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Analyses of oscillators with non-polynomial damping terms

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Abstract

In this paper the properties of the oscillatory motion of the system with non-polynomial damping is investigated. The two limits for damping are the dry friction and linear viscous damping. The mathematical model of the system is a strong nonlinear differential equation with fraction order velocity terms. Using the modified version of He's homotopy perturbation method the approximate analytic solution is obtained. The generating solution is assumed in the form which corresponds to the system with linear viscous damping. Two special cases are considered: first, when the coefficient of the damping force is small and second, when the damping force is close to dry friction. The obtained analytical solutions are compared with numerical ones. They show good agreement.

1. Introduction

In most textbooks, the vibrations of two typical damping systems are considered: the harmonic oscillator with linear viscous damping

$$\ddot{x} + (2\delta)\dot{x} + \omega^2 x = 0 \tag{1}$$

and the harmonic oscillator with dry friction, i.e., with zeroth order damping

$$\ddot{x} + (2\delta)\mathrm{sgn}(\dot{x})\dot{x}^0 + \omega^2 x = 0, \tag{2}$$

with the initial conditions

$$x(0) = A, \quad \dot{x}(0) = 0, \tag{3}$$

where

$$\mathrm{sgn}(\dot{x}) = \begin{bmatrix} -1 \text{ for } \dot{x} < 0 \\ 0 \text{ for } \dot{x} = 0 \\ 1 \text{ for } \dot{x} > 0 \end{bmatrix}, \tag{4}$$

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x is the displacement, t is time, δ is the damping coefficient, $\omega > 0$ is the eigenfrequency, $A > 0$ is the initial displacement and $\dot{x} \equiv dx/dt$, $\ddot{x} \equiv d^2x/dt^2$. The term $\mathrm{sgn}(\dot{x})$ takes into consideration the change of direction of the friction force. Namely, the damping force has the opposite direction to the motion and changes sign with velocity: for $\dot{x} > 0$ the sign is positive, and for $\dot{x} < 0$ the sign is negative. The change of sign is for $\dot{x} = 0$. It means that the motion has to be divided into intervals bounded with the condition that the velocity is zero.

For the case when $\omega > \delta$ the solution of Eq. (1) is oscillatory

$$x = A\exp(-\delta t)\left(\cos(\kappa t) + \frac{\delta}{\kappa}\sin(\kappa t)\right), \tag{5}$$

where

$$\kappa = \sqrt{\omega^2 - \delta^2}. \tag{6}$$

The solution of Eq. (2) for the initial conditions (3) has the form

$$x = (-1)^{n-1}\left(\frac{2\delta}{\omega^2}\right) + \left[A - (2n-1)\left(\frac{2\delta}{\omega^2}\right)\right]\cos\omega t, \tag{7}$$

where $n = 1, 2, \ldots$ represents the number of motion periods between two zero velocities ($\dot{x} = 0$): $n = 1, 3, 5\ldots$ for from right to left when $\dot{x} < 0$ and $n = 2, 4, 6, \ldots$ from left to right when $\dot{x} > 0$. The motion stops for $A - (2n-1)(2\delta/\omega^2) = 0$ when the 'stop region' is reached, i.e., when the initial position is equal or smaller than $|2\delta/\omega^2|$.

However, these two types of damping (linear viscous and dry friction) exist only theoretically. Some indication for this conclusion is mentioned by Hemp [1], who proposed that the damping for a runaway escapement mechanism is between the linear and zero form, i.e., the damping is of fraction order

$$0 < \frac{m}{q} < 1, \quad m < q. \tag{8}$$

The differential equation of motion of the system with fraction order damping force is in general

$$\ddot{x} + \omega^2 x + (2\delta)\mathrm{sgn}(\dot{x})|\dot{x}|^{m/q} = 0 \tag{9}$$

i.e., for the motion from right to left between two consecutive zeros of $\dot{x}$ when $\dot{x} < 0$

$$\ddot{x} + \omega^2 x = (2\delta)|\dot{x}|^{m/q} \tag{10}$$

and for the motion from left to right when $\dot{x} > 0$

$$\ddot{x} + \omega^2 x = -(2\delta)|\dot{x}|^{m/q}, \tag{11}$$

where $|\dot{x}|$ is the absolute velocity of motion.

In this paper the vibrations of the system with fraction order damping are considered.

