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Modification of fundamental vibration modes of circular plates with free edges

W.H. Duan[a], C.M. Wang[b,*], C.Y. Wang[c]

[a]Department of Civil Engineering, National University of Singapore, Kent Ridge, Singapore 119260, Singapore
[b]Engineering Science Programme and Department of Civil Engineering, National University of Singapore, Kent Ridge, Singapore 119260, Singapore
[c]Department of Mathematics, Michigan State University, East Lansing, MI 48824, USA

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Abstract

For an isotropic circular plate with constant thickness and free edges, its fundamental vibration mode takes the form of a twisting mode with two nodal diameters, i.e. $n = 2$. In certain applications, it may be necessary to have an axisymmetric shape for the fundamental vibration mode (i.e. mode shape with no nodal diameter, $n = 0$). In this paper, we show that such an axisymmetric vibration mode can be realized by increasing the bending rigidity of the outer rim of the circular plate by using a larger thickness or by using a material with a larger Young's modulus or both. We also determine the critical flexural rigidity of the outer rim that will trigger this vibration mode change from $n = 2$ type to $n = 0$ type for a given rim width. The ability to alter the mode shape of circular plates with free edges has useful applications in sensing and actuating devices and large pontoon-type floating circular structures.

1. Introduction

Plated structures are employed in many engineering and industrial applications. For example, piezoelectric plates are used in energy harvesting [1], silicon plates for sensing [2–4] and biological analysis [5], pontoon-type very large floating structures [6–8]. For analysis of these aforementioned example applications, the structures may be modeled as plates with free edges. Considering an isotropic circular plate with uniform thickness and free edges, the fundamental mode of vibration takes on a twisting mode with two nodal diameters (i.e. $n = 2$). However, the axisymmetric vibration mode (i.e. $n = 0$) is often desired as the fundamental mode for certain applications. For example, larger responses can be achieved for circular piezoelectric sensors with an axisymmetric mode shape [9]. Therefore, the problem arises on how one can design a circular free-edge plate that has an axisymmetric fundamental mode of vibration (i.e. $n = 0$). By noting that the fundamental mode of vibration is axisymmetric for circular plates

*Corresponding author. Tel.: +65 65162157.
E-mail address: cvewcm@nus.edu.sg (C.M. Wang).

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with clamped edges, we propose to solve the aforementioned problem by using a stepped circular where its outer annular sub-plate (or outer rim) has a larger bending rigidity as compared with the bending rigidity of its inner circular sub-plate. Such a plate design simulates the case of an inner circular sub-plate having a clamped-type boundary condition due to the relatively larger bending rigidity of the outer rim.

Extensive studies have been carried out for the vibration of circular plates with varying thickness. Many of the studies have been well documented in the excellent review papers of Leissa [10–16]. The vibration problems of circular and annular plates with non-linear thickness variations have been solved by approximate solution methods, such as the Rayleigh–Ritz method [17–21], perturbation method [22], the generalized differential quadrature rule (GDQR) [23], as well as by analytical methods [24–27]. The vibration problem of stepped circular plates, based on classical plate theory and improved Mindlin plate theory, has also been studied by numerous investigators [8,28–31].

The aim of the paper is to show that we can modify the fundamental mode of the stepped circular plate from a twisting mode shape to an axisymmetric mode shape by adjusting the rigidity of the edge annular plate. This has an important implication for the design of circular plated structures with free edges. The Mindlin plate theory is applied to describe the dynamic behavior of the circular stepped plate in order to allow for the significant effect of transverse shear deformation in the thickened portion of the plate. The vibration problem is solved analytically. Comparison studies with 3D finite element model (FEM) are carried out to provide an independent check on the analytical results. Extensive exact vibration frequencies are presented for circular stepped plates with free edge. The influences of stepped thickness and stepped material properties on transforming the fundamental mode shape are highlighted.

2. Mathematical modelling

Consider a stepped circular plate of radius a, stepped thickness at $\zeta = r/a = b$, Young's modulus E_1 and E_2, mass density ρ and Poisson's ratio v. The plate edge is completely free. Following the method proposed by Hang et al. [28], the stepped circular plate is decomposed into two sub-plates where the continuity conditions are satisfied at the stepped boundary $\zeta = b$ as shown in Fig. 1. The outer annular sub-plate 1 has a constant thickness $\tau_1 = h_1/a$ while the inner circular sub-plate 2 has a constant thickness $\tau_2 = h_2/a$. The problem at hand is to determine the fundamental frequency and mode shape of the freely vibrating, stepped, circular plates for various combinations of stepped designs.

