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Statistical energy analysis modelling of complex structures as coupled sets of oscillators: Ensemble mean and variance of energy

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Abstract

Expressions are derived for the ensemble means and variances of the subsystem energies of built-up systems comprising two subsystems. The approach is based on the Statistical Energy Analysis of two spring-coupled oscillators and sets of oscillators, or coupled continuous subsystems, described by Mace and Ji [The statistical energy analysis of coupled sets of oscillators, Proceedings of the Royal Society A 1824 (2007)]. The paper focuses on spring coupling, although similar results hold for more general forms of conservative coupling. Randomness is introduced into the system by assuming that the natural frequency spacings in each subsystem conform to certain statistical distributions. A "coupling coefficient parameter" is introduced which, together with the "coupling strength parameter" defined by Mace and Ji (2007), accounts for the statistics of the coupling stiffness. Various approximations and assumptions are made. It is seen that the variance of the excited subsystem depends primarily on the variance of the input power, which in turn depends on the variance of the number of modes of the excited subsystem in the frequency band of excitation and their mode shapes. The variance of the undriven subsystem, on the other hand, depends primarily on the variance of the intermodal coupling coefficients, which in turn depend on the variances of the number of in-band modes of both subsystems and their mode shapes. The cases of Poisson and Gaussian Orthogonal Ensemble natural frequency spacing statistics are considered. Numerical examples of two plates coupled by one or a number of springs are presented.

1. Introduction

Statistical Energy Analysis (SEA) is commonly used for high-frequency modelling of complex systems [2–3]. The whole system is divided into subsystems connected by interfaces and the response is described in terms of the time- and frequency-averaged subsystem energies and input powers. Compared to deterministic Finite Element Analysis (FEA) in which a detailed, computationally expensive model is formed and the system properties are assumed to be known exactly, the SEA approach is very appealing for the following reasons: (1) it is physically simple and easily understood; (2) the effects of changes and modification to parts of the

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structure, and hence design changes, can be easily predicted; and (3) the relevant computational cost is very small. In the past decades, SEA has received much attention and results are available in many publications, e.g. Refs. [2–6]. However, there are also some difficulties in the application of SEA. First of all, SEA is not a systematic procedure in that the SEA equations involve a number of explicit and implicit assumptions and approximations, the validity of many of which are unknown. Secondly, the prediction of coupling loss factors, being the most important SEA parameters, is often problematic. Finally, there is an issue of estimating response variability. These SEA problems have been addressed by many researchers. For example, in Ref. [7], Keane and Price examined the SEA equations with particular reference to the problem of two multimodal subsystems, strongly coupled at a single point. Maxit and Guyader [8] proposed the modal energy distribution analysis (SmEdA) approach, in which it is not assumed that equipartition of modal energies occurs. Mace [9] used a modal approach to investigate the conditions under which a built-up structure can be described by an SEA model and "quasi-SEA" approach was developed.

There has been much interest in the statistics of the energy response of built-up systems as well. The earliest works concerning the variance of the band-averaged energy of built-up system was that of [4,13,14] and summarized by Lyon and DeJong in [2]. This considered coupled subsystems that have Poisson natural frequency statistics. The most comprehensive analysis so far regarding variance prediction of a built-up system is that of Langley and Brown [10,11] and Langley and Cotoni [12] in which Gaussian Orthogonal Ensemble (GOE) natural frequency spacing statistics were assumed. In Ref. [10], the ensemble statistics of the response to harmonic excitation of a single dynamic system were investigated, the system natural frequency spacings being assumed to conform to Poisson, Rayleigh and GOE statistics. Then in Ref. [11], the theory for harmonic excitation was further extended to band-averaged response of the random system. In Ref. [12], a variance prediction method was developed for the energy levels in a general built-up structure within the context of standard SEA, each subsystem being assumed to conform to GOE natural frequency spacing statistics. Both variance theories are based on the first order expansion of the conventional SEA matrix equations, and a brief review of the main results is given in the appendix.

In Ref. [1], the SEA of coupled oscillators and coupled sets of oscillators was revisited. The oscillator properties were assumed to be random and ensemble averages of SEA parameters found. In particular, account was taken of the correlation between the coupling parameters for the oscillator pairs and their energies. Consequently, some of the assumptions of conventional SEA were removed or relaxed. Various observations were made which depart from these earlier works, e.g. the coupling power and coupling loss factor (CLF) are governed by the "strength of connection" and the "strength of coupling" parameters; the CLF is proportional to damping at low damping and independent of damping in the high damping limit, which corresponds to the weak coupling limit; equipartition of energy does not occur as damping tends to zero, except for the case of two oscillators which have identical natural frequencies. It was found that the coupled oscillator theory [1] and conventional SEA [2,4,13,14] are equivalent for weak coupling [1].

In this paper, based on the coupled oscillator theory described in Ref. [1], general formulas are developed in Sections 2 and 3, respectively, for the ensemble mean and variance of the energies of two coupled oscillators and two sets of coupled oscillators. Throughout, spring coupling is assumed, although similar behaviour is seen for general conservative coupling [1]. Randomness in the system is introduced by assuming that the natural frequency spacings of each subsystem conform to certain, known, statistical distributions. The spectral densities of the excitation are also assumed to be random. In deriving the variance expression in Section 3, a "coupling coefficient parameter" is introduced. This, together with the "strength of connection parameter" defined in Ref. [1], accounts for the statistics of the coupling stiffness. Theoretical implementations are made in Section 4 for two classical forms of statistics of dynamic systems, namely Poisson and GOE natural frequency spacing statistics. Numerical examples of two plates coupled by single and a number of springs are given in Section 5. Results are summarized and discussed in Section 6.

2. Statistical energy analysis of two coupled oscillators

Consider two oscillators with masses m_1 and m_2, damping constants c_1 and c_2 and stiffnesses k_1 and k_2 connected by a spring of stiffness k, as shown in Fig. 1. The oscillators are subjected to two, uncorrelated, band-limited, white noise excitations with spectral densities S_{f1} and S_{f2} over a frequency band Ω. The

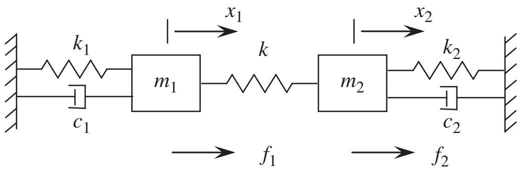

Fig. 1. Two spring-coupled oscillators.

time- and frequency-averaged oscillator energy and input power are [1,2]:

$$P_1 = \frac{1}{\Omega}\frac{\pi S_{f1}}{m_1} \tag{1}$$

$$E_1 = \frac{P_1}{\Delta_1} - \frac{k^2(\Delta_1+\Delta_2)}{m_1 m_2 \Delta_1}\frac{1}{Q}\left(\frac{P_1}{\Delta_1} - \frac{P_2}{\Delta_2}\right) \tag{2}$$

$$Q = (\omega_1^2 - \omega_2^2)^2 + (\Delta_1+\Delta_2)(\omega_2^2\Delta_1 + \omega_1^2\Delta_2) + \frac{k^2(\Delta_1+\Delta_2)^2}{m_1 m_2 \Delta_1 \Delta_2} \tag{3}$$

where $\omega_1 = \sqrt{(k_1+k)/m_1}$ and $\omega_2 = \sqrt{(k_2+k)/m_2}$ are the natural frequencies of oscillators 1 and 2 when the other oscillator is held fixed, and $\Delta_1 = c_1/m_1$ and $\Delta_2 = c_2/m_2$ are the corresponding damping bandwidths of the two oscillators. Throughout, similar expressions hold for oscillator 2 by reversing subscripts. In the above equations, it is assumed that the system damping is not very high, so that $\Delta_{1,2} \ll \Omega$, and that Ω is wide enough to include both natural frequencies ω_1 and ω_2.

In the above, Q is a constant which is determined by the properties of each oscillator and the coupling element between them. Also it is seen that the energy of oscillator 1 comprises 3 components: (1) the energy Δ_1/P_1 when the oscillators are uncoupled; (2) the energy ejected to oscillator 2 through the coupling, which is proportional to the external input power to oscillator 1; (3) the energy injected from oscillator 2 through the coupling, which is proportional to the power input to oscillator 2.

2.1. Ensemble mean and variance

Now assume that the system properties and the excitation spectral densities are not known exactly but are random. The randomness is introduced by assuming the natural frequencies ω_1 and ω_2 are distributed randomly within the frequency band Ω. The mass, damping and coupling stiffness are assumed constant. The excitation spectral densities are also assumed to be random and uncorrelated. (This introduces a contradiction in that, for given mass, the natural frequencies depend on the stiffnesses. However, the sensitivity of the oscillator energies with respect to the stiffnesses are relatively very small compared to the sensitivities with respect to the difference in the natural frequencies, which arises in the term Q in the denominator of Eq. (2), and of the spectral densities, so that negligible errors arise from approximating the stiffnesses by their mid-band values.) The ensemble average input power and oscillator energy are [1]:

$$\overline{P}_1 = \frac{1}{\Omega}\frac{\pi \overline{S}_{f1}}{m_1} \tag{4}$$

$$\overline{E}_1 = \frac{\overline{P}_1}{\Delta_1} - \frac{k^2(\Delta_1+\Delta_2)}{m_1 m_2 \Delta_1} E\left[\frac{1}{Q}\left(\frac{P_1}{\Delta_1} - \frac{P_2}{\Delta_2}\right)\right] \tag{5}$$

Here both $\overline{\bullet}$ and $E[\bullet]$ represent the expectation over the ensemble. Eq. (4) shows that P_1 and P_2 do not depend sensitively on the oscillator properties, so that typically the variations of P_1 and P_2 are relatively very small compared to those of $1/Q$, which depends sensitively on the spacing of the oscillator natural frequencies in particular.