Usually, the systems with polynomial damping force and linear or nonlinear elastic force are considered (see Refs. [2–5]). The oscillator with non-polynomial fraction order elastic force is investigated in the papers [6–12]. The central result for the latter is that a system under the influence of such a force has periodic solutions only when both the numerator $(2m+1)$ and the denominator $(2n+1)$ of the exponent of deflection are odd. If one of them is even, the motion is not oscillatory. The method of harmonic balance is used to calculate the analytical approximation of these periodic solutions [6]. The higher order harmonic balance method combined with numerical procedure (see Refs. [7,8]) has been used to construct an analytical approximation to a system modelled by an $x^{4/3}$ potential. The generalization of the result is done by Hu and Xiong [9]. For the case when the restoring force is close to $\mathrm{sign}(x)$, the small δ-method is applied [10]. Using the advantages of the first integral, the exact analytical expression for the period of periodic solutions of the oscillator equation $\ddot{x} + x^{1/(2n+1)} = 0$ are determined [11]. Waluya and Horssen [12] applied the perturbation method based on integrating factors to approximate first integrals for a generalized nonlinear Van der Pol oscillator equation. The existence, uniqueness, stability and periods of the time-periodic solutions were established straightforwardly.

In this paper the previously mentioned methods and results are applied in order to solve analytically the differential equation (9) with a fraction order damping term. The well known He's homotopy perturbation method (see Refs. [13–17]) is adopted for solving the strong nonlinear differential equation with fraction order velocity. For the case when the coefficient of the damping force is small, i.e., the term with fraction order velocity is small, the method of straightforward expansion [18] is used and the approximate analytic solution is obtained. The special case when the damping force is close to dry friction, i.e., the fraction order of the damping force is a small value, the method of variable amplitude and phase [19] is extended for solving the differential equation. The approximate analytic solutions are compared with numeric 'exact' ones using the Runge–Kutta solving procedure. The differences between solutions are discussed. The properties of the system with fraction order damping are analyzed.

2. System with damping force close to linear viscous damping

The system with fraction order damping force close to linear is considered. The mathematical model of the system is the differential equation (9) where the first time derivative is of fraction order. The approximate solution of the differential equation (9) is obtained using the homotopy perturbation technique [13].

Remark 1. Due to the fact that Eqs. (10) and (11) correspond to Eq. (9) and have the same forms but the opposite signs of the right-hand side terms, in this paper the application of the homotopy perturbation technique is shown only for Eq. (10). The same procedure is evident for Eq. (11).

In view of the homotopy perturbation technique, we can construct the following homotopy for Eq. (10) transforming the variable $x(t)$ to $X(t,p)$

$$\ddot{X} + \omega^2 X + 2\delta\dot{X} = p(2\delta\dot{X} + (2\delta)|\dot{X}|^{m/q}), \tag{12}$$

where $p \in [0,1]$ is the embedding parameter. The initial conditions are

$$X(0,p) = A, \quad \dot{X}(0,p) = 0. \tag{13}$$

In case $p = 0$, Eq. (9) becomes

$$\ddot{X} + \omega^2 X + 2\delta\dot{X} = 0, \tag{14}$$

the solution of which is Eq. (5) for $X(t,0) = x(t)$. For $p = 1$ Eq. (12) turns out to be the original differential equation (10) with fraction order damping, and the solution is

$$X(t,1) = x(t).$$

Remark 2. The homotopy method admits the introduction of the linear operator (14) which describes the physical sense of motion and does not require the solution of the mathematical linear part of Eq. (9) to be the basis function for further approximation. From the physical points of view it is known that the sum of the kinetic and potential energy of the considered system does not keep the same, but decreases. The solution is approximated by a quasi-periodic function on a time interval of half the period of the non-damped motion. The solution should decay.

The solution of Eq. (12) can be written as a power series in p

$$X = x_0 + px_1 + \cdots \tag{15}$$

Substituting Eq. (15) into Eq. (12) and separating the terms with the same order of the parameter p, the following system of differential equations is obtained:

$$p^0 : \ddot{x}_0 + \omega^2 x_0 + 2\delta\dot{x}_0 = 0, \tag{16}$$

$$p^1 : \ddot{x}_1 + \omega^2 x_1 + 2\delta\dot{x}_1 = 2\delta\dot{x}_0 + (2\delta)|\dot{x}_0|^{m/q}, \quad \ldots \tag{17}$$

with the initial conditions for Eq. (16)

$$x_0(0) = A, \quad \dot{x}_0(0) = 0, \tag{18}$$

and for Eq. (17)

$$x_1(0) = 0, \quad \dot{x}_1(0) = 0, \tag{19}$$

According to Eq. (5) and initial conditions (18), the solution of Eq. (10) is

$$x_0 = A\exp(-\delta t)\left(\cos(\kappa t) + \frac{\delta}{\kappa}\sin(\kappa t)\right), \tag{20}$$

where

$$\kappa = \sqrt{\omega^2 - \delta^2}. \tag{21}$$

Using Eq. (20), the first-order deformation equation is

$$\begin{aligned} &\ddot{x}_1 + 2\delta\dot{x}_1 + \omega^2 x_1 + 2\delta A\exp(-\delta t)\frac{\omega^2}{\kappa}\sin(\kappa t) \\ &- (2\delta)\left|-A\frac{\omega^2}{\kappa}\exp(-\delta t)\sin(\kappa t)\right|^{m/q} = 0. \end{aligned} \tag{22}$$