In order to allow for the effects of transverse shear deformation and rotary inertia, especially when having a relatively thickened portion in stepped plates, it is proposed that the Mindlin plate theory be adopted instead of the commonly used classical thin plate theory. Following the work by Mindlin and Deresiewicz [32,33], the rotations ψ_r, ψ_θ and the transverse displacement w (normalized by the radius a) may be expressed as functions of three potentials η_1, η_2 and H in the following manner:

$$\begin{aligned} \psi_r &= (\sigma_1 - 1)\frac{\partial \eta_1}{\partial \zeta} + (\sigma_2 - 1)\frac{\partial \eta_2}{\partial \zeta} + \frac{1}{\zeta}\frac{\partial H}{\partial \theta}, \\ \psi_\theta &= (\sigma_1 - 1)\frac{\partial \eta_1}{\zeta \partial \theta} + (\sigma_2 - 1)\frac{\partial \eta_2}{\zeta \partial \theta} - \frac{\partial H}{\partial \zeta}, \\ w &= \eta_1 + \eta_2. \end{aligned} \tag{1}$$

In view of the modes of vibration of interest in the present case (i.e. $n = 0$ and $n = 2$), three potentials η_1, η_2 and H can be expressed as

$$\begin{aligned} \eta_1 &= [A_1 J_n(\delta_1 r) + B_1 Y_n(\delta_1 r)] \cos n\theta, \\ \eta_2 &= [A_2 I_n(\delta_2 r) + B_2 K_n(\delta_2 r)] \cos n\theta, \\ H &= [A_3 I_n(\omega r) + B_3 K_n(\omega r)] \sin n\theta, \end{aligned} \tag{2}$$

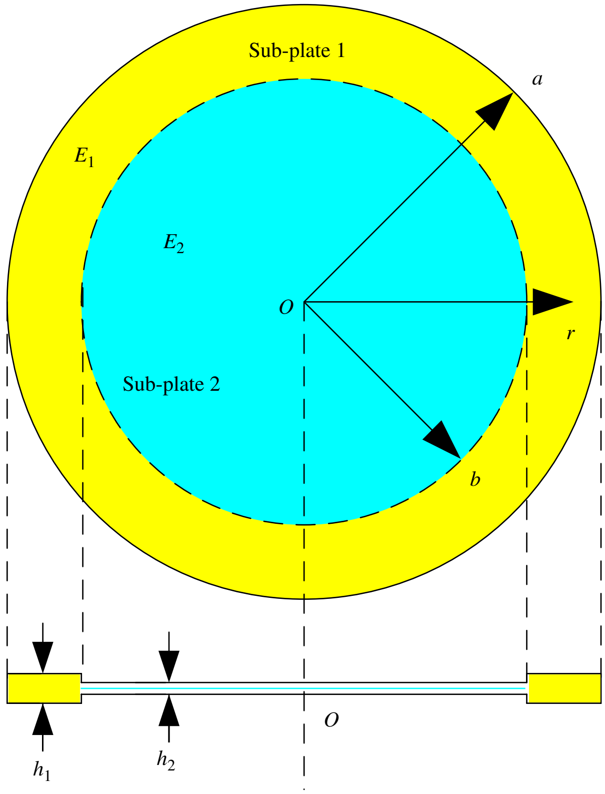

Fig. 1. Stepped circular plate with free edges.

where J, Y are Bessel functions of the first and second kind, respectively; I, K are the modified Bessel functions of the first and second kind, respectively; and the other parameters are

$$\sigma_1, \sigma_2 = (\delta_2^2, \delta_1^2)(R\lambda^2 - S^{-1})^{-1},$$
$$\delta_1^2, \delta_2^2 = \frac{\delta_0^4}{2}\left\{R + S \pm \left[(R-S)^2 + \frac{4}{\lambda^2}\right]^{1/2}\right\},$$

$$\delta_3^2 = \frac{2(R\delta_0^4 - S^{-1})}{1-v}, \quad \lambda^2 = \frac{a^4\rho p^2 h}{D},$$
$$R = \frac{\tau^2}{12}, \quad S = \frac{\tau^2}{6\kappa^2(1-v)}, \tag{3}$$