An approximation is now introduced by assuming that P_1, P_2 and $1/Q$ are statistically independent. Eq. (5) then reduces to

$$\overline{E}_1 = \frac{\overline{P}_1}{\Delta_1} - \frac{k^2(\Delta_1+\Delta_2)}{m_1 m_2 \Delta_1} E\left[\frac{1}{Q}\right]\left(\frac{\overline{P}_1}{\Delta_1} - \frac{\overline{P}_2}{\Delta_2}\right) \tag{6}$$

From Eqs. (2) and (6), it follows that

$$E_1 - \overline{E}_1 = \frac{P_1 - \overline{P}_1}{\Delta_1} - \frac{k^2(\Delta_1+\Delta_2)}{m_1 m_2 \Delta_1}\left\{\frac{1}{Q}\left(\frac{P_1}{\Delta_1} - \frac{P_2}{\Delta_2}\right) - E\left[\frac{1}{Q}\right]\left(\frac{\overline{P}_1}{\Delta_1} - \frac{\overline{P}_2}{\Delta_2}\right)\right\} \tag{7}$$

Eq. (7) can be re-written as

$$\begin{aligned} E_1 - \overline{E}_1 = {} & \left\{1 - \frac{k^2(\Delta_1+\Delta_2)}{m_1 m_2 \Delta_1} E\left[\frac{1}{Q}\right]\right\}\left(\frac{P_1 - \overline{P}_1}{\Delta_1}\right) + \frac{k^2(\Delta_1+\Delta_2)}{m_1 m_2 \Delta_1} E\left[\frac{1}{Q}\right]\left(\frac{P_2 - \overline{P}_2}{\Delta_2}\right) \\ & - \frac{k^2(\Delta_1+\Delta_2)}{m_1 m_2 \Delta_1}\left(\frac{P_1}{\Delta_1} - \frac{P_2}{\Delta_2}\right)\left(\frac{1}{Q} - E\left[\frac{1}{Q}\right]\right) \end{aligned} \tag{8}$$

Hence Eq. (8), after some manipulation, yields

$$\begin{aligned} \mathrm{Var}[E_1] = {} & \left[1 - \frac{k^2(\Delta_1+\Delta_2)}{m_1 m_2 \Delta_1} E\left[\frac{1}{Q}\right]\right]^2 \mathrm{Var}\left[\frac{P_1}{\Delta_1}\right] + \left\{\frac{k^2(\Delta_1+\Delta_2)}{m_1 m_2 \Delta_2} E\left[\frac{1}{Q}\right]\right\}^2 \mathrm{Var}\left[\frac{P_2}{\Delta_2}\right] \\ & + \left[\frac{k^2(\Delta_1+\Delta_2)}{m_1 m_2 \Delta_1}\right]^2 \left\{\left(\frac{\overline{P}_1}{\Delta_1} - \frac{\overline{P}_2}{\Delta_2}\right)^2 + \mathrm{Var}\left[\frac{P_1}{\Delta_1}\right] + \mathrm{Var}\left[\frac{P_2}{\Delta_2}\right]\right\} \mathrm{Var}\left[\frac{1}{Q}\right] \end{aligned} \tag{9}$$

Here Var[•] represents the variance.

In Ref. [1], two parameters were introduced to describe the coupling dynamics. The strength of coupling is defined by

$$\gamma^2 = \frac{\kappa_{12}^2}{\Delta_1 \Delta_2} \tag{10}$$

while

$$\kappa_{12}^2 = \frac{k^2}{m_1 m_2 \omega^2} \tag{11}$$

is the strength of connection and ω is the centre frequency of the frequency band Ω. Weak coupling corresponds to $\gamma^2 \ll 1$ [1]. Broadly, γ^2 quantifies the effects of damping and the relative effects of energy dissipation and energy exchange between the oscillators and, below, the sets of oscillators. The strength of connection κ_{12}^2 indicates whether the modes of the coupled system are localized within one or other oscillator (or set of oscillators) or are global. See Ref. [1] for further discussion. An approximation to Eq. (9) can be made for the case of weak coupling, which is the case of most practical importance and the case which will be considered in detail in this paper. Since P_1 and P_2 are uncorrelated then the effects of the individual excitations can be superposed. It is sufficient then to consider the case where $P_2 = 0$. If $k^2/(m_1 m_2 Q) \ll 1$ (which is always the case if the coupling is weak, i.e. $\gamma^2 \ll 1$) and if $\mathrm{Var}[P_1]$ is small compared to $\overline{P}_1^2$, then variances of the energies of oscillators 1 and 2, by Eq. (9), can be approximated, respectively, as

$$\mathrm{Var}[E_1] \approx \frac{\mathrm{Var}[P_1]}{\Delta_1^2} + \left[\frac{k^2(\Delta_1+\Delta_2)}{m_1 m_2 \Delta_1}\right]^2 \left(\frac{\overline{P}_1}{\Delta_1}\right)^2 \mathrm{Var}\left[\frac{1}{Q}\right] \tag{12}$$

$$\mathrm{Var}[E_2] \approx \left[\frac{k^2(\Delta_1+\Delta_2)}{m_1 m_2 \Delta_1} E\left[\frac{1}{Q}\right]\right]^2 \frac{\mathrm{Var}[P_1]}{\Delta_1^2} + \left[\frac{k^2(\Delta_1+\Delta_2)}{m_1 m_2 \Delta_2}\right]^2 \left(\frac{\overline{P}_1}{\Delta_1}\right)^2 \mathrm{Var}\left[\frac{1}{Q}\right] \tag{13}$$

The terms on the right-hand side of Eq. (2) are statistically independent. For the excited oscillator, it is seen from Eq. (12) that variations of the force spectral density S_{f1} across the ensemble contribute to the first term, while the second term arises from variations in $1/Q$ which in turn arise from variations in the natural frequencies and their spacing. The variance of the energy of the excited oscillator (Eq. (12)) depends primarily on that of the input power so that the second term can usually be neglected. However, the variance of the unexcited oscillator's energy (Eq. (13)) depends primarily on the variations in $1/Q$, and the first term can therefore usually be neglected. As a result, the variance expressions in Eqs. (12) and (13) reduce approximately to

$$\mathrm{Var}[E_1] \approx \frac{\mathrm{Var}[P_1]}{\Delta_1^2} \tag{14}$$

$$\mathrm{Var}[E_2] \approx \left[\frac{k^2(\Delta_1+\Delta_2)}{m_1 m_2 \Delta_2}\right]^2 \left(\frac{\overline{P}_1}{\Delta_1}\right)^2 \mathrm{Var}\left[\frac{1}{Q}\right] \tag{15}$$

3. Two coupled sets of oscillators

In this section, the results for two oscillators are extended to the case of two sets of oscillators and expressions for the mean and variance of the energies and input powers developed. Later, each set of oscillators represents the modes of vibration of a subsystem, and these modes become coupled when the subsystems are physically connected.

The sets of oscillators are coupled by springs as shown in Fig. 2. The jth oscillator in set a (with mass m_j^a, damping c_j^a and stiffness k_j^a) and the kth oscillator in set b (with mass m_k^b, damping c_k^b and stiffness k_k^b) are connected by a spring of stiffness k_{jk}^{ab} which is in general random. The system is subjected to statistically independent, white noise excitations with spectral densities S_j^a and S_k^b for oscillators j and k over the frequency band Ω. For coupled multimodal subsystems these represent the modal excitations for modes j and k of subsystems a and b. The total input power and energy for each set, found by summing the contributions for each oscillator, from Eqs. (1) and (2), can be written as [1]

$$P_a = \sum_{j=1}^{N_a} P_j^a = \sum_{j=1}^{N_a} \left[\frac{1}{\Omega}\frac{\pi S_j^a}{m_j^a}\right] \tag{16}$$

$$E_a = \sum_{j=1}^{N_a} E_j^a \tag{17}$$

Here N_a and N_b are the numbers of the oscillators (modes) in sets a and b whose natural frequencies are within the frequency band Ω. N_a and N_b are of course random variables: their means depend on the modal densities and their variances depend on the natural frequency spacing statistics.

To determine the energies, assumptions must be made about the coupling power between each oscillator pair. In Ref. [1], two approaches were suggested: in one the coupling power is assumed proportional to the

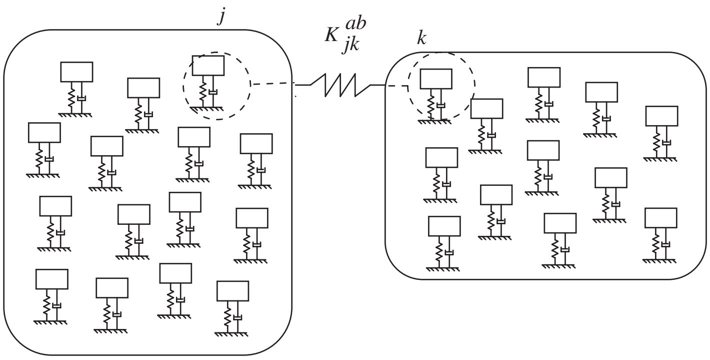

Fig. 2. Two spring-coupled sets of oscillators.

difference of the actual oscillator energies, in the other that it is proportional to the difference in their "blocked" energies. Both approaches lead to expressions which are identical if the strength of coupling is weak or if the strength of connection is weak. The latter, "blocked" approach assumes expressions for the oscillator energies equivalent to Eq. (2), giving

$$E_a = \sum_{j=1}^{N_a} E_j^a = \sum_{j=1}^{N_a} \left[\frac{P_j^a}{\Delta_j^a} - \sum_{k=1}^{N_b} \frac{(k_{jk}^{ab})^2 (\Delta_j^a + \Delta_k^b)}{m_j^a m_k^b Q_{jk}^{ab} \Delta_j^a} \left(\frac{P_j^a}{\Delta_j^a} - \frac{P_k^b}{\Delta_k^b} \right) \right] \tag{18}$$

where (cf. Eq. (3))

$$Q_{jk}^{ab} = [(\omega_j^a)^2 - (\omega_k^b)^2]^2 + (\Delta_j^a + \Delta_k^b)[(\omega_k^b)^2 \Delta_j^a + (\omega_j^a)^2 \Delta_k^b] + \frac{(k_{jk}^{ab})^2 (\Delta_j^a + \Delta_k^b)^2}{m_j^a m_k^b \Delta_j^a \Delta_k^b} \tag{19}$$

is determined by the properties of the jth and kth oscillators in sets a and b. In deriving Eq. (18) it is assumed that the interaction of any pair of oscillators is not affected by the presence of the others, i.e., the two sets of oscillators are weakly connected or coupled. Furthermore, out-of-band modes are neglected.