Introducing the Fourier series expression (see Ref. [20]) for the function

$$\left[\exp\left(\left(\frac{q}{m} - 1\right)\delta t\right)\sin(\kappa t)\right]^{m/q} = \frac{a_0}{2} + \sum_{k=1}^{\infty}(a_k\cos(k\kappa t) + b_k\sin(k\kappa t)), \tag{23}$$

into Eq. (22) it follows

$$\begin{aligned} &\ddot{x}_1 + 2\delta\dot{x}_1 + \omega^2 x_1 - \exp(-\delta t)\left[a_0\delta\left|-A\frac{\omega^2}{\kappa}\right|^{m/q} + 2\delta a_1\left|-A\frac{\omega^2}{\kappa}\right|^{m/q}\cos(\kappa t)\right. \\ &+ 2\delta\left(-A\frac{\omega^2}{\kappa} + b_1\left|-A\frac{\omega^2}{\kappa}\right|^{m/q}\right)\sin(\kappa t) \\ &\left.+(2\delta)\left|-A\frac{\omega^2}{\kappa}\right|^{m/q}\sum_{k=2}^{\infty}(a_k\cos(k\kappa t) + b_k\sin(k\kappa t))\right] = 0, \end{aligned} \tag{24}$$

where the coefficients a_0, a_k and b_k depend on the fraction m/q. We obtain the solution of Eq. (24) with Eq. (19)

$$\begin{aligned} x_1 = &\frac{2\delta}{\kappa^2}\left|-A\frac{\omega^2}{\kappa}\right|^{m/q}\exp(-\delta t)\left[\frac{a_0}{2}(1 - \cos(\kappa t))\right. \\ &\left.+\sum_{k=2}^{\infty}\left(\frac{a_k}{1-k^2}(\cos(k\kappa t) - \cos(\kappa t)) + \frac{b_k}{1-k^2}(\sin(k\kappa t) - k\sin(\kappa t))\right)\right] \\ &-\frac{\delta}{\kappa}\left(A\frac{\omega^2}{\kappa} - b_1\left|-A\frac{\omega^2}{\kappa}\right|^{m/q}\right)\exp(-\delta t)\left(t\cos(\kappa t) - \frac{1}{\kappa}\sin(\kappa t)\right) \\ &+\frac{\delta}{\kappa}a_1\left|-A\frac{\omega^2}{\kappa}\right|^{m/q}\exp(-\delta t)t\sin(\kappa t). \end{aligned} \tag{25}$$

Using Eqs. (20) and (25), the solution in the first approximation is determined

$$x = \exp(-\delta t)\left\{A\cos(\kappa t) + \frac{\delta}{\kappa^2}\left|-A\frac{\omega^2}{\kappa}\right|^{m/q} a_0(1-\cos(\kappa t)) - \frac{\delta}{\kappa}\left(A\frac{\omega^2}{\kappa} - b_1\left|-A\frac{\omega^2}{\kappa}\right|^{m/q}\right)\left(t\cos(\kappa t) - \frac{1}{\kappa}\sin(\kappa t)\right) + \frac{\delta}{\kappa}\sin(\kappa t)\left[A + a_1 t\left|-A\frac{\omega^2}{\kappa}\right|^{m/q}\right]\right\}. \tag{26}$$

Analyzing the relations obtained, the following is obvious:

1. For the special case when the damping parameter is zero ($\delta = 0$), the solution (26) simplifies to
$$x = A\cos\omega t,$$
which represents the well known solution of a harmonic oscillator for the initial conditions (3).
2. If $m/q = 1$, the damping is the linear function of velocity, the solution (26) transforms to Eq. (5) which corresponds to the linear viscous damping system (1).
3. The homotopy perturbation method uses the imbedding parameter p as a small parameter and only few iteration are enough for asymptotic solution.
4. Due to straightforward expansion (15) the approximate solution (26) contains the so-called secular term with the factor $t\sin(\kappa t)$. Because of secular term, expansion (15) is not quasi-periodic. Thus x_1 does not provide a small correction to x_0.
5. Using the physical point of view the improvement of the approximation is necessary. The expansion for ω and δ is introduced
$$\omega_0^2 = \omega^2 + p\omega_1 + \dots, \quad \delta_0 = \delta + p\delta_1 + \dots, \tag{27}$$

