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Vibration damping using four-layer sandwich

Binod P. Yadav*

SECAB Institute of Engineering and Technology, 424, Nauraspur, Bagalkot Road, Bijapur 586101, Karnataka, India

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Abstract

This paper discusses vibration damping using four-layer sandwich beam. The present work deals with the analysis of vibration of the primary system having a mass and rubber spring mounted on a four-layer viscoelastic simply supported symmetrically arranged sandwich beam. The equation of motion of a general four-layer with alternate elastic layer and viscoelastic layer simply supported sandwich beam is first derived using the method of equilibrium of forces and beam theory. The above differential equation has been solved for harmonically force excited sandwich beam by applying suitable boundary conditions to get the impedance of the sandwich beam. This impedance is then combined with the impedance of the primary system to obtain the expression for the response of harmonically excited mass and then the expression for transmissibility is obtained. The effectiveness of geometrical and physical parameters in minimizing response and transmissibility for central mounting of the primary system is evaluated.

1. Introduction

There is a large body of literature on damping in composite materials where many reseachers have evaluated a material's specific damping capacity. Baburaj and Matswzaki [1] and the references therein give an account of research in this area. A more extensive list of references is given in Adhikari [2]. Sandwich structures have been the subject of many investigations. A large amount of literature has been devoted to the development of theories for conventional sandwich structures and to the study of their static and dynamic behavior. A detailed review of this work is given in the paper written by Noor et al. [3]. The free vibration analysis of plates made up of two elastic layers with a thin viscoelastic damping layer was first investigated by Cupial and Niziol [4].

Sandwich structures with elastic faces and viscoelastic cores are nowadays of importance in aircraft and spacecraft structures. They are being used more and more where high strength and low weights are desired and also where damping is required to dissipate vibrational energy. Situations also arise when precision instruments or vibrating machines are installed on such structures with flexible mounts, and attempts are being made to provide effective vibration isolation by the use of flexible mounts in conjunction with viscoelastic sandwich structures [5]. The arrangement can provide a sufficient amount of damping to combat the menace of resonance. At higher frequencies even very small amplitudes of vibration can give rise to appreciable noise

*Mobile: +919972148019; fax: +918352277353.
E-mail address: byadavp@yahoo.co.in

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Nomenclature

A_j	complex constants
b	width of beam
B_j	A_j/F_o, $j = 1,2, \ldots, 6$
d_1	distance between neutral axes of first and third elastic layers $= h_1 + h_2$
d_2	distance between neutral axis of third elastic layer and end of the fourth plastic layer
D_t	overall bending stiffness of elastic layers about their neutral axes
E_j	Young's modulus of jth layer
f	frequency in rad/s
f_c	frequency in cps
F	exciting force
F_0	exciting force at junction point
F_1	force transmitted to support
G	in-phase shear modulus of viscoelastic material
G^*	complex shear modulus of viscoelastic material $= G(1 + \mathrm{i}\beta)$
g	shear parameter $= G^*/Eh_1h_2$
h_j	thickness of jth layer
i	$\sqrt{-1}$
K	dynamic stiffness of rubber material
l	length of beam
m	main vibrating mass
M	total bending moment
p	transverse loading per unit length of beam
P_j	longitudinal forces in jth layer
T	transmissibility
t	time variable
u_j	longitudinal displacement of neutral axes of jth layers
u	longitudinal displacement of any point in the core
W	transverse displacement of beam, function of x
w	transverse displacement of beam, function of x and t
y_3	displacement response of primary system, function of x and t
Y_1	geometrical parameter $= Eh_1bd_1^2/D_t$
Y_2	geometrical parameter $= Eh_1bd_2^2/D_t$
Y	geometrical parameter $= Eh_1bd^2/D_t$
Y_3	displacement response of primary system, function of x
Z	impedance of sandwich beam
Z_{eq}	equivalent impedance of the system using sandwich beam
μ	mass of sandwich beam per unit length
γ, γ'	shear strain
τ, τ'	shear stress
β	loss factor of core material
δ	loss factor of rubber
σ_j	complex roots of characteristic Eq. (31)

levels, and this situation occurs in many machines. It may then be necessary to reduce the vibration of the machinery by the use of damping materials.

In this paper the problem of a primary vibration excitation system in contact with a four-layer sandwich beam is considered. Many investigators [6–13] have reported work on the analysis of flexural vibrations of sandwich beams. To simplify the analysis these investigators have taken into consideration only the strain energy due to bending and longitudinal deformation of the elastic faces and that due to shear deformation of the viscoelastic layer. In the work reported here, both the vibration response of a flexibly supported mass attached to a four-layer sandwich beam at its center and the force transmissibility provided by the complete system have been computed, as both these aspects are important from the point of view of vibration control.

The equation of motion of bending vibration of a sandwich beam is derived and used as the starting point. The dynamic stiffness of the beam with respect to the junction point is first determined. This is combined with that of the primary system to obtain the resultant dynamic stiffness, which in turn is used for finding the expressions for the response of the primary system and for the transmissibility.

2. Problem formulation

Fig. 1 shows the model system, which consists of a flexibly supported excitation system attached to the center of a simply supported four-layered sandwich beam consisting of alternate layers of an elastic and stiff material such as metal and a high damping viscoelastic material such as plastic. The vibrating excitation

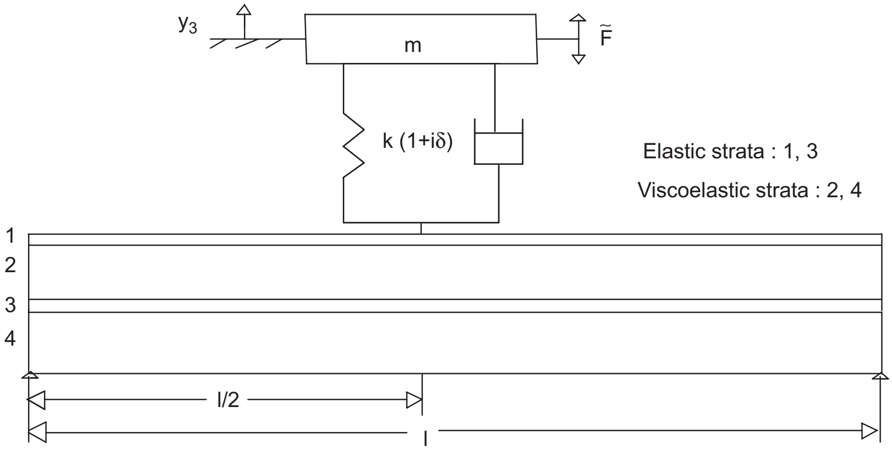

Fig. 1. Primary vibratory system mounted on four-layer sandwich beam.