in which κ^2 is the shear correction factor, p the natural circular frequency, $D = Eh^3/[12(1-v^2)]$ the flexural rigidity, and τ the non-dimensional thickness of circular plate normalized by the radius a. The plate stress resultants are given in terms of the plate displacements by

$$M_r = \frac{D}{a}\left[\frac{\partial\psi_r}{\partial\zeta} + \frac{v}{\zeta}\left(\psi_r + \frac{\partial\psi_\theta}{\partial\theta}\right)\right], \quad M_\theta = \frac{D}{a}\left[v\frac{\partial\psi_r}{\partial\zeta} + \frac{1}{\zeta}\left(\psi_r + \frac{\partial\psi_\theta}{\partial\theta}\right)\right],$$
$$M_{r\theta} = \frac{D}{2a}(1-v)\left[\frac{\partial\psi_\theta}{\partial\zeta} + \frac{1}{\zeta}\left(\frac{\partial\psi_r}{\partial\theta} - \psi_\theta\right)\right],$$
$$Q_r = \kappa^2 Gh\left(\psi_r + \frac{\partial w}{\partial\zeta}\right), \quad Q_\theta = \kappa^2 Gh\left(\psi_\theta + \frac{1}{\zeta}\frac{\partial w}{\partial\theta}\right). \tag{4}$$

For the vibrations of a plate with free edges, the natural boundary conditions are

$$M_r = 0, \quad M_{r\theta} = 0, \quad Q_r = 0 \quad \text{at } \zeta = 1. \tag{5}$$

The continuity conditions at the stepped boundary are given by

$$\left.\begin{array}{lll} \psi_{r1} = \psi_{r2}, & \psi_{\theta 1} = \psi_{\theta 2}, & w_1 = w_2 \\ M_{r1} = M_{r2}, & M_{\theta 1} = M_{\theta 2}, & Q_{r1} = Q_{r2} \end{array}\right\} \text{at } \zeta = b, \tag{6}$$

where subscript 1 denotes the quantities belonging to the outer annular sub-plate and subscript 2 for the inner circular sub-plate 2. By substituting the displacements Eq. (1) and the stress resultants Eq. (4) into Eqs. (5) and (6), a set of homogeneous system of equations is obtained and the equations may be written as

$$[K]_{9\times 9}\{\Psi\}_{9\times 1} = \{0\}_{9\times 1}, \tag{7}$$

where $\{\Psi\}$ is the vector of nine unknown coefficients. The elements of the matrix $[K]$ for asymmetric vibration mode ($n\neq 0$) and axisymmetric vibration mode ($n = 0$) are given in the paper by Ref. [28]. The frequency parameter λ of the entire plate is evaluated by setting the determinant of $[K]$ to zero and then solving the characteristic equation by a root finding algorithm, such as the bisection method [34].

3. Results and discussions

We consider various plate designs that involve changing the stepped thickness ratio $\alpha = \tau_1/\tau_2$, and the stepped modulus of elasticity $\beta = E_1/E_2$. The stepped location b is kept at 0.8, a value that is relatively large so that a more sensitive response of the inner sub-plate 2 may be realized when the plate is used as a sensor. The constant thickness $\tau_2 = 0.005$ is prescribed for the inner circular sub-plate 2 while the bending rigidity of the outer annular sub-plate 1 is changed. The Poisson ratio $v = 0.3$ and Mindlin's correction factor $\kappa^2 = \pi^2/12$ are used in all calculations.

The variations of the frequencies for the two vibration modes ($n = 0$ and 2) of circular stepped plates with respect to the thickness ratio α are shown in Fig. 3. The thickness τ_1 of the outer annular sub-plate 1 varies from 0.005 to 0.06 m, and hence the stepped thickness ratio α varies from 1 to 12. It should be noted that the radius to thickness ratio of sub-plate 2 is 140 while the radius to thickness ratio of sub-plate 1 varies from 60 to 5. Both radius to thickness ratios are within the assumptions of the Mindlin plate theory. It is seen that when the stepped thickness ratio $\alpha = 1$, the value of frequency parameter $\lambda = 9.00$ with $n = 0$ vis-à-vis the frequency parameter $\lambda = 5.35$ with $n = 2$. Therefore, the fundamental mode is a twisting mode shape with $n = 2$ instead of an axisymmetric mode shape with $n = 0$. For stepped thickness ratio α varying from 1 to 5.64, the mode shape is a twisting mode shape. For $\alpha > 5.64$, the mode shape becomes axisymmetric. Therefore, there is a critical value for the stepped thickness ratio α in which the mode shape could be either twisting-type or axisymmetric-type.