i.e.,

$$\omega^2 = \omega_0^2 - p\omega_1 - \dots, \quad \delta = \delta_0 - p\delta_1 - \dots, \tag{28}$$

where ω_1 and δ_1 are frequency and damping correction parameters, respectively. Substituting Eqs. (15) and (28) into Eq. (12) and separating the terms with the same order of the parameter p, the following system of differential equations is obtained:

$$p^0: \quad \ddot{x}_0 + \omega_0^2 x_0 + 2\delta_0\dot{x}_0 = 0, \tag{29}$$

$$p^1: \quad \ddot{x}_1 + \omega_0^2 x_1 + 2\delta_0\dot{x}_1 = \omega_1 x_0 + 2(\delta_1 + \delta_0)\dot{x}_0 + (2\delta_0)|\dot{x}_0|^{m/q}. \quad \dots \tag{30}$$

The assumption of the solution of Eq. (29) in the form (20) and substitution of Eq. (28) into Eq. (30) leads to the following second-order differential equation:

$$\ddot{x}_1 + 2\delta_0\dot{x}_1 + \omega_0^2 x_1 - \exp(-\delta_0 t)\left[a_0\delta_0\left|-A\frac{\omega_0^2}{\kappa_0}\right|^{m/q} + \left(\omega_1 A + 2\delta_0 a_1\left|-A\frac{\omega_0^2}{\kappa_0}\right|^{m/q}\right)\cos(\kappa_0 t)\right.$$

$$+\left(\omega_1 A\frac{\delta_0}{\kappa_0} - 2(\delta_1+\delta_0)A\frac{\omega_0^2}{\kappa_0} + 2\delta_0 b_1\left|-A\frac{\omega_0^2}{\kappa_0}\right|^{m/q}\right)\sin(\kappa_0 t)$$
$$+(2\delta_0)\left|-A\frac{\omega_0^2}{\kappa_0}\right|^{m/q}\sum_{k=2}^{\infty}(a_k\cos(k\kappa_0 t)+b_k\sin(k\kappa_0 t))\Bigg] = 0. \tag{31}$$

Eliminating the secular terms in Eq. (31) the correction parameters are determined

$$\omega_1 = -\left(\frac{2\delta}{A}\right)\left|-A\frac{\omega^2}{\kappa}\right|^{m/q}a_1, \quad \delta_1 = -\delta + \frac{\kappa\delta}{A\omega^2}\left|-A\frac{\omega^2}{\kappa}\right|^{m/q}\left(b_1 - a_1\frac{\delta}{\kappa}\right). \tag{32}$$

Applying the homotopy perturbation procedure and using Eq. (32) the general form of parameter corrections for Eq. (9) are obtained

$$\omega_1 = -(-1)^{n-1}\left(\frac{2\delta}{A}\right)\left|-A\frac{\omega^2}{\kappa}\right|^{m/q}a_1, \quad \delta_1 = -\delta + (-1)^{n-1}\frac{\kappa\delta}{A\omega^2}\left|-A\frac{\omega^2}{\kappa}\right|^{m/q}\left(b_1 - a_1\frac{\delta}{\kappa}\right), \tag{33}$$

where $n = 1, 2, 3, \ldots : n = 1, 3, 5, \ldots$ for Eq. (10) and $n = 2, 4, 6, \ldots$ for Eq. (11). According to Eq. (27) and the relations (33), the frequency and damping parameters are in the first approximation

$$\omega_0 = \sqrt{\omega^2\left[1 - 2p(-1)^{n-1}a_1\left(\frac{\delta}{\kappa}\right)\left(\frac{\kappa}{A\omega^2}\right)^{1-m/q}\right]}, \tag{34}$$

$$\delta_0 = (-1)^{n-1}\frac{\delta p}{\kappa}\left(\frac{\kappa}{A\omega^2}\right)^{1-m/q}(\kappa b_1 - \delta a_1). \tag{35}$$

For $p = 1$ we obtain

$$\omega_0^2 \approx \omega^2\left[1 - 2(-1)^{n-1}a_1\left(\frac{\delta}{\kappa}\right)\left(\frac{\kappa}{A\omega^2}\right)^{1-m/q}\right], \tag{36}$$

$$\delta_0 \approx (-1)^{n-1}\frac{\delta}{\kappa}\left(\frac{\kappa}{A\omega^2}\right)^{1-m/q}(\kappa b_1 - \delta a_1) \tag{37}$$

and the approximate value of κ_0 is

$$\kappa_0 = \sqrt{\omega^2\left[1 - 2a_1\left(\frac{\delta}{\kappa}\right)\left(\frac{\kappa}{A\omega^2}\right)^{1-m/q}\right] - \left(\frac{\delta}{\kappa}\right)^2\left(\frac{\kappa}{A\omega^2}\right)^{2(1-m/q)}(\kappa b_1 - \delta a_1)^2}. \tag{38}$$