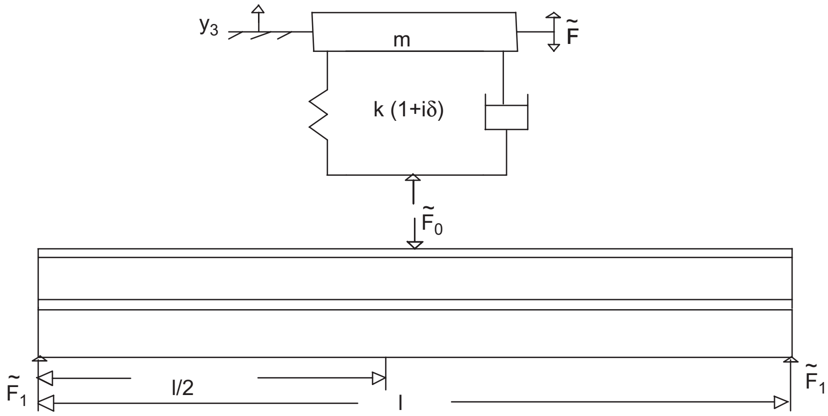

Fig. 2. Free body diagram.

system, here designated as the primary system, consists of a mass and rubber spring whose dynamic characteristic is defined by the equation $K^* = K(1+i\delta)$, where K and δ are the stiffness and the loss factor, respectively. A four-layered sandwich beam has elastic layers of thickness h_1 and plastic layers of thickness h_2. The elastic layers have Young's modulus E. The plastic layers have shear modulus $G(1+i\beta)$, β being the loss factor of the viscoelastic material. Fig. 2 shows the free body diagrams of the primary system and the sandwich beam. The concentrated harmonic force F_0 is acting on the beam. The sandwich beam is analyzed for its dynamic stiffness with respect to F_0.

3. The equation of a four-layered sandwich beam

The differential equation of longitudinal displacement has previously been derived by Di Taranto [7]. The analysis we present now leads to the equation of transverse displacement.

Geometry of a four-layered sandwich beam of width b is shown in Fig. 3. The analysis is developed for a model of symmetric layers composed of two identical elastic layers of thickness h_1 with Young's modulus E and two identical viscoelastic layers of thickness h_2 with shear modulus $G^* = G\,(1+i\beta)$. The deflected face of the beam is shown in Fig. 4.

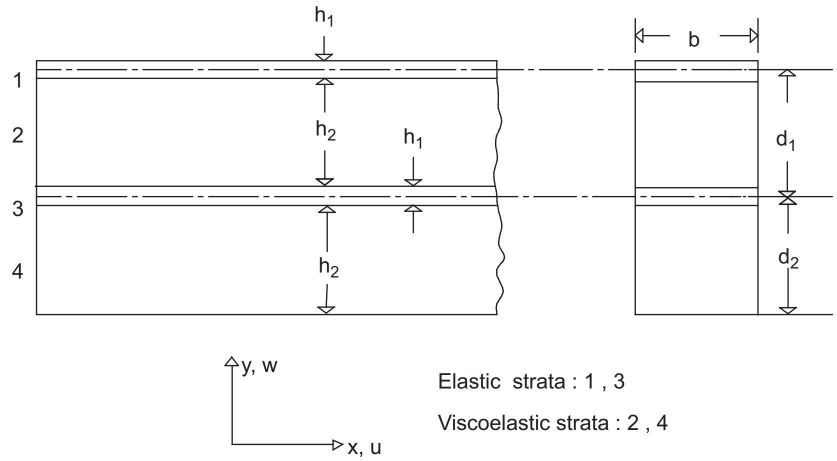

Fig. 3. Beam geometry (dimensions and coordinate systems).

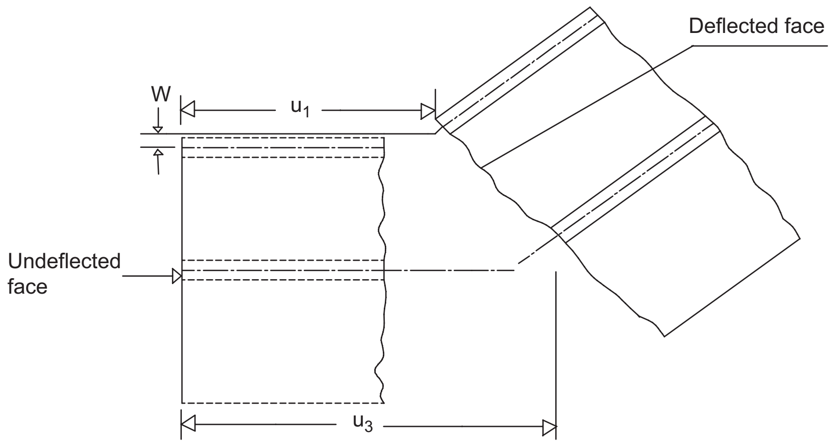

Fig. 4. Displacement configuration (deflected beam).

It will be assumed that shear strains in the elastic plates are negligible, and that longitudinal direct stresses in the plastic plates are negligible. Transverse direct strains in both plastic plates and elastic plates are also neglected, so that the transverse displacements w of all points on a cross-section xx are equal.