Now assume that the system properties are not known exactly but are random. The randomness is introduced by assuming that the natural frequencies ${\omega_j}^a$ and ${\omega_k}^b$ of each set conform to certain, known, statistical distributions over the frequency band Ω while the mass and damping of each oscillator are deterministic. Again, this introduces a mild approximation. For simplicity, it is further assumed that each oscillator in set a has the same mass m_a and damping bandwidth Δ_a, i.e.,

$$m_j^a = m_a; \quad \Delta_j^a = \Delta_a \tag{20),(21}$$

Similar assumptions apply to set b. In a similar manner to before, an approximation is introduced by assuming that $P_{a,b}$, $({k_{jk}}^{ab})^2$ and ${Q_{jk}}^{ab}$ are statistically independent variables. Then the ensemble mean energy in each set, from Eq. (18), is given by

$$\overline{E}_a = \frac{\overline{P}_a}{\Delta_a} - \frac{(\Delta_a + \Delta_b)}{\Delta_a} E\left[\sum_{j=1}^{N_a} \sum_{k=1}^{N_b} \frac{(k_{jk}^{ab})^2}{m_a m_b Q_{jk}^{ab}} \right] \left(\frac{\overline{P}_a}{\Delta_a} - \frac{\overline{P}_b}{\Delta_b} \right) \tag{22}$$

The variance of the energy can be found by evaluating $E[{E_a}^2]$.

Some simplifications are now introduced to illustrate the factors that contribute to the variance. If it is further assumed that the coupling is weak, then

$$\gamma^2 = \frac{\kappa^2}{\Delta_a \Delta_b} \ll 1 \tag{23}$$

where

$$\kappa^2 = \frac{E[(k_{jk}^{ab})^2]}{m_a m_b \omega^2} \tag{24}$$

is the strength of connection between the two sets of oscillators. This assumption allows simplifications to be made in a manner similar to that leading to Eqs. (12) and (13). If only set a is excited, then $P_b = 0$ and the variances of the energies reduce to

$$\mathrm{Var}[E_a] \approx \frac{\mathrm{Var}[P_a]}{\Delta_a^2} \tag{25}$$

$$\mathrm{Var}[E_b] \approx \left(\frac{\Delta_a + \Delta_b}{\Delta_b} \right)^2 \left(\frac{\overline{P}_a}{\Delta_a} \right)^2 \mathrm{Var}\left[\sum_{j=1}^{N_a} \sum_{k=1}^{N_b} \frac{(k_{jk}^{ab})^2}{m_a m_b Q_{jk}^{ab}} \right] \tag{26}$$

Here, for simplicity, the contribution of $\mathrm{Var}[P_a]$ to $\mathrm{Var}[E_b]$ has been neglected (cf. Eqs. (14) and (15)) as is usually the case in practice.

It is seen that, in general, the statistics of the energy of each subsystem are functions of the statistics of the input powers $P_{a,b}$ and those of the coupling coefficients $(k_{jk}^{ab})^2 \big/ Q_{jk}^{ab}$, which behave as a kind of "energy-transfer-function" at the interface between the subsystems. The statistics of the input powers depend on the number of oscillators in the frequency band Ω (which in turn depends on the modal density), the natural frequency spacing distributions and the spectral densities of the excitation. In particular, Poisson and Gaussian Orthogonal Ensemble (GOE) statistics can be used to model the natural frequency spacing distributions for various limiting cases (e.g. Refs. [2,11]). This issue will be further considered in Section 4.1. Once again, if only one set of oscillators is excited, the variance of the excited set depends primarily on that of the input power (Eq. (25)), while that of the unexcited set depends primarily on the variance of the coupling coefficients (Eq. (26)).

3.1. Statistics of the input power

The input power (Eq. (16)) depends on the number of oscillators in the band and on the spectral densities S_j^a. For coupled multimodal subsystems these are the number of modes in the band and the spectral densities of the modal excitations. The modal excitations depend of course on the spectral density of the physical forces applied to the subsystem and the mode shapes.

Assuming that these are uncorrelated, it follows that

$$\overline{P}_a = \frac{1}{\Omega}\frac{\pi}{m_a}E[N_a]E[S_j^a] = \frac{\pi n_a}{m_a}\bar{S}_j^a \tag{27}$$

$$E[P_a^2] = \left(\frac{1}{\Omega}\frac{\pi}{m_a}\right)^2 E[N_a^2]E[S_j^{a2}] \tag{28}$$

where it is noted that $E[N_{a,b}] = n_{a,b}\Omega$. For a single point, white, random force, with a spectral density S_{ff}, applied to a multimodal subsystem the modal excitations are such that

$$\bar{S}_j^a = S_{ff}E[\phi_j^{a2}], \quad E[S_j^{a2}] = S_{ff}^2 E[\phi_j^{a4}] \qquad (29),(30)$$

where $\phi_j{}^a$ is the mode shape at the excitation point of the jth mode of subsystem a. For spatially random rain-on-the-roof excitation (i.e., spatially uncorrelated excitation whose spectral density is proportional to the mass density,) with a spectral density S_{ff},

$$\overline{S}_j^a = S_{ff}, \quad E[S_j^{a2}] = S_{ff}^2 \qquad (31),(32)$$

3.2. Statistics of the coupling coefficients

From Eqs. (22), (25) and (26), it is seen that the mean and variance of the energies depend on the statistics of $k_{jk}{}^{ab}$, $Q_{jk}{}^{ab}$ and the number of modes $N_{a,b}$ in the frequency band Ω. To estimate the mean and variance, various simplifications and approximations are made. First, define

$$\Delta = (\Delta_j^a + \Delta_k^b)/2 \tag{33}$$

Combining Eq. (33) with Eqs. (19)–(21), and noting that $\omega_j{}^a + \omega_k{}^b \approx 2\omega$, yields

$$Q_{jk}^{ab} \approx 4\omega^2\left[(\omega_j^a - \omega_k^b)^2 + \left(\Delta^2 + \frac{(k_{jk}^{ab})^2}{m_a m_b \omega^2}\right)\right] \tag{34}$$

It then follows that

$$\sum_{j=1}^{N_a}\sum_{k=1}^{N_b}\frac{(k_{jk}^{ab})^2}{m_a m_b Q_{jk}^{ab}} \approx \frac{1}{4m_a m_b \omega^2}\sum_{j=1}^{N_a}\sum_{k=1}^{N_b}\frac{(k_{jk}^{ab})^2}{(\omega_j^a - \omega_k^b)^2 + (\Delta^2 + \kappa^2)} \tag{35}$$

where as an approximation, the terms $k_{jk}{}^{ab}$ in the denominator have been replaced by their expectation, Eq. (24). Eq. (35) indicates that the statistics of the coupling coefficients depend not only on the numbers of oscillators in the frequency band and the coupling stiffnesses but also on the natural frequencies and, in particular, the differences between them.

If correlation between $k_{jk}{}^{ab}$ and the natural frequencies is neglected, then the mean of the coupling coefficients (Eq. (35)) can be approximated as

$$E\left[\sum_{j=1}^{N_a}\sum_{k=1}^{N_b}\frac{(k_{jk}^{ab})^2}{m_a m_b Q_{jk}^{ab}}\right] = \frac{E[(k_{jk}^{ab})^2]}{4m_a m_b \omega^2}E\left[\sum_{j=1}^{N_a}\sum_{k=1}^{N_b}\frac{1}{(\omega_j^a-\omega_k^b)^2+(\Delta^2+\kappa^2)}\right] = \frac{\kappa^2}{4}E[N_a]E[N_b]\iint_{\Omega}\frac{p(\omega_j^a,\omega_k^b)\,\mathrm{d}\omega_j^a\,\mathrm{d}\omega_k^b}{(\omega_j^a-\omega_k^b)^2+(\Delta^2+\kappa^2)} \quad (36)$$

where $p(\omega_j{}^a,\omega_k{}^b)$ is the joint probability density function of $\omega_j{}^a$ and $\omega_k{}^b$. Given that $\omega_j{}^a$ and $\omega_k{}^b$ are natural frequencies of completely different subsystems, it is assumed that they are uncorrelated and uniformly distributed within the frequency band Ω. Consequently, the joint probability density function can be written as

$$p(\omega_j^a,\omega_k^b) = p(\omega_j^a)p(\omega_k^b) = \frac{1}{\Omega^2} \quad (37)$$

The double integral is evaluated by substituting $\delta\omega = (\omega_j{}^a - \omega_k{}^b)$ and integrating with respect to $\omega_k{}^b$ and then $\delta\omega$. If the damping is light enough, the limits of integration in the second integral can be replaced by $(-\infty, +\infty)$. Eq. (36) finally gives

$$E\left[\sum_{j=1}^{N_a}\sum_{k=1}^{N_b}\frac{(k_{jk}^{ab})^2}{m_a m_b Q_{jk}^{ab}}\right] \approx \frac{\pi n_a n_b \Omega \kappa^2}{4\sqrt{\Delta^2+\kappa^2}} \quad (38)$$