Using Eq. (20) with Eqs. (36)–(38), the solution in the first approximation is

$$x = A\exp\left(-t\left[(-1)^{n-1}\frac{\delta}{\kappa}\left(\frac{\kappa}{A\omega^2}\right)^{1-m/q}(\kappa b_1 - \delta a_1)\right]\right)$$
$$\times\left(\cos\left\{t\left[\omega^2 - 2(-1)^{n-1}a_1\left(\frac{\delta\omega^2}{\kappa}\right)\left(\frac{\kappa}{A\omega^2}\right)^{1-m/q} - \left(\frac{\delta}{\kappa}\right)^2\left(\frac{\kappa}{A\omega^2}\right)^{2(1-m/q)}(\kappa b_1 - \delta a_1)^2\right]^{1/2}\right\}\right.$$

$$+\frac{\delta}{\kappa}\frac{(-1)^{n-1}(\kappa b_1-\delta a_1)}{\left[\omega^2-2(-1)^{n-1}a_1\left(\frac{\delta\omega^2}{\kappa}\right)\left(\frac{\kappa}{A\omega^2}\right)^{1-m/q}-\left(\frac{\delta}{\kappa}\right)^2\left(\frac{\kappa}{A\omega^2}\right)^{2(1-m/q)}(\kappa b_1-\delta a_1)^2\right]^{1/2}}\left(\frac{\kappa}{A\omega^2}\right)^{1-m/q}$$
$$\times\sin\left\{t\left[\omega^2-2(-1)^{n-1}a_1\left(\frac{\delta\omega^2}{\kappa}\right)\left(\frac{\kappa}{A\omega^2}\right)^{1-m/q}-\left(\frac{\delta}{\kappa}\right)^2\left(\frac{\kappa}{A\omega^2}\right)^{2(1-m/q)}(\kappa b_1-\delta a_1)^2\right]^{1/2}\right\}\Bigg). \tag{39}$$

Analyzing the relations it can be concluded:

1. The frequency of vibrations depends on the initial amplitude A; the exception is for $\kappa/A\omega^2 = 1$ when the frequency is

$$\kappa_0=\sqrt{\omega^2\left[1-2a_1\left(\frac{\delta}{\kappa}\right)\right]-\left(\frac{\delta}{\kappa}\right)^2(\kappa b_1-\delta a_1)^2}. \tag{40}$$

2. The frequency of vibration depends on the fraction order m/q.
3. The parameter of amplitude decrease δ_0 depends on the initial amplitude and the fraction order of the damping force (see Eq. (37)). For $\kappa/A\omega^2 = 1$, the amplitude decrease depends only on the value of m/q:

$$\delta_0 \approx \delta\left(b_1-\frac{\delta}{\kappa}a_1\right). \tag{41}$$

4. If $m/q = 1$, the damping is the linear function of velocity, and the Fourier coefficients (23) are

$$a_1 = 0, \quad b_1 = 1, \tag{42}$$

and the correction parameters (33) are

$$\omega_1 = 0, \quad \delta_1 = 0. \tag{43}$$

The solution (39) transforms to Eq. (5) which corresponds to the linear viscous damping system (1).
5. For $m/q = 0$, when $(\sin(\kappa_0 t))^0 = 1$, the Fourier coefficients (23) are zero ($a_1 = 0$, and $b_1 = 0$), and the correction parameters are

$$\omega_1 = 0, \quad \delta_1 = -\delta. \tag{44}$$

The vibration parameters are $\omega_0 = \omega$ and $\delta_0 = 0$, and the motion corresponds to the case of dry friction (4).

3. System with small damping coefficient

It is of special interest to analyze the system with small damping. For the case when the coefficient of the damping force is small, i.e.,

$$2\delta \ll 1, \tag{45}$$

the differential equation (9) transforms to the differential equation with small nonlinearity

$$\ddot{x}+\omega^2 x+\varepsilon\,\mathrm{sgn}(\dot{x})|\dot{x}|^{m/q}=0, \tag{46}$$

i.e.,

$$\ddot{x}+\omega^2 x=\pm\varepsilon|\dot{x}|^{m/q}, \tag{47}$$

where

$$\varepsilon = 2\delta, \tag{48}$$

and the upper (+) sign correspond to ($\dot{x}<0$) and the lower sign (−) to ($\dot{x}>0$). The sign changes for $\dot{x}=0$. In order to solve Eq. (46), the method of straightforward expansion is introduced.