The longitudinal x-wise displacements of the mid-planes of the elastic plates are u_1 and u_3. The longitudinal displacement component of any point in the viscoelastic layer is u.

Now the shear strain, γ_1, in the second layer, which is viscoelastic, is given by

$$\gamma_1 = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial y} \tag{1}$$

Relative displacement of the two faces of the second layer in the longitudinal direction is

$$u_1 - u_3 + h_1 \frac{\partial w}{\partial x}$$

Hence,

$$\frac{\partial u}{\partial y} = \frac{1}{h_2} \left\{ (u_1 - u_3) + h_1 \frac{\partial w}{\partial x} \right\} \tag{2}$$

Shear strain in fourth layer, which is also viscoelastic is given as

$$\gamma_2 = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial y} \tag{3}$$

Relative displacement of the two faces of the fourth layer in the longitudinal direction is

$$u_3 + \frac{h_1}{2}\frac{\partial w}{\partial x}$$

Hence,

$$\frac{\partial u}{\partial y} = \frac{1}{h_2}\left[u_3 + \frac{h_1}{2}\frac{\partial w}{\partial x}\right] \tag{4}$$

Substituting Eq. (2) into Eq. (1) we get

$$\gamma_1 = \frac{\partial w}{\partial x} + \frac{u_1 - u_3}{h_2} + \frac{h_1}{h_2}\frac{\partial w}{\partial x}$$

or

$$\gamma_1 = \frac{u_1 - u_3}{h_2} + \frac{d_1}{h_2}\frac{\partial w}{\partial x} \tag{5}$$

where $d_1 = h_1 + h_2$.

Substituting Eq. (4) into Eq. (3) we get

$$\gamma_2 = \frac{\partial w}{\partial x} + \frac{u_3}{h_2} + \frac{h_1}{2h_2}\frac{\partial w}{\partial x}$$

or

$$\gamma_2 = \frac{u_3}{h_2} + \frac{d_2}{h_2}\frac{\partial w}{\partial x} \tag{6}$$

where $d_2 = h_2 + (h_1/2)$.

The shear stress in the second layer is, therefore, given by

$$\tau_1 = G^*\gamma_1 = G^*\left[\frac{d_1}{h_2}\frac{\partial w}{\partial x} + \frac{u_1 - u_3}{h_2}\right] \tag{7}$$

The shear stress in the fourth layer is, therefore, given by

$$\tau_2 = G^*\gamma_2 = G^*\left[\frac{d_2}{h_2}\frac{\partial w}{\partial x} + \frac{u_3}{h_2}\right] \tag{8}$$

At any section of the beam, the shear forces and bending moments are shown in Fig. 5. The shear force on first face layer is

$$S_1 = D_1\frac{\partial^3 w}{\partial x^3} \tag{9}$$

where $D_1 = Ebh_1^3/12$.

The shear force on third face layer is

$$S_3 = D_3\frac{\partial^3 w}{\partial x^3} \tag{10}$$

where $D_3 = Ebh_1^3/12$.

The shear force S_2 is associated with the second layer shear stress. This shear stress must be considered to act uniformly between the mid-plane of first and third layers. Actually, it is constant over the depth of second layer and varies linearly to zero across the thickness of first and third layers. Thus the shear is equivalent to be uniform between the mid-planes of first and third layers.

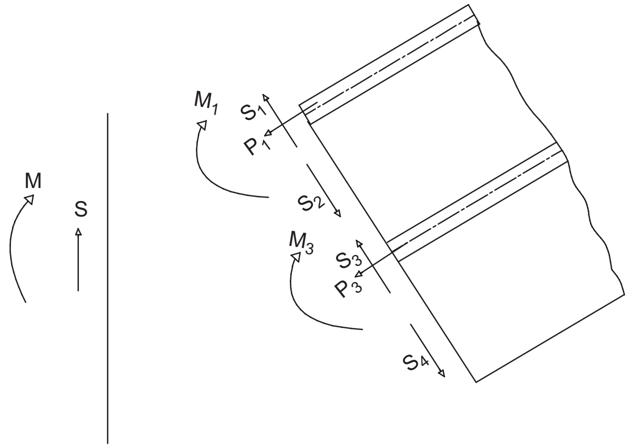

Fig. 5. Equilibrium of beam (forces and moments acting on a section).

Hence,

$$S_2 = \tau_1 b d_1 \tag{11}$$

Similarly, the shear force S_4 is associated with the fourth layer shear stress. This shear stress must be considered to act uniformly between the mid-plane of the third layer.

Hence,

$$S_4 = \tau_2 b d_2 \tag{12}$$

The total shear forces is given by

$$S = S_1 - S_2 + S_3 - S_4 \tag{13}$$

The minus sign is included because the direction of S_2 and S_4 (Fig. 5) is opposite to the direction of S_1 and S_3.

Hence,

$$S = D_1 \frac{\partial^3 w}{\partial x^3} - \tau_1 b d_1 + D_3 \frac{\partial^3 w}{\partial x^3} - \tau_2 b d_2$$

or

$$S = (D_1 + D_3)\frac{\partial^3 w}{\partial x^3} - G^* b d_1 \left[\frac{d_1}{h_2}\frac{\partial w}{\partial x} + \frac{u_1 - u_3}{h_2}\right] - G^* b d_2 \left[\frac{d_2}{h_2}\frac{\partial w}{\partial x} + \frac{u_3}{h_2}\right] \tag{14}$$

Now the transverse loading, p, on the beam is related to S by $p = \partial S/\partial x$. Hence, by differentiating Eq. (14), putting $D_1 + D_3 = D_t$, and re-arranging, we obtain

$$p = D_t \frac{\partial^4 w}{\partial x^4} - \frac{G^* b d_1^2}{h_2}\frac{\partial^2 w}{\partial x^2} - \frac{G^* b d_1}{h_2}\left[\frac{\partial u_1}{\partial x} - \frac{\partial u_3}{\partial x}\right] - \frac{G^* b d_2^2}{h_2}\frac{\partial^2 w}{\partial x^2} - \frac{G^* b d_2}{h_2}\frac{\partial u_3}{\partial x} \tag{15}$$