A similar approach can be used to estimate the variance of the coupling coefficients. The variance is, by definition,

$$\mathrm{Var}\left[\sum_{j=1}^{N_a}\sum_{k=1}^{N_b}\frac{(k_{jk}^{ab})^2}{m_a m_b Q_{jk}^{ab}}\right] = E\left[\left(\sum_{j=1}^{N_a}\sum_{k=1}^{N_b}\frac{(k_{jk}^{ab})^2}{m_a m_b Q_{jk}^{ab}}\right)^2\right] - E\left[\sum_{j=1}^{N_a}\sum_{k=1}^{N_b}\frac{(k_{jk}^{ab})^2}{m_a m_b Q_{jk}^{ab}}\right]^2 \quad (39)$$

Now,

$$E\left[\left(\sum_{j=1}^{N_a}\sum_{k=1}^{N_b}\frac{(k_{jk}^{ab})^2}{m_a m_b Q_{jk}^{ab}}\right)^2\right] = E\left[\sum_{j=1}^{N_a}\sum_{k=1}^{N_b}\sum_{m=1}^{N_a}\sum_{n=1}^{N_b}\frac{(k_{jk}^{ab})^2}{m_a m_b Q_{jk}^{ab}}\frac{(k_{jk}^{ab})^2}{m_a m_b Q_{mn}^{ab}}\right] \quad (40)$$

and the quadruple sum can be divided into two parts, in the first of which the indices j and m are equal, while for the second they are different, i.e.

$$E\left[\left(\sum_{j=1}^{N_a}\sum_{k=1}^{N_b}\frac{(k_{jk}^{ab})^2}{m_a m_b Q_{jk}^{ab}}\right)^2\right] = E\left[\sum_{j=1}^{N_a}\sum_{k=1}^{N_b}\left(\frac{k_{jk}^{ab2}}{m_a m_b Q_{jk}^{ab}}\right)^2\right] + E\left[\sum_{j=1}^{N_a}\sum_{k=1}^{N_b}\frac{k_{jk}^{ab2}}{m_a m_b Q_{jk}^{ab}}\sum_{\substack{m=1\\m\neq j}}^{N_a}\sum_{\substack{n=1\\n\neq k}}^{N_b}\frac{k_{jk}^{ab2}}{m_a m_b Q_{mn}^{ab}}\right] \quad (41)$$

Following lines similar to the above, and ignoring correlation between $\omega_j{}^a$ and $\omega_m{}^a$ ($m\neq j$), $\omega_k{}^b$ and $\omega_n{}^b$ ($n\neq k$) and $k_{jk}{}^{ab}$ and $k_{mn}{}^{ab}$ ($m\neq j$, $n\neq k$), leads to

$$\mathrm{Var}\left[\sum_{j=1}^{N_a}\sum_{k=1}^{N_b}\frac{(k_{jk}^{ab})^2}{m_a m_b Q_{jk}^{ab}}\right] = E[N_a]E[N_b]E\left[\left(\frac{(k_{jk}^{ab})^2}{m_a m_b Q_{jk}^{ab}}\right)^2\right] + (E[N_a^2]E[N_b^2] - E[N_a]^2E[N_b]^2 - E[N_a]E[N_b])E\left[\frac{(k_{jk}^{ab})^2}{m_a m_b Q_{jk}^{ab}}\right]^2 \quad (42)$$

This finally yields

$$\mathrm{Var}\left[\sum_{j=1}^{N_a}\sum_{k=1}^{N_b}\frac{(k_{jk}^{ab})^2}{m_a m_b Q_{jk}^{ab}}\right] = \frac{\pi n_a n_b \Omega \kappa^4 \sigma_{ab}}{4(2\sqrt{\Delta^2+\kappa^2})^3} \quad (43)$$

where

$$\sigma_{ab} = \frac{E[(k_{jk}^{ab})^4]}{E[(k_{jk}^{ab})^2]^2} + \frac{2\pi\sqrt{\Delta^2+\kappa^2}}{\Omega}\left(\frac{E[N_a^2]E[N_b^2] - E[N_a]^2E[N_b]^2}{E[N_a]E[N_b]} - 1\right) \tag{44}$$

is the coupling coefficient parameter. The variance of the coupling coefficients is thus caused by two effects: the first concerns the statistics of the coupling spring stiffnesses ${k_{jk}}^{ab}$ while the second depends on the number and variance of modes in the band. If the second term is much smaller than the first (which is often the case because, typically, $\Delta \ll \Omega$ and $\kappa^2 \ll \Delta^2$ for weak coupling) then

$$\sigma_{ab} \approx \frac{E[(k_{jk}^{ab})^4]}{E[(k_{jk}^{ab})^2]^2} \tag{45}$$

3.3. Energy means and variances

An expression for the ensemble mean of the energy can be obtained by substituting Eq. (38) into Eq. (22), to give

$$\overline{E}_a = \frac{\overline{P}_a}{\Delta_a} - \frac{\pi n_a n_b \Omega \kappa^2}{2\sqrt{\Delta^2+\kappa^2}}\left(\frac{\overline{P}_a}{\Delta_a} - \frac{\overline{P}_b}{\Delta_b}\right) \tag{46}$$

The variances of the energies, with $P_b = 0$, from Eqs. (25), (26), (38) and (43), can be obtained as

$$\mathrm{Var}[E_a] \approx \frac{\mathrm{Var}[P_a]}{\Delta_a^2} \tag{47}$$

$$\mathrm{Var}[E_b] \approx \frac{\pi n_a n_b \Omega \kappa^4 \sigma_{ab}}{\left(2\sqrt{\Delta^2+\kappa^2}\right)^3}\left(\frac{\overline{P}_a}{\Delta_a}\right)^2 \tag{48}$$

The mean and variance of the input power are given by Eqs. (27) and (28). The variance of the input power depends on the expected number of modes of the excited subsystem within the band and the spectral densities of the excitations. Note that in Eq. (47) the variance associated with the coupling parameter σ_{ab} has been neglected, while in Eq. (48) the variance associated with the input power has been neglected, for the reasons given in deriving the approximations of Eqs. (25) and (26).

In summary, the variance of the energy of the driven subsystem (Eq. (47)) depends primarily on the variance of the input power, which in turn depends on the number of modes of the driven subsystem and the excitation spectral densities (Eq. (28)), while the variance of the energy of the undriven subsystem (Eq. (48)) depends primarily on the variance of the coupling parameter which in turn depends on the variance of the coupling stiffnesses and the number of modes in each subsystem. It is worth noting that Eqs. (46)–(48) are derived without specific subsystem natural frequency spacing statistics having been assumed. Also the two subsystems are allowed to possess different natural frequency statistics.

3.4. Two coupled continuous subsystems

A continuous subsystem can be regarded as a set of oscillators, each oscillator corresponding to one of the modes of the subsystem. For two subsystems joined by a spring of stiffness K, the coupling stiffness between the jth and kth modes of subsystems a and b is [1]

$$k_{jk}^{ab} = K\phi_j^a(x_I^a)\phi_k^b(x_I^b) \tag{49}$$

Here ${\phi_j}^a({x_I}^a)$ and ${\phi_k}^b({x_I}^b)$ are the jth and kth mass-normalized mode shapes of subsystems a and b at the interface locations ${x_I}^a$ and ${x_I}^b$. If ${x_I}^a$ and ${x_I}^b$ are assumed to be chosen randomly and are statistically independent then, ignoring the second term in Eq. (44), the coupling parameter σ_{ab} becomes

$$\sigma_{ab} = \frac{E[\phi_a^4(x_I^a)]}{E[\phi_a^2(x_I^a)]^2}\frac{E[\phi_b^4(x_I^b)]}{E[\phi_b^2(x_I^b)]^2} \tag{50}$$

If only subsystem a is excited, the relative variance of the energy of subsystem b, from Eqs. (46) and (48), is

$$r_{E_b}^2 = \frac{\mathrm{Var}[E_b]}{E[E_b]^2} \approx \frac{1}{\Omega}\frac{1}{\pi n_a n_b\left(2\sqrt{\Delta^2+\kappa^2}\right)}\sigma_{ab} \tag{51}$$

which by ignoring the second term in Eq. (44), becomes

$$r_{E_b}^2 \approx \frac{1}{\Omega}\frac{1}{\pi n_a n_b\left(2\sqrt{\Delta^2+\kappa^2}\right)}\frac{E[\phi_a^4(x_I^a)]}{E[\phi_a^2(x_I^a)]^2}\frac{E[\phi_b^4(x_I^b)]}{E[\phi_b^2(x_I^b)]^2} \tag{52}$$

In Ref. [2] (Eq. (4.2.14)), it is shown that, if subsystems a and b are joined at a single point and subsystem a is excited by a point force applied at a random point $x_s{}^a$, then the relative variance of the mean square velocity response of the receiving subsystem b at a randomly chosen observation point $x_r{}^b$ is

$$r_{V_b^2}^2 \approx \frac{1}{\Omega}\frac{2}{\pi n_a n_b(\Delta_a+\Delta_b)} \times \frac{E[\phi_a^4(x_r^b)]}{E[\phi_a^2(x_r^b)]^2}\frac{E[\phi_b^4(x_s^a)]}{E[\phi_b^2(x_s^a)]^2}\frac{E[\phi_a^4(x_I^a)]}{E[\phi_a^2(x_I^a)]^2}\frac{E[\phi_b^4(x_I^b)]}{E[\phi_b^2(x_I^b)]^2} \tag{53}$$

$x_I{}^a$ and $x_I{}^b$ are the coupling points on subsystems a and b. In deriving Eq. (53), both subsystems were assumed to conform to Poisson natural frequency spacing statistics. If the source subsystem a is subjected to rain-on-the-roof forcing, the space-averaged mean square response of the receiving subsystem b becomes

$$r_{V_b^2}^2 \approx \frac{1}{\Omega}\frac{2}{\pi n_a n_b(\Delta_a+\Delta_b)}\frac{E[\phi_a^4(x_I^a)]}{E[\phi_a^2(x_I^a)]^2}\frac{E[\phi_b^4(x_I^b)]}{E[\phi_b^2(x_I^b)]^2} \tag{54}$$

It is seen that Eq. (52) reduces to Eq. (54) for the case of weak coupling, for which $\Delta^2\kappa^2$, apart from a factor of 2.

