelength through the velocity c of the wave propagation along the string and the angular frequency $\omega_n = 2\pi f_n$, one arrives at the condition

$$\omega_n = \frac{n\pi c}{l}, \quad f_n = \frac{nc}{2l}. \tag{19}$$

An "effective" propagation velocity c_{eff} can be introduced in such a way that the propagation time of a wave over the string's length with velocity c_{eff} equals to the real time of a wave's propagation:

$$\frac{l}{c_{\text{eff}}} = \sum_{i=1}^{N} \frac{\Delta h_i}{c_i} \tag{20}$$

or

$$c_{\text{eff}} = \left(\sum_{i=1}^{N} \frac{\Delta h_i^*}{c_i}\right)^{-1}, \tag{21}$$

where $\Delta h_i^* = h_i^* - h_{i-1}^*, \quad h^* = h/l$. Substituting c_{eff} for c in Eq. (19) results in

$$\omega_{n\ \text{eff}} = \frac{n\pi c_{\text{eff}}}{l}, \quad f_{n\ \text{eff}} = \frac{nc_{\text{eff}}}{2l}. \tag{22}$$

3. Experimental validation of the theory

To validate this theory, experiments on forced vibrations of a partially immersed fiber have been carried out. An optical method utilizing a forward light scattering pattern [5] has been used to detect small (< 1.0 μm)

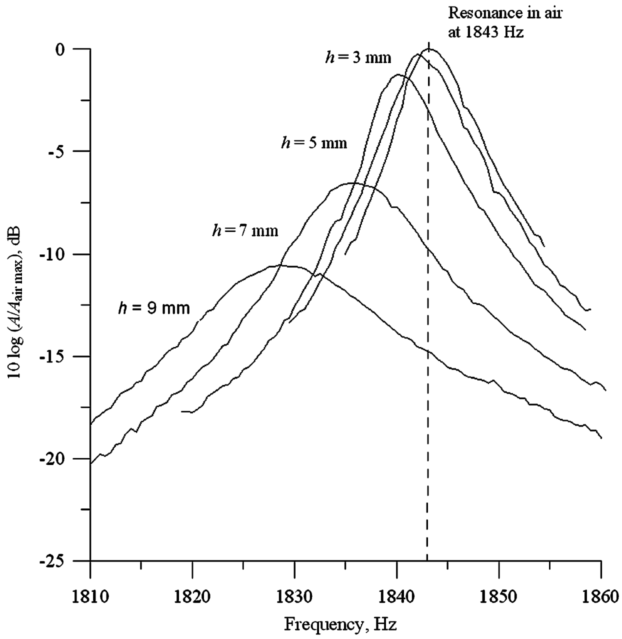

Fig. 1. Vibrational amplitude of partially immersed fiber versus external frequency for different depths of liquid solution (36% glycerol—water solution).

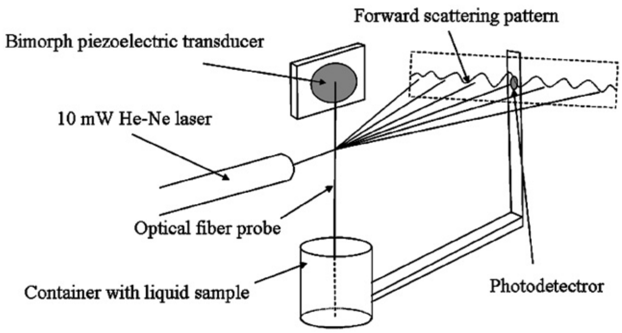

Fig. 2. A block diagram of the device for measuring the vibrational amplitude by using forward light scattering pattern.

amplitude vibrations of the fiber. A block diagram of the setup is shown in Fig. 1. It consists of a 10 mW He–Ne linearly polarized laser and a single mode optical fiber probe. One end of the optical fiber probe is fixed to a bimorph piezoelectric transducer; other end is clamped and secured to the bottom of a container with liquid. Light from a horizontally placed laser is incident normal to the optical fiber probe with density $\rho = 2200\,\mathrm{kg/m^3}$ (diameters of the core and cladding are equal to 10 μm and 125 μm, respectively) and results in a light pattern scattering from the fiber. The bimorph piezoelectric transducer is mounted on a three axis translation stage which allows one to calibrate this device and also to adjust the tension of the optical fiber probe. Maximum displacement responses of the fiber with respect to the driving frequencies are obtained by setting the fiber in motion with the bimorph piezoelectric transducer driving at frequencies near the fiber's resonance. Required driving frequencies are supplied by using a function generator and an amplifier. The displacement of the oscillating fiber is measured based on the observed intensity variation created by the fiber's displaced forward light scattering pattern. This intensity variation is detected by a PIN diode, converted to an electrical signal, subsequently analyzed by using a dynamic signal analyzer, and the data stored in a general purpose computer via a data acquisition system.

Table 1
Experimental values (f_{exp1}) of the resonance frequency for different depths of 36% water-glycerol solution and theoretical predictions of the first mode (f_1) according to Eq. (17)

Liquid depth, mm	f_{n1}, Hz	f_{exp}, Hz	Ψ_{exp1}
3	1842	1842	3.177
5	1841	1840	3.199
7	1837	1836	3.218
9	1830	1829	3.230

Ψ_{exp1} is the dimensionless experimental value of angular frequency ($\Psi_1 = \omega_1 l/c_{eff}$); the dimensionless effective angular frequencies is equal to π. The density of solution $\rho_s = 1085.42\,kg/m^3$ the tensile force $F = 1.11\,N$, the speed of wave propagation over the fiber $c = 202.77\,m/s$, $l = 0.055\,m$.

Fig. 2 shows several measured resonance curves shifts as depth of the fiber probe in a liquid solution (36% glycerol—water solution) increases. Eq. (22) allows one to give an *a-priori* explanation of the frequency response observed in these experiments: if either immersion depth or density of liquid increases, the effective velocity decreases and causes leftward resonance frequency shift.

Table 1 summarizes predictions of the first normal mode for a partially immersed fiber in 36% glycerol-water solution for different depths. As is seen, theoretical predictions according to the analytical solution and effective model agree qualitatively and quantitatively well with experiments.

4. Conclusions

Summarizing, a family of transcendental equations which describe the normal modes of a wide range of mechanical systems with *N*-piecewise constant properties have been derived analytically. The transcendental equations relate thickness of layers and their elastic moduli; it makes possible experimental measurement of thin film thickness and elastic moduli. An effective model of the wave propagation in such systems gives an *a-priori* qualitative explanation of the natural frequencies change and allows one to predict them quantitatively.

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