Let the net longitudinal forces in each of the face plates be denoted by P_1 and P_3. These forces have their lines of action in the mid-planes of the face plates and are related to the longitudinal displacements by

$$P_1 = Ebh_1 \frac{\partial u_1}{\partial x}, \quad P_3 = Ebh_1 \frac{\partial u_3}{\partial x}. \tag{16}$$

Since there can be no resultant longitudinal force on the whole section, $P_1 = P_3$

or

$$Ebh_1 \frac{\partial u_1}{\partial x} = -Ebh_1 \frac{\partial u_3}{\partial x} \tag{17}$$

Furthermore, considering the physical nature of the system and its displacements, it is obvious that we can also write

$$Ebh_1 u_1 = -Ebh_1 u_3 \tag{18}$$

Eq. (15) can therefore be written in the alternative form

$$D_t \frac{\partial^4 w}{\partial x^4} - \frac{G^* Eh_1 bd_1^2}{Eh_1 h_2} \frac{\partial^2 w}{\partial x^2} - \frac{G^* Eh_1 bd_2^2}{Eh_1 h_2} \frac{\partial^2 w}{\partial x^2} + \frac{2G^* Eh_1 bd_1^2}{Eh_1 h_2 d_1} \frac{\partial u_3}{\partial x} - \frac{G^* Eh_1 bd_2^2}{Eh_1 h_2 d_2} \frac{\partial u_3}{\partial x} = p \tag{19}$$

In subsequent equations, it is convenient to introduce the symbols

$$g = \frac{G^*}{Eh_1 h_2} \tag{20}$$

$$Y_1 = \frac{Eh_1 bd_1^2}{D_t} \tag{21}$$

and

$$Y_2 = \frac{Eh_1 bd_2^2}{D_t} \tag{22}$$

Using these in Eq. (19), we may re-arrange it into the form

$$\frac{\partial^4 w}{\partial x^4} - gY_1 \frac{\partial^2 w}{\partial x^2} - gY_2 \frac{\partial^2 w}{\partial x^2} + \frac{2gY_1}{d_1} \frac{\partial u_3}{\partial x} - \frac{gY_2}{d_2} \frac{\partial u_3}{\partial x} = \frac{p}{D_t} \tag{23}$$

Next, consider the longitudinal equilibrium of a lengthwise element, δ_x, of the lower face plate (see Fig. 6). It is evident that

$$\delta P_3 = -(\tau_1 + \tau_2) b \delta_x$$

or

$$\frac{\partial P_3}{\partial x} = -(\tau_1 + \tau_2) b \tag{24}$$

Substitute into this the expressions for τ_1, τ_2 and P_3 from Eqs. (7), (8) and (16) and use Eq. (18) to eliminate

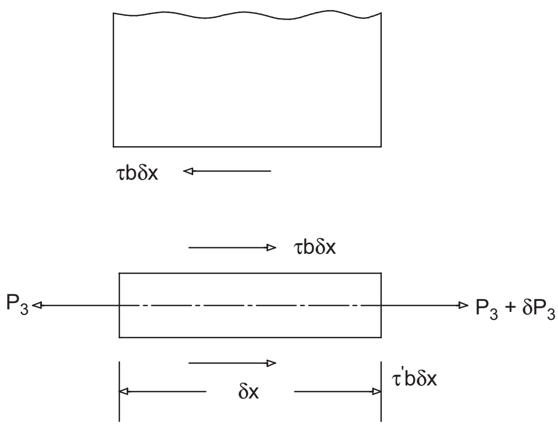

Fig. 6. Longitudinal equilibrium of lower face element.

u_1. After re-arrangement, Eq. (24) then becomes

$$\frac{\partial^2 u_3}{\partial x^2} - g u_3 = -g(d_1 + d_2)\frac{\partial w}{\partial x} \tag{25}$$

Eqs. (23) and (24) constitute the simplest pair of differential equations relating the displacements w and u_3 to the applied loading p. On eliminating u_3 from this pair, a single sixth-order differential equation in w is obtained, viz.

$$\frac{\partial^6 w}{\partial x^6} - g[1 + (Y_1 + Y_2)]\frac{\partial^4 w}{\partial x^4} + g^2[(Y_1 + Y_2) - \left(\frac{2Y_1}{d_1} - \frac{Y_2}{d_2}\right)(d_1 + d_2)]\frac{\partial^2 w}{\partial x^2} = \frac{1}{D_t}\left[\frac{\partial^2 p}{\partial x^2} - pg\right] \tag{26}$$

4. Method of solution

For harmonic motion one can assume that

$$w(x,t) = W(x)\exp(\mathrm{i}ft) \tag{27}$$

Then the inertia force p has the form

$$p = -\mu\partial^2 w(x,t)/\partial t^2 = -\mu W f^2 \exp(\mathrm{i}ft) \tag{28}$$

Substitution of w and p from Eqs. (27) and (28) into Eq. (26) gives

$$\frac{\mathrm{d}^6 W}{\mathrm{d}x^6} - g[1 + (Y_1 + Y_2)]\frac{\mathrm{d}^4 W}{\mathrm{d}x^4} + g^2[(Y_1 + Y_2) - \left(\frac{2Y_1}{d_1} - \frac{Y_2}{d_2}\right)(d_1 + d_2)]\frac{\mathrm{d}^2 W}{\mathrm{d}x^2} - s\frac{\mathrm{d}^2 W}{\mathrm{d}x^2} + sgW = 0 \tag{29}$$

which is a simple linear differential equation of the sixth order. Hence a solution of the form

$$W(x) = A\exp(\sigma x) \tag{30}$$

can be assumed. Substitution of Eq. (30) into Eq. (29) yields the characteristic equation

$$\sigma^6 - g[1 + (Y_1 + Y_2)]\sigma^4 + g^2[(Y_1 + Y_2) - \left(\frac{2Y_1}{d_1} - \frac{Y_2}{d_2}\right)(d_1 + d_2)]\sigma^2 - s\sigma^2 + sgW = 0 \tag{31}$$

which is cubic in σ^2. The roots can be exactly determined [14]. The method of finding roots is described in Appendix A. The complete solution of differential equation (29) can then be expressed as

$$W(x) = \sum_{j=1}^{6} A_j \exp(\sigma_j x) \tag{32}$$

The constants A_j, $j = 1, 2, \ldots, 6$, are to be obtained by application of the boundary conditions of the beam.