In Ref. [12], the ensemble variances of the energies of built-up SEA subsystems were estimated under the assumption of GOE natural frequency spacing statistics, the main results being summarized in Appendix A. In the case of a single point coupling, the relative variance of the energy of the receiver subsystem is given by

$$r_{E_b}^2 \approx \frac{1}{\Omega n_b}\left(\alpha\frac{E[\phi_b^4]}{E[\phi_b^2]^2} - 1\right) \tag{55}$$

where $\alpha = 2$ if no average is taken over the coupling locations before the ensemble average is performed, and $\alpha = 1$ if an average is taken over many coupling locations before ensemble averaging.

It is seen that Eqs. (52) and (55) are dissimilar in form. This is mainly because, in deriving Eq. (45) and hence Eq. (52), it is assumed that the contribution of out-of-band modes can be neglected so that the frequency range of integration for those modes which lie inside Ω can be extended to $(0, \infty)$, while Eq. (55) applies when the system has a modal overlap larger than 0.6 [11]. These issues will be further discussed in Section 4.4.

4. Poisson and GOE natural frequency spacing statistics

From Eqs. (46)–(48), it is seen that the statistics of the energies depend on the statistics of the input powers (Eqs. (27) and (28)) and those of the coupling parameters (Eqs. (24) and (45)), which in turn depend on the statistics of the number of modes in the frequency band Ω and the statistics of the mode shapes of the subsystems. In this section, the current approach is applied to coupled subsystems whose natural frequency spacings are assumed to conform to two common forms for dynamic systems, namely Poisson [2] and GOE [17] natural frequency spacing statistics. Poisson statistics are typical of separable geometries [2], while GOE statistics are believed to be asymptotically realistic for systems with irregularity [17]. Other forms of natural frequency spacing statistics, e.g. semi-Poisson statistics [18], are not discussed here. Spatially random, rain-on-the-roof excitation, as often encountered in SEA, is considered, for simplicity. For other forms of excitation, Eqs. (46)–(48) are directly applicable, though.

4.1. Definitions of Poisson and GOE statistical distributions

For Poisson distribution, the probability density function of the natural frequency spacings is [16]

$$p(\delta_\omega) = n\mathrm{e}^{-n\delta_\omega}, \quad \delta_\omega > 0 \tag{56}$$

where n is the modal density.

For a dynamic system with Poisson natural frequency spacing statistics, the ensemble mean and variance of the number of modes lying inside the frequency band Ω are given by [16]

$$E[N] = n\Omega \tag{57}$$

$$\mathrm{Var}[N] = n\Omega \tag{58}$$

Under the GOE assumption, the spacing of successive eigenvalues has a probability density function [15]

$$p(\delta_\omega) = \left(\frac{\pi}{2}\delta_\omega\right)\mathrm{e}^{-((\pi/2)\delta_\omega)^2} \tag{59}$$

For a dynamic system with GOE natural frequency spacing statistics, $E[N]$ remains the same as that given in Eq. (57), while the ensemble variance becomes [15]

$$\mathrm{Var}[N] = 2\ln(n\Omega)/\pi^2 + 0.44 \tag{60}$$

4.2. Random subsystems with Poisson natural frequency spacing statistics

The ensemble mean and variance of the input power, with subsystem a subjected to rain-on-the-roof excitation S_f^a, from Eqs. (27), (28), (57) and (58), are:

$$\overline{P}_a = \frac{\pi S_f^a}{m_a} n_a \tag{61}$$

$$\mathrm{Var}[P_a] = \frac{n_a}{\Omega}\left(\frac{\pi S_f^a}{m_a}\right)^2 \tag{62}$$

Combining Eqs. (61) with Eq. (46), the mean energy of subsystem a is given by

$$\overline{E}_a = \frac{\pi S_f^a}{m_a \Delta_a} n_a - \frac{\pi n_a n_b \kappa^2}{2\sqrt{\Delta^2 + \kappa^2}}\left(\frac{\pi S_f^a}{m_a \Delta_a} - \frac{\pi S_f^b}{m_b \Delta_b}\right) \tag{63}$$

The energy variance for the driven subsystem a, from Eqs. (47) and (62), is

$$\mathrm{Var}[E_a] \approx \frac{n_a}{\Omega}\left(\frac{\pi S_f^a}{m_a \Delta_a}\right)^2 \tag{64}$$

while the energy variance of the undriven subsystem b, from Eq. (48) and (62), is

$$\mathrm{Var}[E_b] \approx \frac{\pi n_a n_b \Omega \kappa^4 \sigma_{ab}}{\left(2\sqrt{\Delta^2 + \kappa^2}\right)^3}\left(\frac{\pi S_f^a}{m_a \Delta_a}\right)^2 \tag{65}$$

In the above equations, the coupling parameters κ^2 and σ_{ab} are defined in Eqs. (24) and (45), respectively, and depend on the statistics of k_{jk}^{ab}. Generally, k_{jk}^{ab} are functions of the subsystem mode shapes at the interfaces and the coupling stiffness of the interfaces. For example, for two subsystems coupled by N_I discrete springs, ${k_{jk}}^{ab}$ between the jth and the kth modes of subsystems a and b is given by [19]

$$k_{jk}^{ab} = \sum_{n_I=1,N_I} \phi_j^a(x_{n_I}^a) K_{n_I} \phi_k^b(x_{n_I}^b) \tag{66}$$

where K_{nI} is the stiffness of the n_Ith spring. Substituting Eq. (66) into Eqs. (24) and (45), respectively, yields

$$\kappa^2 \approx \frac{1}{M_a M_b \omega_c^2} \left(\sum_{n_I=1}^{N_I} K_{n_I}^2 \right) \tag{67}$$

$$\sigma_{ab} \approx 3 + \frac{1}{N_I} \left(\frac{E[K_{n_I}^4]}{E[K_{n_I}^2]^2} \frac{E[\phi_a^4]}{E[\phi_a^2]^2} \frac{E[\phi_b^4]}{E[\phi_b^2]^2} - 3 \right) \tag{68}$$

Clearly σ_{ab} depends on the modal shape statistics $E[\phi^4]/E[\phi^4]^2$ of each subsystem. For example, $E[\phi^4]/E[\phi^4]^2 = 2.25$ for a two-dimensional dynamic system with sinusoidal mode shapes [2]. Meanwhile, Eq. (68) indicates that, for many coupling points, i.e., large N_I, σ_{ab} is asymptotic to 3 regardless of the exact subsystem modal shape statistics, which in turn implies that the variance of the power exchanged between the two coupled subsystems tends to be independent of the exact mode shape statistics of each subsystem.

4.3. Subsystems with GOE natural frequency spacing statistics

Combining Eqs. (57) and (60) with (27) and (28), gives

$$\overline{P}_a = \frac{\pi S_f^a}{m_a} n_a \tag{69}$$

$$\text{Var}[P_a] = \frac{1}{\Omega^2 \pi^2} [2 \ln(n_a \Omega) + 0.44\pi^2] \left(\frac{\pi S_f^a}{m_a} \right)^2 \tag{70}$$

Comparing Eqs. (61) and (62) with (69) and (70), it is seen that the mean of the power is independent of the subsystem natural frequency statistics, while the variance of the power depends on the subsystem natural frequency statistics. The variance based on the assumption of GOE natural frequency statistics [17] is much smaller than that from the assumption of Poisson statistics.

Substituting Eqs (69) and (70) into (46)–(48), leads to

$$\overline{E}_a = \frac{\pi S_f^a}{m_a \Delta_a} n_a - \frac{\pi n_a n_b \kappa^2}{2\sqrt{\Delta^2 + \kappa^2}} \left(\frac{\pi S_f^a}{m_a \Delta_a} - \frac{\pi S_f^b}{m_b \Delta_b} \right) \tag{71}$$

$$\text{Var}[E_a] \approx \frac{1}{\Omega^2} \left[\frac{2 \ln(n_a \Omega)}{\pi^2} + 0.44 \right] \left(\frac{\pi S_f^a}{m_a \Delta_a} \right)^2 \tag{72}$$

$$\text{Var}[E_b] \approx \frac{\pi n_a n_b \Omega \kappa^4 \sigma_{ab}}{\left(2\sqrt{\Delta^2 + \kappa^2} \right)^3} \left(\frac{\pi S_f^a}{m_a \Delta_a} \right)^2 \tag{73}$$

Note that, κ^2 and σ_{ab} are the same expressions as those given in Eqs. (67) and (68) for the case of multiple spring couplings, apart from the fact that $E[\phi^4]/E[\phi^2]^2$ in Eq. (68) equals 2.75 for a two-dimensional system with GOE mode shape statistics [17]. ($E[\phi^4]/E[\phi^2]^2 \approx 2.25$ for a two-dimensional system with Poisson statistics.) This suggests that the variance of the energy exchanged between the coupled subsystems predicted from Poisson statistics and that from GOE statistics may differ only slightly as the number of coupling springs increases.