Remark 3. The procedure applied for solving the differential equation (47) with positive sign is the same as for the other differential equation with negative sign. In the paper the solving procedure for the differential equation with positive sign is introduced.

Assuming the solution of Eq. (47) in the form of series

$$x = x_0 + \varepsilon x_1 + \ldots, \quad \omega_0^2 = \omega^2 + \varepsilon\omega_1 + \ldots, \tag{49}$$

and by substituting into Eq. (47), the following system of differential equations is obtained:

$$\varepsilon^0: \quad \ddot{x}_0 + \omega_0^2 x_0 = 0, \tag{50}$$

$$\varepsilon^1: \quad \ddot{x}_1 + \omega_0^2 x_1 = \omega_1 x_0 + |\dot{x}_0|^{m/q},$$

$$\ldots. \tag{51}$$

For the initial conditions (18) the solution for x_0 is

$$x_0 = A\cos(\omega_0 t). \tag{52}$$

For $\varepsilon = 0$ the frequency is $\omega = \omega_0$. Substituting Eq. (52) into Eq. (51), the linear nonhomogeneous differential equation is obtained

$$\ddot{x}_1 + \omega_0^2 x_1 = \omega_1 A\cos(\omega_0 t) + |-A\omega_0\sin(\omega_0 t)|^{m/q}. \tag{53}$$

Transforming the trigonometric function [20]

$$\sin^{m/q}(\omega_0 t) = \frac{a_0'}{2} + \sum_{k=1}^{\infty} a_k' \cos(k\omega_0 t), \tag{54}$$

where a_0' and a_k' are coefficients which depend on (m/q), and substituting into Eq. (53) yields

$$\ddot{x}_1 + \omega_0^2 x_1 = \frac{a_0^*}{2} + (\omega_1 A + a_1^*)\cos(\omega_0 t) + \sum_{k=2}^{\infty} a_k^* \cos(k\omega_0 t), \tag{55}$$

where

$$a_0^* = a_0'|-A\omega_0|^{m/q}, \quad a_1^* = a_1'|-A\omega_0|^{m/q}, \quad a_k^* = a_k'|-A\omega_0|^{m/q}. \tag{56}$$

Eliminating the secular term the frequency correction is obtained

$$\omega_1 = -\frac{a_1^*}{A}. \tag{57}$$

The solution of Eq. (55) is

$$x_1 = K_1 + K_2\cos\omega_0 t + K_3\sin\omega_0 t + \sum_{k=2}^{\infty}(C_k^*\cos(k\omega_0 t)), \tag{58}$$

where

$$K_1 = \frac{a_0^*}{2\omega_0^2}, \quad C_k^* = \frac{a_k^*}{\omega_0^2(1-k^2)}, \tag{59}$$

and K_2 and K_3 are arbitrary constants calculated according to initial conditions (19)

$$K_2 = -K_1 - \sum_{k=2}^{\infty} C_k^*, \quad K_3 = 0.$$

Based on Eqs. (49), (52) and (58) with Eq. (59) the solution of Eq. (47) in the first approximation is

$$x = \left(A - \frac{\varepsilon a_0^*}{2\omega^2} - \varepsilon \sum_{k=2}^{\infty} \frac{a_k^*}{\omega^2(1-k^2)} \right) \cos \omega t + \frac{\varepsilon a_0^*}{2\omega^2} + \sum_{k=2}^{\infty} \varepsilon \frac{a_k^*}{\omega^2(1-k^2)} \cos(k\omega t), \tag{60}$$

where

$$\omega_0 = \sqrt{\omega^2 - \varepsilon \frac{a_1^*}{A}}. \tag{61}$$

In general, the solution of Eq. (46) in the first approximation is

$$x = (A_0 + \varepsilon K_2) \cos \omega_0 t \pm \varepsilon K_1 + \varepsilon K_3 \sin \omega_0 t \pm \sum_{k=2}^{\infty} \varepsilon C_k^* \cos(k\omega_0 t). \tag{62}$$

where A_0 is the initial amplitude which is different for all intervals of motion between two consecutive zero velocities and

$$\omega_0 = \sqrt{\omega^2 - \varepsilon \frac{a_1' | - A_0 \omega_0 |^{m/q}}{A_0}}. \tag{63}$$

For

$$\frac{m}{q} = \frac{2N}{2M+1}, \quad M = 1, 2, 3, \ldots, \quad N = 1, 2, 3, \ldots,$$

the sin terms in Fourier series [20] are zero. Using only the first three terms of Fourier expansion the approximate solution is obtained. In general, the solution in the first approximation is

$$x = A_0 \cos \omega_0 t + \varepsilon A_1 \cos \omega_0 t + (-1)^{n-1} \varepsilon (K_1 + C_2^* \cos 2\omega_0 t) \tag{64}$$

and the corresponding time derivative is

$$\dot{x} = -\omega_0 [(A_0 + \varepsilon A_1) + 4(-1)^{n-1} \varepsilon C_2^* \cos \omega_0 t] \sin \omega_0 t, \tag{65}$$

where A_0 and A_1 depend on the initial conditions. Analyzing the relation (65), it is obvious that the velocity is zero for $\sin \omega_0 t = 0$, i.e.,

$$T = \frac{(n-1)\pi}{\omega_0} \approx \frac{(n-1)\pi}{\omega}. \tag{66}$$

It means that the time limits for one direction of motion are $(n-1)\pi/\omega$ and $n\pi/\omega$.