4.1. Boundary conditions

The beam can be imagined to be comprised of identical halves, each of which is acted upon by one-half of the applied force F_0 at the junction point (Fig. 7). The center of the beam can now conveniently be taken as the origin (Fig. 8).

The expressions for P_1, M, S and u_1 in terms of w and its derivatives are obtained as follows.

Since there can be no longitudinal force on both the face plates, i.e., the first face plate and the third face plate.

$$\text{Hence,}\quad P_1 + P_3 = 0 \quad \text{or} \quad P_1 = -P_3$$

$$\text{or}\quad Eh_1 b\frac{\partial u_1}{\partial x} = -Eh_1 b\frac{\partial u_3}{\partial x} \tag{33}$$

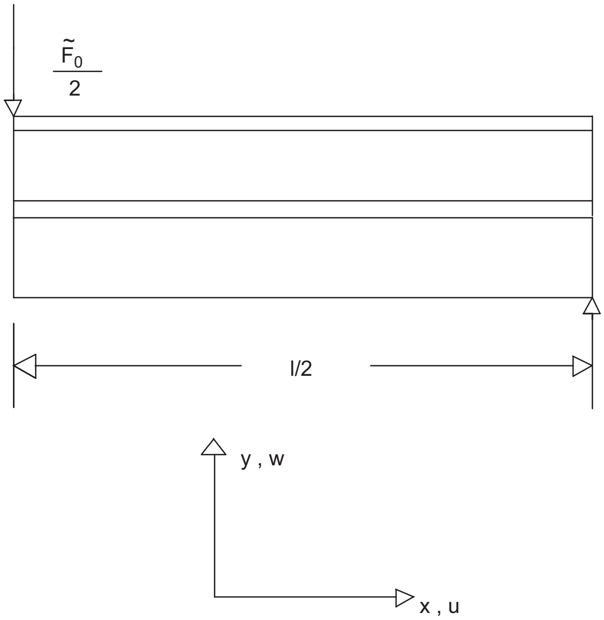

Fig. 7. Identical right half of sandwich beam.

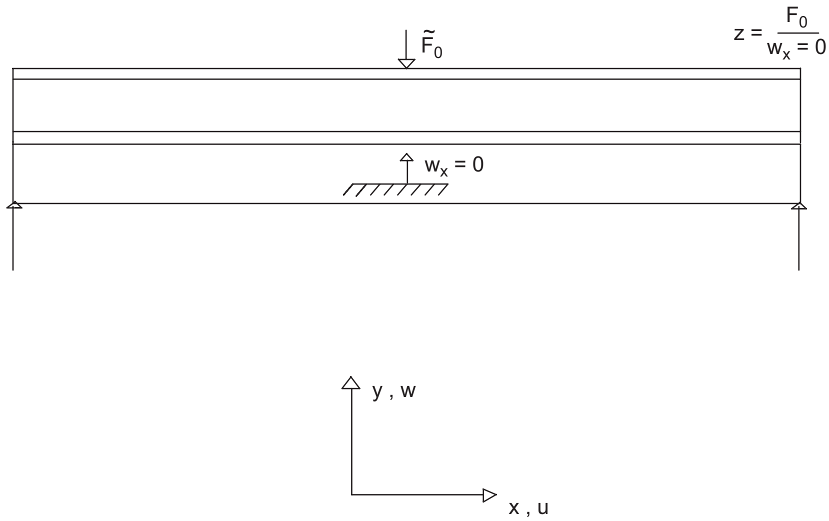

Fig. 8. Driving point impedance.

Eqs. (15), (16) and (33) can readily be manipulated to show that

$$P_1 = \frac{D_t}{g(2d_1 - d_2)}\left[\frac{\partial^4 w}{\partial x^4} - g(Y_1 + Y_2)\frac{\partial^2 w}{\partial x^2} - Ws\right] \tag{34}$$

The total bending moment, M, acting on the section can be split into four components analogous to those of the shear force.

(a) and (b) moments M_1 and M_3 associated with the flexural stiffness D_1 and D_3 of the top and bottom face plates, i.e.

$$M_1 = D_1\frac{\partial^2 w}{\partial x^2}, \quad M_3 = D_3\frac{\partial^2 w}{\partial x^2} \tag{35a, b}$$

(c) Moment M_2 associated with the equal and opposite forces, P_1 and P_3 which act along the mid-planes of the face plates, i.e.

$$M_2 = P_3 d_1 = -\frac{D_t}{g(2d_1 - d_2)}\left[\frac{\partial^4 w}{\partial x^4} - g(Y_1 + Y_2)\frac{\partial^2 w}{\partial x^2} - Ws\right] d_1 \tag{35c}$$

(d) Moment M_4 associated with the force P_3 which act along the mid-plane of the face plate, i.e.

$$M_4 = -P_3 d_2 = \frac{D_t}{g(2d_1 - d_2)}\left[\frac{\partial^4 w}{\partial x^4} - g(Y_1 + Y_2)\frac{\partial^2 w}{\partial x^2} - Ws\right] d_2 \tag{35d}$$

Then

$$\begin{aligned} M &= M_1 + M_2 + M_3 + M_4 \\ &= \frac{D_t(d_1 - d_2)}{g(2d_1 - d_2)}\left[\frac{-\partial^4 w}{\partial x^4} + g\left\{\frac{(Y_1 + Y_2)(d_1 - d_2) + (2d_1 - d_2)}{(d_1 - d_2)}\right\}\frac{\partial^2 w}{\partial x^2} + Ws\right] \end{aligned} \tag{36}$$