Another observation is that the variation of the stepped thickness ratio α has a larger effect on the frequency value associated with $n = 0$ than the frequency value associated with $n = 2$. For example, at the beginning where α is near to 1, the rate of increase in the frequency values with $n = 2$ and $n = 0$ are similar whereas for the case when α becomes large (say 5), the rate of increase in the frequency associated with $n = 0$ becomes slower as compared with that of frequency with $n = 2$. This frequency trends result in having an intersection point between the curves belonging to frequencies associated with $n = 0$ and $n = 2$.

To provide an independent check on the aforementioned analytical results, we analyzed the stepped circular plates by using ABAQUS 6.4. The finite element meshes of the circular plates with stepped thickness and stepped Young's modulus are shown in Fig. 2. The circular plate is modeled by using 20-node solid elements (C3D20R). Two mesh designs are considered to show the convergence of results. For the circular plate with stepped thickness, the total numbers of elements for these two meshes are 1920 and 2360, and the corresponding numbers of nodes are 10,645 and 13,075, respectively. For the circular plate with stepped Young's modulus, the total numbers of elements for the two meshes are 793 and 1656 and the corresponding

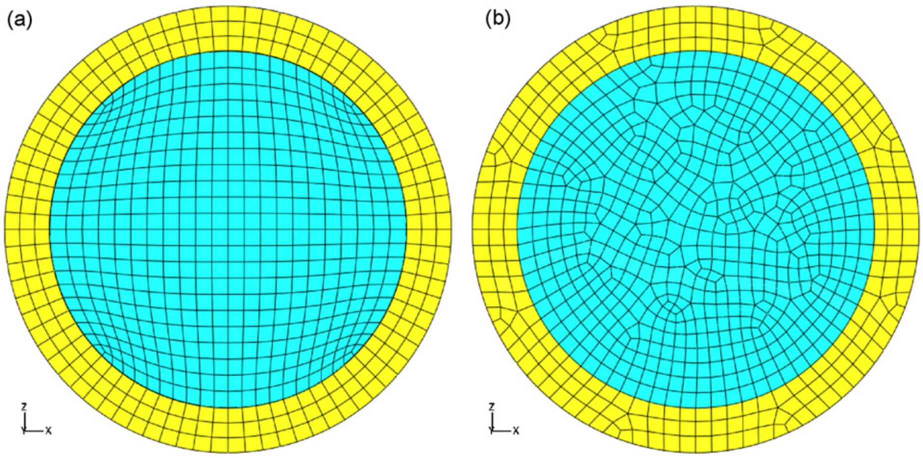

Fig. 2. Finite element discretization of circular plates with (a) stepped thickness (1920 elements) and (b) stepped Young's modulus (793 elements).

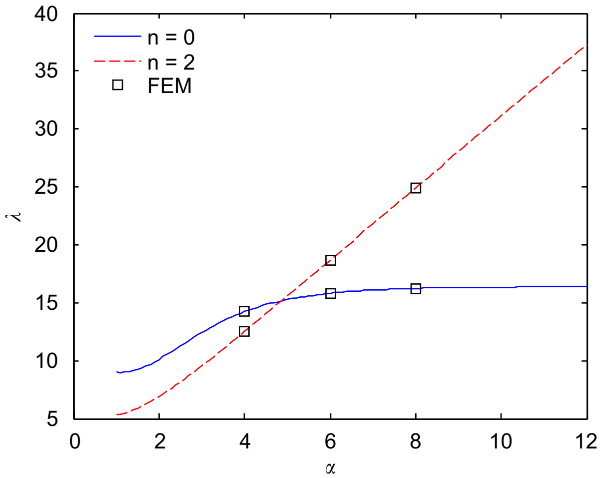

Fig. 3. Variations of frequency parameter λ (associated with $n=0$ and $n=2$) with respect to stepped thickness ratio α.

numbers of nodes are 5774 and 11,910, respectively. The finer mesh design will suffice in producing converged results since the two mesh designs yield vibration frequencies that are within 0.1% of each other.