Substituting the lower time limit in Eq. (64), the boundary position $x_{(n-1)b}$ is

$$x_{(n-1)b} = (-1)^{n-1} [A_0 + \varepsilon A_1 + \varepsilon (K_1 + C_2^*)]. \tag{67}$$

Separating the terms with ε^0 and ε^1, the following is obtained:

$$x_{0(n-1)b} = (-1)^{n-1} A_0, \quad x_{1(n-1)b} = (-1)^{n-1} (A_1 + K_1 + C_2^*). \tag{68}$$

Using these relations, the arbitrary constants are defined

$$A_0 = (-1)^{n-1} x_{0(n-1)b}, \quad A_1 = (-1)^{n-1} x_{1(n-1)b} - (K_1 + C_2^*). \tag{69}$$

For the other limit value of time, the boundary position is

$$x_{nb} = (-1)^{n-1}(-A_0 - \varepsilon A_1 + \varepsilon(K_1 + C_2^*)]$$
$$= (-1)^{2n-1}x_{0(n-1)b} - \varepsilon x_{1(n-1)b} + 2\varepsilon(-1)^{n-1}(K_1 + C_2^*) \tag{70}$$

and the values with ε^0 and ε^1 are

$$x_{0nb} = (-1)^{2n-1}x_{0(n-1)k}, \quad x_{1nk} = -x_{1(n-1)k} + 2(-1)^{n-1}(K_1 + C_2^*). \tag{71}$$

The values (71) are the initial values for A_0 and A_1 for the following n intervals of motion. Thus, for the first interval of motion the initial conditions (3) are

$$A_0 = A, \quad A_1 = -(K_1 + C_2^*). \tag{72}$$

The motion is

$$x = A\cos\omega_0 t - \varepsilon(K_1 + C_2^*)\cos\omega_0 t + \varepsilon(K_1 + C_2^*\cos 2\omega_0 t). \tag{73}$$

According to Eq. (70), the final position at $t = \pi/\omega$ is

$$x_b = -A + 2\varepsilon(K_1 + C_2^*). \tag{74}$$

Using the previous consideration and the initial conditions (70), it can be concluded that the general form of the constants is

$$A_0 = x_b - \varepsilon[(-1)^{n-1} - (2n-1)](K_1 + C_2^*), \quad A_1 = -(2n-1)(K_1 + C_2^*) \tag{75}$$

and the motion is

$$x = A_0\cos\omega_0 t - \varepsilon(2n-1)(K_1 + C_2^*)\cos\omega_0 t + \varepsilon(-1)^{n-1}(K_1 + C_2^*\cos 2\omega_0 t), \tag{76}$$

with the initial conditions

$$x_b = (-1)^{n-1}[A - 2(n-1)\varepsilon(K_1 + C_2^*)], \quad \dot{x}_b = 0, \tag{77}$$

for $n = 1, 2$. For the relation (76), the maximal displacements are

$$A, \quad -\left[A - 2\varepsilon K_1(1 - \frac{2}{3}\frac{a_2'}{a_0'})\right], \quad A - 4\varepsilon K_1\left(1 - \frac{2}{3}\frac{a_2'}{a_0'}\right), \quad -\left[A - 4\varepsilon K_1\left(1 - \frac{2}{3}\frac{a_2'}{a_0'}\right)\right], \ldots. \tag{78}$$

So, the maximal amplitudes decrease in arithmetic progression with difference of $2\varepsilon K_1(1 - \frac{2a_2'}{3a_0'})$ for half of the period of vibration ($t = \pi/\omega$). The amplitude decrease decrement is

$$D = (A\omega_0)^{m/q}\frac{2\delta a_0'}{\omega_0^2}\left(1 - \frac{2}{3}\frac{a_2'}{a_0'}\right). \tag{79}$$

Due to the results obtained, it can be concluded:

1. The amplitude decrease depends on: the initial amplitude A, the degree of the damping force m/q, the coefficient of damping 2δ and the frequency ω_0.
2. The period of vibration does not depend on the order of the damping function for the case when the damping coefficient is small.
3. For larger values of the damping coefficient, the amplitude decreases faster.
4. For the same fraction order, the higher the value of the initial amplitude, the faster the vibration amplitude decreases.
5. For $A\omega_0 = 1$, the amplitude decrement is approximately a linear function of m/q.
6. The decrease depends on the frequency of the system $\sim 1/\omega_0^{2-m/q}$: for $\omega_0 > 1$ the amplitude decrease is slower for smaller values of fraction order; for $\omega_0 = 1$ the change of the fraction order has no influence on the amplitude decrease, and for $\omega_0 < 1$ the decrease is faster for smaller fraction order.