Since the total shear force, S, is given by $\partial M/\partial x$, it follows that

$$S = \frac{D_t(d_1 - d_2)}{g(2d_1 - d_2)}\left[-\frac{\partial^5 w}{\partial x^5} + g\left\{\frac{(Y_1 + Y_2)(d_1 - d_2) + (2d_1 - d_2)}{(d_1 - d_2)}\right\}\frac{\partial^3 w}{\partial x^3} + \frac{\partial W}{\partial x}s\right] \tag{37}$$

Now, $P_1 = Eh_1 b(\partial u_1/\partial x)$ hence from Eq. (34),

$$\frac{D_t}{g(2d_1 - d_2)}\left[\frac{\partial^4 w}{\partial x^4} - g(Y_1 + Y_2)\frac{\partial^2 w}{\partial x^2} - Ws\right] = Eh_1 b\frac{\partial u_1}{\partial x}$$

or

$$\frac{\partial u_1}{\partial x} = \frac{D_t}{g^2(2d_1 - d_2)Eh_1 b}\left[g\frac{\partial^4 w}{\partial x^4} - g^2(Y_1 + Y_2)\frac{\partial^2 w}{\partial x^2} - gWs\right] \tag{38}$$

Eqs. (29) and (38) can readily be manipulated to show that

$$u_1 = \frac{D_t}{g^2 Eh_1 b(2d_1 - d_2)}\left[\frac{\partial^5 w}{\partial x^5} - g(Y_1 + Y_2)\frac{\partial^3 w}{\partial x^3} - \left\{g^2 s + 2g^2 Y_1\left(\frac{d_1 + d_2}{d_1}\right) - g^2 Y_2\left(\frac{d_1 + d_2}{d_2}\right)\right\}\frac{\partial w}{\partial x}\right] \tag{39}$$

The possible boundary conditions for a sandwich beam free at one end and simply supported at the other end are as follows:

at $x = 0$ (at center)

(i) shear force $= F_0/2$, (ii) slope $= \mathrm{d}W/\mathrm{d}x = 0$, (iii) $u_1 = 0$;

at $x = l/2$ (at right end)

(iv) deflection $= W = 0$, (v) bending moment $= 0$, (vi) $P_1 = -P_3 = 0$.

Applying the above six boundary conditions, with the help of Eqs. (34) and (36)–(39), one obtains finally a matrix equation of the form

$$[C]\{B\} = \{H\} \tag{40}$$

where $[C]$ is a square matrix of dimension 6×6. $\{B\}$ and $\{H\}$ are column matrices. The elements of these matrices are, for $j = 1, 2, \ldots, 6$,

$$C_{1j} = -\sigma_j^5 + g\left[\frac{(Y_1 + Y_2)(d_1 - d_2) + (2d_1 - d_2)}{(d_1 - d_2)}\right]\sigma_j^3$$

$$C_{2j} = \sigma_j$$

$$C_{3j} = \sigma_{j5} - g(Y_1 + Y_2)\sigma_j^3$$
$$C_{4j} = \exp\left(\sigma_j \frac{l}{2}\right)$$
$$C_{5j} = \sigma_j^2 \exp\left(\sigma_j \frac{l}{2}\right)$$
$$C_{6j} = \sigma_j^4 \exp\left(\sigma_j \frac{l}{2}\right)$$
$$B_j = A_j/F_0, \quad H_j = \frac{g(2d_1 - d_2)}{2D_t(d_1 - d_2)}, \quad j = 1$$
$$= 0, \quad j \neq 1$$

Eq. (40) can be solved for $B_1, B_2, \ldots, B_6$. The beam solution then can be written as

$$\frac{W(x)}{F_0} = \sum_{j=1}^{6} B_j \exp(\sigma_j x)$$

4.2. Response of primary system

With the solution for the beam thus obtained, the primary system with its spring and damper can now be considered as attached to the center of the beam (see Figs. 1 and 2).

In Fig. 1 the beam can be replaced by its dynamic stiffness as shown in Figs. 9 and 10. The dynamic stiffness of the beam at the junction point, which may be defined as the ratio of force to displacement is given by

$$Z = F_0/W(0) = 1 \Bigg/ \sum_{j=1}^{6} B_j \tag{41}$$

The equivalent dynamic stiffness of the system (see Fig. 10) is given by

$$Z_{\text{eq}} = 1 \Bigg/ \left\{ \sum_{j=1}^{6} B_j + \frac{1}{K(1 + i\delta)} \right\} \tag{42}$$

The equation of motion for the system as shown in Fig. 10 is

$$m\ddot{y}_1 + Z_{\text{eq}} y_1 = F \exp(ift) \tag{43}$$

Since the motion is harmonic, y_1 may be assumed to be of the form

$$y_1 = Y_1 \exp(ift) \tag{44}$$

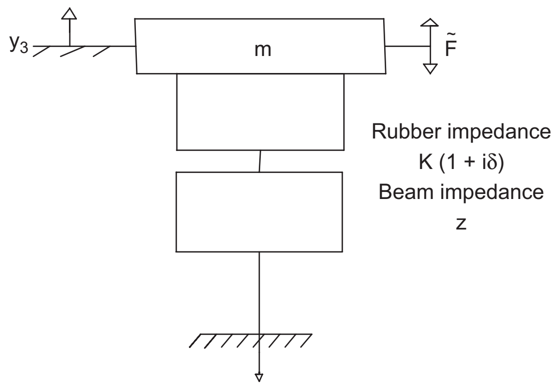

Fig. 9. System with rubber impedance and beam impedance.

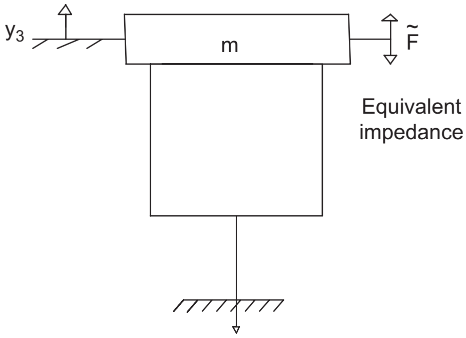

Fig. 10. Rubber impedance and beam impedance combined.