4.4. Estimate of σ_{ab} for subsystems with large modal bandwidths

Strictly speaking, σ_{ab} in Eq. (45) is derived by assuming each subsystem mode to have a small modal bandwidth and that Ω is large enough so that the response can be well approximated by only considering the modes whose natural frequencies lie within the frequency band Ω. However, if this is not the case, then out-of-band modes might contribute significantly. In this case, the estimate of σ_{ab} given in Eq. (45) will unavoidably cause certain "cut-off" errors in that the effect of the out-of-band modes has been neglected, which in turn can cause the energy variance to be underestimated. This is the cause of the differences between Eqs. (52) and (55) in Section 3.4.

In this situation, therefore, σ_{ab} should be modified for large modal overlap (e.g. $n\Delta > 1$) and small Ω cases instead of using the direct calculation of Eq. (45). This is achieved using the results of other established theories [2,11–12]. Details are given below.

From the coupled oscillator theory [1], the coupling loss factor η_{ab} [2] is given by

$$\omega n_a \eta_{ab} \approx \frac{\pi \kappa^2 n_a n_b}{2\sqrt{\Delta^2 + \kappa^2}} \Delta \tag{74}$$

The relative variance of η_{ab}, by the current theory, can be found from the previous expressions for energies and input powers, and is given by

$$r^2_{\eta_{ab}} \approx \frac{1}{\Omega} \frac{\sigma_{ab}}{\pi n_a n_b \left(2\sqrt{\Delta^2 + \kappa^2}\right)} \tag{75}$$

Under the assumptions of Poisson and GOE natural frequency spacing statistics [17], the relative variances of η_{ab}, by Lyon and DeJong [2] (Eq. (12.3.7)) and Langley and Cotoni [11] (Eq. (39)), respectively, are:

$$r^2_{\eta_{ab}} \approx \frac{1}{\pi(n_a\Delta_a + n_b\Delta_b) + \Omega(n_a + n_b)} \frac{E[\phi_a^4]}{E[\phi_a^2]^2} \frac{E[\phi_b^4]}{E[\phi_b^2]^2} \tag{76}$$

$$r^2_{\eta_{ab}} \approx \frac{(\alpha_{ab} - 1)}{\pi n_a \Omega} \left\{ \pi - 2\tan^{-1}\left(\frac{1}{B_a}\right) - \frac{\ln(1 + B_a^2)}{B_a} \right\} + \frac{\ln(1 + B_a^2)}{(\pi n_a \Omega)^2} \tag{77}$$

Both Eqs. (76) and (77) assume high modal overlap (e.g. $\pi n \gg 1$). In Eq. (77), $B_a = \Omega/\Delta_a$, and α_{ab} is a parameter which depends on the mode shape statistics of the two subsystems and the nature of the coupling. A compendium of values α_{ab} has been derived for a wide range of subsystem couplings and is given in Ref. [12]. For example, if there are N_I distinct connection points then

$$\alpha_{ab} \approx 2 + \frac{2}{N_I} \left(\frac{E[\phi_b^4]}{E[\phi_b^2]^2} - 1 \right) \tag{78}$$

In-line with the results given in Eqs. (75)–(77), σ_{ab}, under the assumptions of Poisson and GOE natural frequency spacing statistics, can be estimated, respectively, as

$$\sigma_{ab} \approx \pi n \Delta \frac{E[\phi_a^4]}{E[\phi_a^2]^2} \frac{E[\phi_b^4]}{E[\phi_b^2]^2} \quad (\pi n \Delta > 1, \ \text{Poisson}) \tag{79}$$

$$\sigma_{ab} \approx \pi n \Delta [2(\alpha_{ab} - 1)] \quad (\pi n \Delta > 1, \ \text{GOE}) \tag{80}$$

where $n = (n_a + n_b)/2$.

5. Numerical examples

In this section, examples of two plates coupled either by one or two springs, as shown in Fig. 3, are considered. The driven plate subsystem is assumed to have Poisson natural frequency statistics while the undriven plate has GOE statistics. The ensemble mean and variance of energy of each plate are predicted by the present theory and compared with results from Monte Carlo simulations.

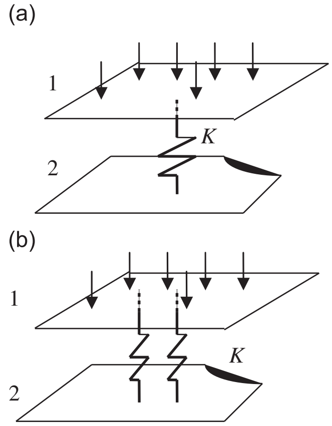

Fig. 3. Two spring-coupled plates: (a) by a single spring; (b) by two springs.

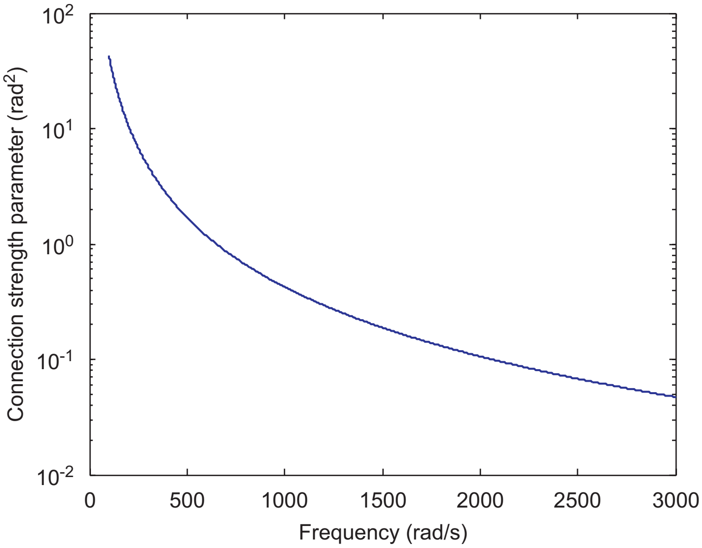

Fig. 4. The connection strength parameter κ^2 (Eq. (24)) for both single- and two-spring coupling models.

5.1. Description of the systems

In Fig. 3, plate 1 is rectangular with simply supported edges and an area of $0.32\,\mathrm{m}^2$, and plate 2 is a rectangular plate with an area of $0.23\,\mathrm{m}^2$ but with a curved region of area $0.02\,\mathrm{m}^2$ cut off at one corner, equivalent in size to a hole of diameter 0.16 m.

The material is perspex with Young's modulus $10^8\,\mathrm{N/m}^2$, density $10^3\,\mathrm{kg/m}^3$, Poisson ratio 0.38 and material damping loss factor 0.01. The modal densities of plates 1 and 2 are 0.08 and 0.06 modes/rad/s, respectively. The frequency range considered is up to 3000 rad/s, which contains approximately 240 and 180 modes of plates 1 and 2, respectively, and gives a modal overlap factor up to 2.4 for plate 1 and up to 1.8 for plate 2. The spring stiffness is 500 N/m in Fig. 3a while in Fig. 3b the stiffness of each spring is 353.6 N/m. (In this case the coupling strengths are the same for Figs. 3a and b.) The corresponding strength of connection and strength of coupling parameters κ^2 (Eq. (24)) and γ^2 (Eq. (23)) are shown in Figs. 4 and 5, respectively. Both decrease as frequency increases. In Fig. 5 it is seen that $\gamma^2 \approx 0.1$ at about 500 rad/s, so that the plates can be regarded as being weakly coupled for frequencies above 500 rad/s.

The systems are randomized by perturbating the geometry of each subsystem, i.e. by varying the dimensions of the plate while keeping the plate area, and hence the modal density remains constant. For plate 1, the aspect

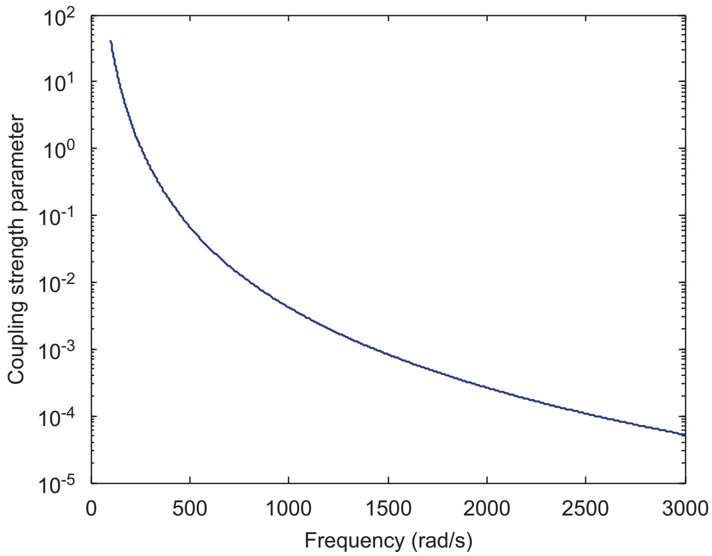

Fig. 5. The coupling strength parameter γ^2 (Eq. (23)) for both single- and two-spring coupling models.