A comparison study between the analytical and finite element results is shown in Fig. 3. Three stepped thickness ratios, i.e. $\alpha = 4$, 6, and 8, are considered. It can be seen that the vibration frequencies are in good agreement (less than 1% difference), thereby verifying the correctness of the analytical results. The mode shapes for plates with $\alpha = 4$, 8 are shown in Fig. 4. It is clearly seen that the vibration modes have changed from a twisting mode $n = 2$ for $\alpha = 4$ to an axisymmetric mode $n = 0$ for $\alpha = 8$, confirming that the fundamental mode of the circular stepped plates can be altered by adjusting the rigidity of the outer rim of the plate.

The effect of having stepped Young's modulus on the variations of frequency associated with the twisting mode and the axisymmetric mode are shown in Fig. 5. The Young's modulus ratio β is varied from 1 to 60. A comparison between the analytical and finite element results for three stepped Young's modulus ratios (i.e. $\beta = 20$, 30 and 40) are considered. The frequencies are in good agreement (less than 1% difference) and these results further verify the correctness of the analytical solutions. The critical value of stepped Young's modulus ratio where fundamental mode switching occurs is $\beta = 27.33$.

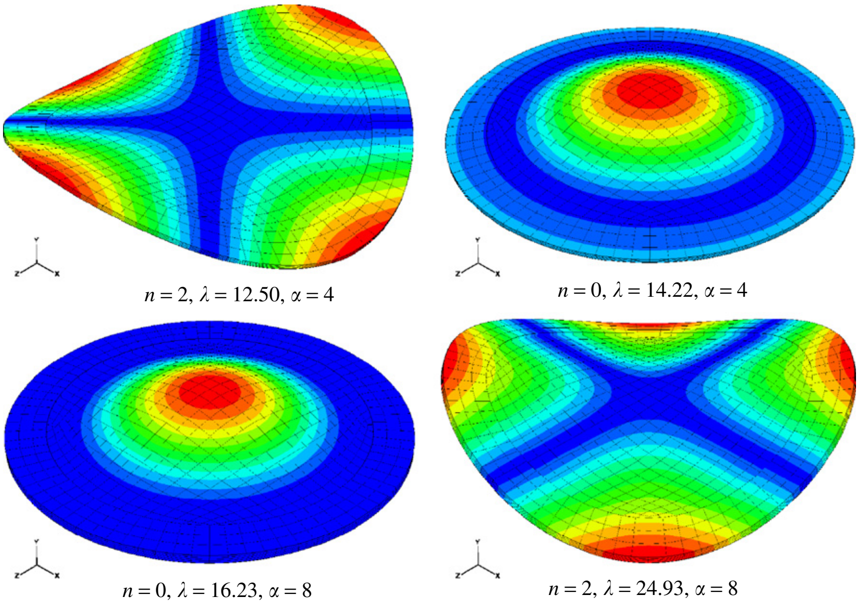

Fig. 4. Mode shape of stepped plate from FEM simulation.

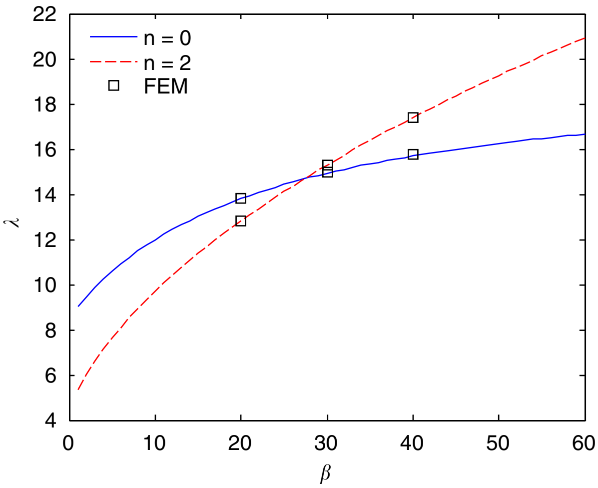

Fig. 5. Variations of frequency parameter λ (associated with $n = 0$ and $n = 2$) with respect to stepped Young's modulus.

4. Concluding remarks

We have shown that the fundamental mode of a circular free-edge plate may be altered from a twisting mode to an axisymmetric one by either having a larger thickness for the outer edge or by having a larger Young's modulus for the outer edge. Both these methods essentially increase the flexural rigidity of the outer edge of the plate, creating an artificial "clamped-typed" boundary condition to the inner circular sub-plate. The ability to modify the mode shape has an important implication for the design of circular plates with free edges.

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