Substituting Eq. (44) in Eq. (42) gives

$$Y_1/F = 1/(-mf^2 + Z_{eq}) \tag{45}$$

Combining Eqs. (42) and (45) then yields

$$Y_1/F = \left\{\sum_{j=1}^{6} B_j + \frac{1}{K(1+i\delta)}\right\} \Big/ \left\{-mf^2 \sum_{j=1}^{6} B_j - \frac{mf^2}{K(1+i\delta)} + 1\right\} \tag{46}$$

which is the response of the primary system to an exciting force of unit amplitude.

4.3. Transmissibility

With reference to Fig. 10 the exciting force at the junction of the beam and the primary system can be expressed in the form

$$F_0/F = Z_{eq}/(-mf^2 + Z_{eq}) \tag{47}$$

From Eq. (37) for the shear force at any section of the beam, the right-hand support force F_1 can be obtained by substituting Eq. (32) into it and putting $x = l/2$. This gives

$$\frac{F_1}{F_0} = \frac{D_t(d_1 - d_2)}{g(2d_1 - d_2)} \sum_{j=1}^{6} \left[\left\{-\sigma_j^5 + g\left(\frac{(Y_1 + Y_2)(d_1 - d_2) + (2d_1 - d_2)}{(d_1 - d_2)}\right)\sigma_j^3 + (\mu f^2/D_t)\sigma_j\right\} B_j \exp(\sigma_j l/2)\right] \tag{48}$$

The transmissibility, which may be defined as the ratio of the total dynamic force transmitted at the end support to the impressed force [15], is given by

$$T = 2F_1/F \tag{49}$$

Combining Eqs. (47)–(49), and simplifying, finally yields

$$T = \frac{2\dfrac{D_t(d_1 - d_2)}{g(2d_1 - d_2)} \displaystyle\sum_{j=1}^{6} \left[\left\{-\sigma_j^5 + g\left(\frac{(Y_1 + Y_2)(d_1 - d_2) + (2d_1 - d_2)}{(d_1 - d_2)}\right)\sigma_j^3 + \left(\frac{\mu f^2}{D_t}\right)\sigma_j\right\} B_j \exp(\sigma_j l/2)\right]}{\left[-mf^2 \displaystyle\sum_{j=1}^{6} B_j - \frac{mf^2}{K(1+i\delta)} + 1\right]} \tag{50}$$

5. Results and discussion

Theoretical results deduced from Eqs. (46) and (50) are plotted in Figs. 11 and 12. In each case the response and transmissibility have been plotted against the frequency. Figs. 11 and 12 show respectively plots

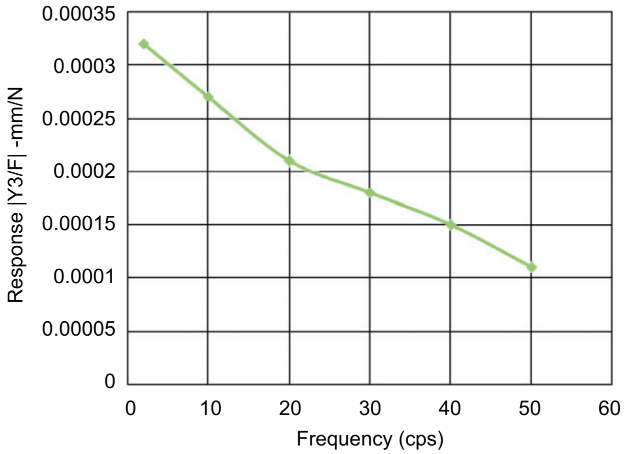

Fig. 11. Variation of response $|Y_3/F|$ with frequency for four-layer beam.

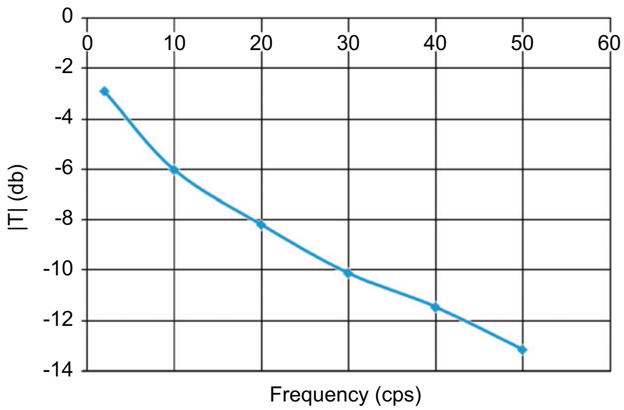

Fig. 12. Variation of transmissibility $|T|$ with frequency for four-layer beam.

of the theoretical curves representing the variation of the response of the primary system and the variation of the transmissibility provided by the complete system with frequency. The beam used is a MS-PVC–MS-PVC sandwich beam. Young's modulus of M.S. $= 1.97 \times 10^5$ MPa. The other parameters were $h_1 = 1$ mm, $h_2 = 10$ mm, $l = 500$ mm and $b = 80$ mm. The dynamic properties of rubber and PVC were taken from experimentally obtained values [16,17] and these are given in Appendix B. The main mass was taken as 1.80 kg. It is observed from Figs. 11 and 12 that the response $|Y_3/F|$ and transmissibility $|T|$ decrease as the frequency (cps) increases. The rate of decrease is slow for higher values of frequency (cps).

6. Conclusions

It can be concluded that a four-layered sandwich beam having a configuration symmetrical with respect to both geometrical and physical parameters provides a minimum response to the primary system as well as minimum transmissibility of the excitation force to the support. Further it is found that response and transmissibility decrease as the frequency (cps) increases. But the rate of decrease is slow for higher values of frequency (cps).