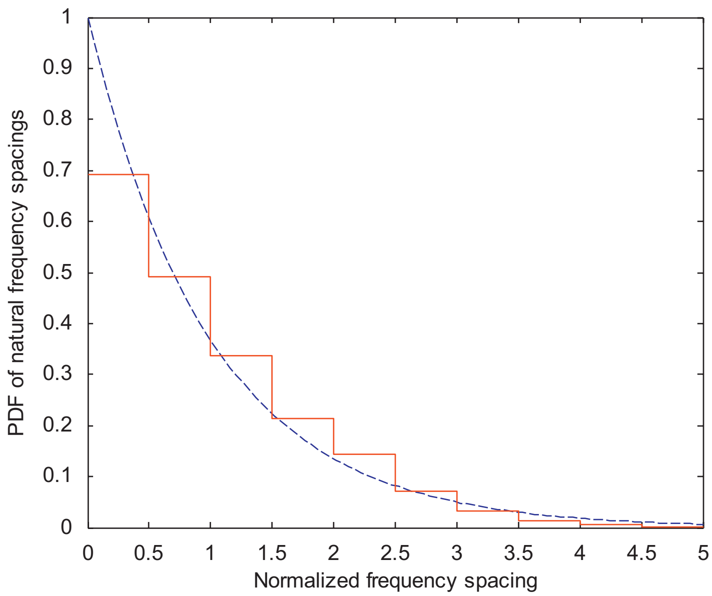

Fig. 6. Probability density function (PDF) of the natural Frequency spacing distribution of plate 1: ——, simulation; - - - -, Poisson distribution.

ratio (length/width) is varied with a uniform probability density function between 0.8 and 1.2, and, for plate 2, the curved cut-off is randomly chosen over the plate surface. Thirty samples are generated for each plate ensemble, giving an ensemble of 900 coupled systems. It is known [19] that integrable dynamics generally leads to Poisson statistics while completely chaotic dynamics to GOE. Consequently the natural frequency spacing statistics of plate 1 are expected to conform to Poisson statistics while those of plate 2 ensemble approximately to GOE. The natural frequency spacing statistics of plates 1 and 2 (normalized by the mean natural frequency spacing of the corresponding plate) are shown in Figs. 6 and 7, which are not inconsistent with these assumptions.

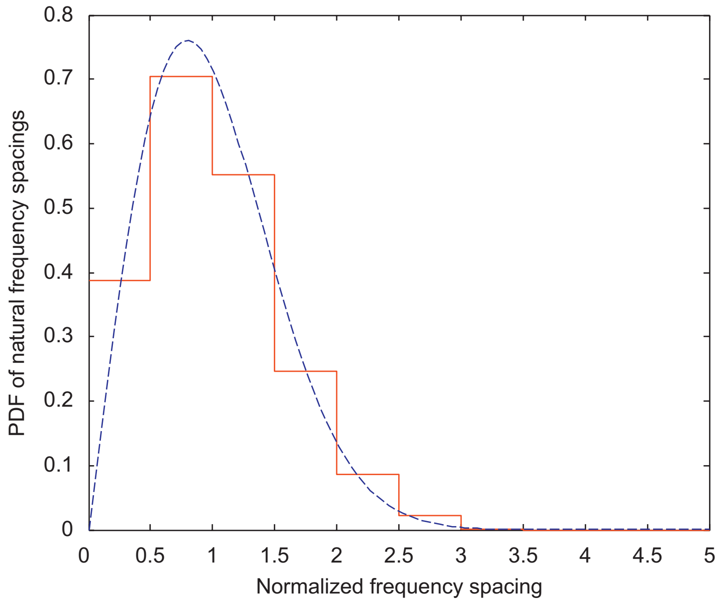

Fig. 7. Probability density function (PDF) of natural frequency spacing distribution of plate 2: ——, simulation; - - - -, Rayleigh distribution.

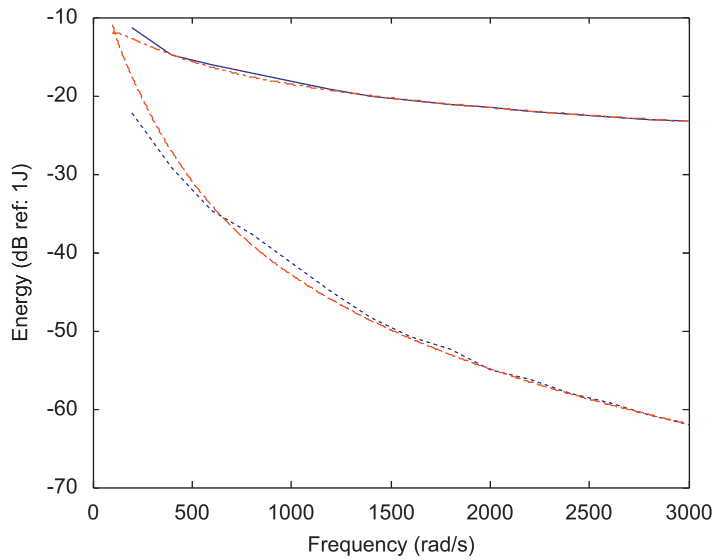

Fig. 8. Ensemble mean of energy for Case 1 with a single-spring coupling. Plate 1: ——, Monte-Carlo simulation; - · - · -, theoretical prediction; plate 2: · · · · · · · · · · · ·, Monte-Carlo simulation; - - - -, theoretical prediction.

Plate 1 is assumed to be subjected to rain-on-the-roof forcing of unit power spectral density over a frequency bandwidth $\Omega = 0.1\omega_c$, where ω_c is the centre frequency of the band Ω.

5.2. A single spring coupling

Fig. 8 shows the mean energies of the two plate ensembles predicted by the present theory and by the Monte Carlo simulations. It is seen that the two sets of results agree very well for the weak coupling region (above 500 rad/s).

Fig. 9 shows the relative variances predicted by the present theory and the Monte Carlo simulations. It is seen, as expected, that, for plate 1, the Poisson-based prediction (Eq. (64)) agrees significantly better with the Monte Carlo solution than the GOE-based one (Eq. (72)), while for plate 2, the opposite is true. This suggests that the variance of the energy of each subsystem is largely dominated by the properties of that subsystem. The relative variance of the receiver plate is larger than that of the source plate. Note that, for the source plate, the relative variance for Poisson statistics is much larger than that for GOE statistics.

Figs. 10 and 11 show the ensemble mean and variance of energy of the plates when the system damping loss factor is 0.001. (The maximum modal overlap factors of both plates are less than 0.2.) Weak coupling occurs

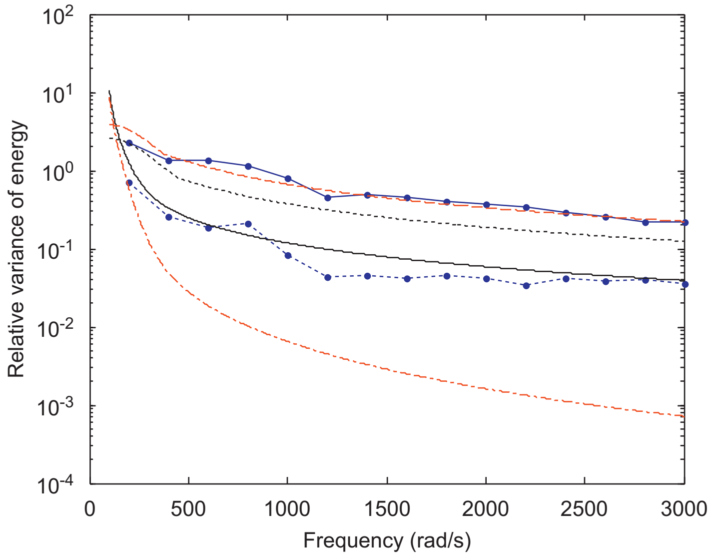

Fig. 9. Relative variance of energy for Case 1 with a single-spring coupling. Plate 1: ·····•·····, Monte-Carlo simulation; ——, Poisson-based prediction; - · - · -, GOE-based prediction. Plate 2: ——•——, Monte-Carlo simulation; · · · · · · · · · · · ·, Poisson-based prediction; - - - -, GOE-based prediction.

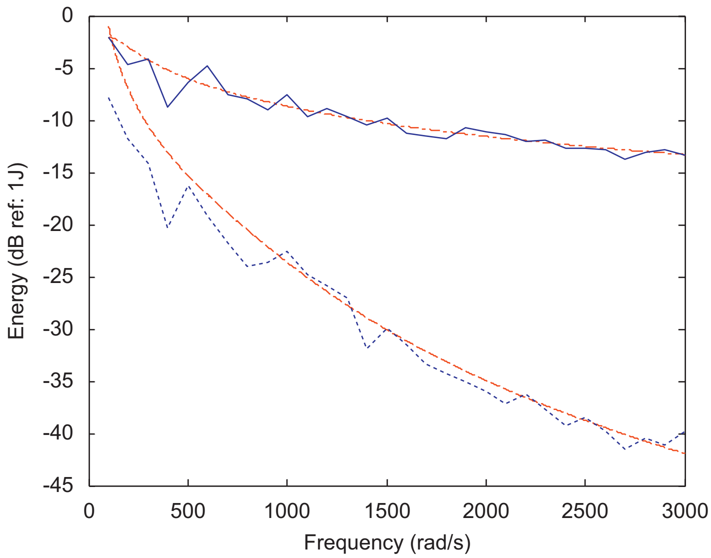

Fig. 10. Ensemble mean of energy for Case 1 with a low damping $\eta = 0.001$. Plate 1: ——, Monte-Carlo simulation; - · - · -, theoretical prediction. Plate 2: · · · · · · · · · · · ·, Monte-Carlo simulation; - - - -, theoretical prediction.

for frequencies above 1500 rad/s. The present theory provides reasonably accurate predictions for the ensemble mean energy even for the frequencies below 1500 rad/s. For the variance of the energy, however, the accuracy is less good unless the coupling is weak. In Fig. 11 the variances are predicted based on the estimate of σ_{ab} from Eq. (68) rather than that from Eqs. (79) and (80). For the receiver plate, the variance is dominated by the coupling coefficient σ_{ab}, while for the source plate, its variance is mainly affected by the excitation type and the statistics of source plate itself but relatively much less affected by the couplings with other subsystems.