Appendix A. Expressions for roots of Eq. (31)

Substituting $\sigma^2 = R$ in Eq. (31) gives

$$R^3 + \alpha_1 R^2 + \alpha_2 R + \alpha_3 = 0 \tag{A.1}$$

$$\alpha_1 = -g(1 + Y_1 + Y_2)$$
$$\alpha_2 = g^2(Y_1 + Y_2) - g^2\left[\frac{2Y_1}{d_1} - \frac{Y_2}{d_2}\right](d_1 + d_2) - \mu f^2/D_t$$
$$\alpha_3 = (\mu f^2/D_t)g$$

with $R = X_1 - (\alpha_1/3)$, Eq. (A.1) becomes

$$X^3 + 3a_1 X + 2b_1 = 0$$
$$3a_1 = (-\alpha_1^2/3), \quad 2b_1 = (2/27)\alpha_1^3 - (\alpha_1\alpha_2/3) + \alpha_3 \tag{A.2}$$

Eq. (A.2) is a standard form cubic equation [14]. The roots are:

$$X_1 = U + V, \quad X_2 = \varepsilon_1 U + \varepsilon_2 V, \quad X_3 = \varepsilon_2 U + \varepsilon_1 V \qquad \text{(A.3–A.5)}$$

$$U = \left(-b_1 + \sqrt{a_1^3 + b_1^2}\right)^{1/3}, \quad V = \left(-b_1 - \sqrt{a_1^3 + b_1^2}\right)^{1/3}, \quad \varepsilon_{1,2} = -\frac{1}{2} \pm i\frac{\sqrt{3}}{2} \qquad \text{(A.6–A.8)}$$

The roots of Eq. (A.1) are

$$R_1 = U + V - \alpha_1/3, \quad R_2 = \varepsilon_1 U + \varepsilon_2 V - \alpha_1/3, \quad R_3 = \varepsilon_2 U + \varepsilon_1 V - \alpha_1/3 \qquad \text{(A.9–A.11)}$$

Hence the roots of Eq. (31) are

$$\sigma_1 = \sqrt{R_1}, \quad \sigma_2 = -\sigma_1, \quad \sigma_3 = \sqrt{R_2} \qquad \text{(A.12–A.14)}$$

$$\sigma_4 = -\sigma_3, \quad \sigma_5 = \sqrt{R_3}, \quad \sigma_6 = -\sigma_5 \qquad \text{(A.15–A.17)}$$

Appendix B. Dynamic properties of rubber and PVC materials

Dynamic stiffness and loss factor of rubber [16] at 30 °C: $K = 850.0 + 28.3f_c$ (N/cm), for $f_c \leqslant 45$Hz; $K = 2500$ N/cm, $f_c > 45$Hz; $\delta = 0.126$

Shear modulus and loss factor of PVC [17] at 30 °C: $G = 420.0 + 2.5f_c$ (N/cm^2) and $\beta = 0.24 + 0.00125f_c$ for $f_c \leqslant 80$ Hz; $G = 570.0 + 0.667f_c$ (N/cm^2) and $\beta = 0.28 + 0.00075f_c$ for $f_c > 80$ Hz.

References

[1] V. Baburaj, Y. Matsukai, A study on the material damping of thin angle-ply laminated plates, *Journal of Sound and Vibration* 172 (1994) 415–419.

[2] S. Adhikari, Damping Models for Structural Vibrations, PhD Thesis, Cambridge University Engineering Department, Cambridge, UK, 2000.

[3] A.K. Noor, W.S. Burton, C.W. Bert, Computational models for sandwich panels and shells, *Applied Mechanics Reviews* 49 (3) (1996) 155–199.

[4] P. Cupial, J. Niziol, Vibration and damping analysis of a three-layered composite plate with a viscoelastic mid-layer, *Journal of Sound and Vibration* 183 (1) (1995) 99–114.

[5] P. Grootenhuis, D. J. Ewins, Multidirectional vibration isolation, presented at the Vibration Isolation Symposium of the Society of Environmental Engineers, Imperial College, London.

[6] D.J. Mead, The double skin damping configuration, University of Southampton, A.A.S.U. Report no. 160, 1962.

[7] R.A. Ditaranto, Theory of vibratory bending for elastic and viscoelastic layered finite-length beams, *Journal of Applied Mechanics* 32 (1965) 881–886.

[8] R.A. Ditaranto, J.R. McGraw, Vibrating bending of damped laminated plates, *Journal of Engineering for Industry, Transactions of the American Society of Mechanical Engineers* 91 (1969) 1081–1090.

[9] E.M. Kerwin Jr, Damping of flexural waves by constrained viscoelastic layers, *Journal of the Acoustical Society of America* 31 (1959) 952–962.
[10] Y.Y. Yu, Damping of flexural vibrations of sandwich plates, *Journal of Aerospace Sciences* 29 (1962) 790–803.
[11] D. Ross, E.E. Ungar, E.M. Kerwin, *Damping of flexural vibrations by means of viscoelastic laminae, Colloquium on Structural Damping*, ASME, New York, 1959.
[12] D.J. Mead, S. Markus, The forced vibration of a three-layer, damped sandwich beam with arbitrary boundary conditions, *Journal of Sound and Vibration* 10 (1969) 163–175.
[13] B.P. Yadav, R.C. Das Vikal, Vibration of excitation system supported flexibly on transverse beam, *Journal of the Institution of Engineers (India)* 80 (1999) 84–89.
[14] C.J. Tranter, *Techniques of Mathematical Analysis*, The English University Press Limited, London, 1957, pp. 131–132.
[15] J.C. Snowdon, *Vibration and Shock in Damped Mechanical System*, Wiley, New York, 1968, p. 225.
[16] R.C. Das Vikal, Two Stage Isolation System Using Viscoelastic Materials, M.Tech. Thesis, Indian Institute of Technology, Delhi, 1976.
[17] N.T. Asnani, Analysis of Multilayer Beams with Constrained Viscoelastic Layer, PhD Thesis, Indian Institute of Technology, Delhi, 1971.