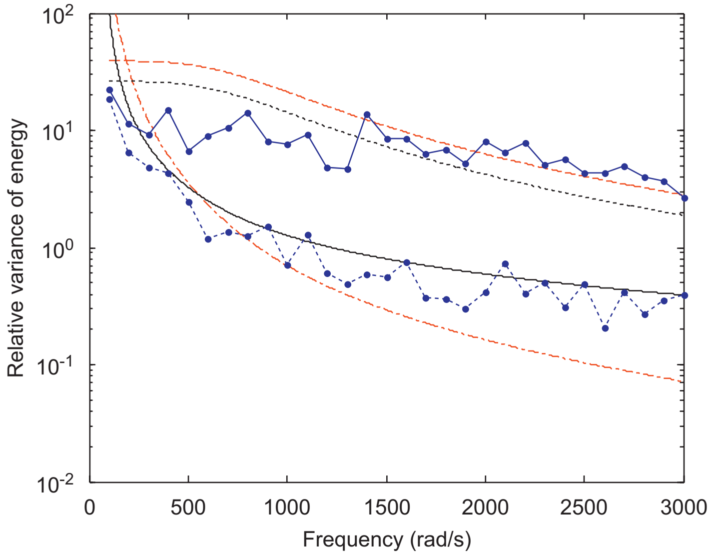

Fig. 11. Relative varaince of energy for Case 1 with a low damping $\eta = 0.001$. Plate 1: ·····●·····, Monte-Carlo simulation; ——, Poisson-based prediction; - · - · -, GOE-based prediction. Plate 2: ——●——, Monte-Carlo simulation; · · · · · · · · · · · ·, Poisson-based prediction; - - - -, GOE-based prediction.

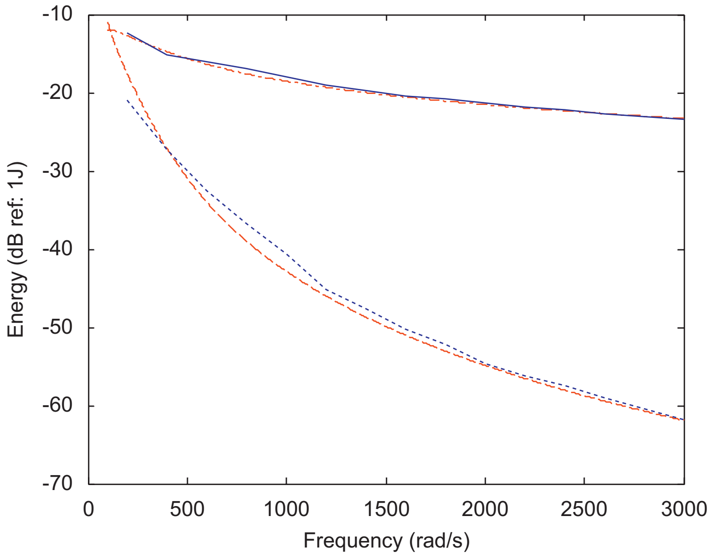

Fig. 12. Ensemble mean of energy for Case 2 with two coupling springs. Plate 1: ——, Monte-Carlo simulation; - · - · -, theoretical prediction. Plate 2: · · · · · · · · · · · ·, Monte-Carlo simulation; - - - -, theoretical prediction.

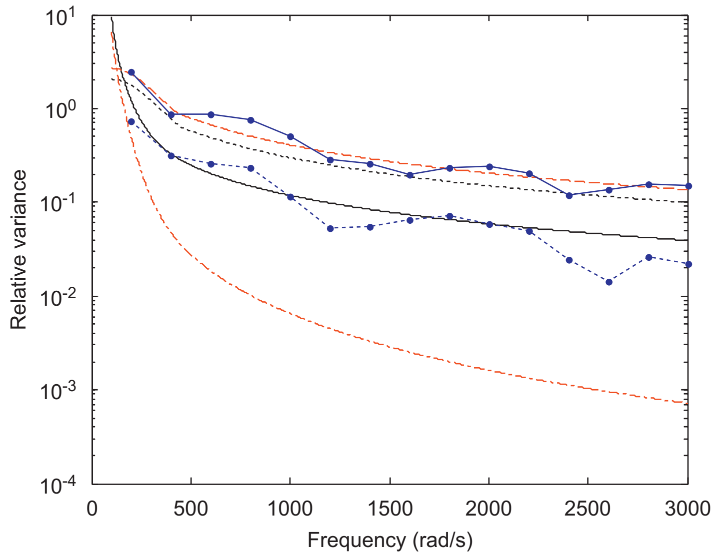

Fig. 13. Relative variance of energy for Case 2 with two coupling springs. Plate 1: ······•·····, Monte-Carlo simulation; ——, Poisson-based prediction; - · - · -, GOE-based prediction. Plate 2: —•—, Monte-Carlo simulation; · · · · · · · · · · · ·, Poisson-based prediction; - - - -, GOE-based prediction.

5.3. *Multiple spring connections*

From Eq. (67), the connecting strength for two subsystems connected by N_I springs is equivalent to that when the two subsystems are connected by a single spring with stiffness $K_{\text{equi}} = \sqrt{\sum_{n_I=1,N_I} K_{n_I}^2}$. Thus the connection strength for Fig. 3b is equal to that for Fig. 3a.

Figs. 12 and 13 show the ensemble mean and variance of the plate energies for the multi-coupling case. Comparing Fig. 12 with Fig. 8, it is seen that the ensemble mean energies are almost the same because the connection strengths for the two cases are equal. Comparing Fig. 13 with Fig. 9, however, it is seen that the relative variances in Fig. 13 are smaller than those in Fig. 9 because there are more coupling points.

6. Concluding remarks

In this paper, expressions were derived for the ensemble means and variances of the subsystem energies of systems comprising two subsystems. The approach is based on the Statistical Energy Analysis of two spring-coupled oscillators and sets of oscillators, or coupled continuous subsystems, described in Ref. [1]. Randomness was introduced into the system by assuming that the natural frequency spacings in each subsystem conform to certain statistical distributions. A coupling coefficient parameter (Eq. (45)) was introduced which, together with the coupling strength parameter (Eq. (24)) defined in Ref. [1], accounts for the statistics of the coupling stiffness. Various approximations and assumptions were made, in particular, for the estimation of the coupling coefficient parameter in Section 3.2.

The theory shows that the variance of the excited subsystem depends primarily on the variance of the input power, which in turn depends on the variance of the number of modes of the excited subsystem in the frequency band of excitation and their mode shapes. The variance of the undriven subsystem, on the other hand, depends primarily on the variance of the intermodal coupling coefficients, which in turn depend on the variances of the number of in-band modes of both subsystems and their mode shapes. The system may have a very low damping (and hence a low modal overlap), provided the weak coupling condition in Eq. (23) is met. Numerical examples were presented.

One of the main assumption of the presented theory was that, for a given bandwidth of excitation Ω, only those modes whose natural frequencies lie within Ω contribute to the response, and their contribution can be estimated by extending Ω to the range of infinity. In this case, the resulting equations become relatively very simple, which helps to shed insight into the underlying behaviour. In principle, however, out-of-band modes and the fact that Ω has a finite bandwidth could both be involved, although the results become substantially more complicated.

Finally, the analysis can be extended to cases of arbitrary interface coupling, and results will be reported elsewhere.

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Appendix

In this appendix some results from two major existing variance predicting theories in Refs. [2,12] are briefly reviewed.

For a built-up system, the power–energy relation, when frequency-averaged, is

$$\mathbf{P} = \mathbf{D}\mathbf{E} \tag{A.1}$$

where **P** and **E** are vectors of frequency-averaged input power, and subsystem energy, respectively, and **D** is a matrix of damping and coupling loss factors.

In Ref. [2], under the assumption of Poisson natural frequency statistics, the relative variance of the input power for rain-on-the-roof forcing, is ([2], Eq. (12.3.5))

$$r^2_{P_k} \approx \frac{1}{\pi n_k \Delta_k + n_k \Omega} \tag{A.2}$$

And the relative variance of the coupling loss factor entry is (Ref. [2], Eq. (12.3.7))

$$r^2_{\eta_{,ks}} \approx \left[\frac{1}{\pi(n_k\Delta_k + n_s\Delta_s) + \Omega(n_k + n_s)}\right] \frac{E[\phi_k^4(x_I^k)]}{E[\phi_k^2(x_I^k)]^2} \frac{E[\phi_s^4(x_I^s)]}{E[\phi_s^2(x_I^s)]^2} \tag{A.3}$$

Under the assumption of GOE natural frequency statistics, the relative variance of the input power for rain-on-the-roof forcing, is ([12], Eq. (A.2))

$$r^2_{P_k} \approx \frac{1}{(\Omega n_k)^2}\left[\frac{2\ln(\Omega n_k)}{\pi^2} + 0.44\right] \tag{A.4}$$

The relative variance of the coupling loss factor is ([12], Eq. (39))

$$r^2_{\eta_{ks}} \approx \frac{1}{\Omega}\frac{(\alpha_{ks} - 1)}{\pi n_k}\left\{\pi - 2\tan^{-1}\left(\frac{1}{B_k}\right) - \frac{\ln(1 + B_k)}{B_k}\right\} + \frac{1}{(\pi n_k \Omega)^2}\ln(1 + B_k^2) \tag{A.5}$$

All the symbols in Eqs. (A.2)–(A.5) have the same meaning as in the previous sections of this paper, except that $\Delta_{k,s}$ correspond to the effective in-situ loss factors of the relevant subsystems [2,12].

It is worth noting that Eq. (A.3) is based on a heuristic modification for the effect of band averaging under the assumption of Poisson natural frequency spacing statistics [2]. This contrasts to Eq. (A.5), which is developed rigorously under the assumption of GOE natural frequency spacing statistics [12].

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