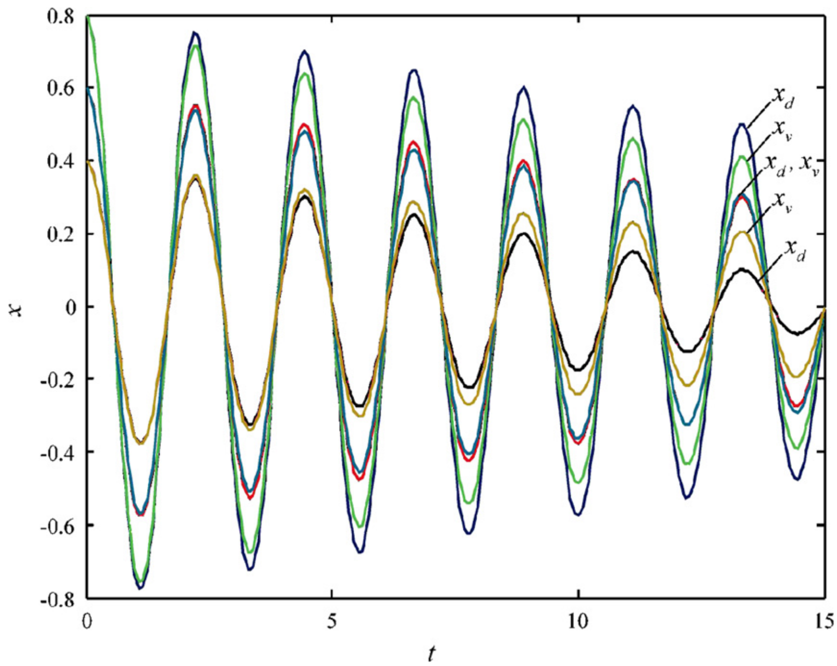

Fig. 1. The time history diagrams for dry friction ($x_d - t$) and viscous damping ($x_v - t$) for the following initial conditions: $A = 0.8$, $A = 0.6$ and $A = 0.4$.

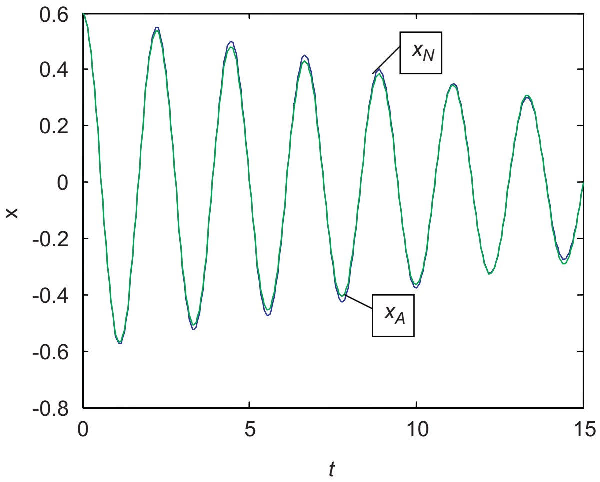

Fig. 2. Time-history diagrams obtained analytically ($x_A - t$) and numerically ($x_N - t$).

7. Besides, from Eq. (76) and its first time derivative, it can be concluded that the motion $x(t)$ and velocity $\dot{x}(t)$ change their directions at the same time.
8. For

$$A\omega_0 = 1.6971, \tag{80}$$

the decrease decrement is approximately the same for viscous damping ($m/q = 1$) and dry friction ($m/q = 0$). For $A\omega_0 > 1.6971$ the amplitude decrease is faster for viscous damping than for the dry friction. For $A\omega_0 < 1.6971$ the damping decrease is faster for dry friction than for viscous damping.

3.1. Examples

To prove the results obtained, two examples are provided:

1. In Fig. 1. the time history diagrams for the system with dry friction x_d and viscous damping x_v are plotted. Using the relation (76) and the parameter values $\varepsilon = 2\delta = 0.1$ and $\omega = 2\sqrt{2}$, the motion for various initial values ($A = 0.4$, 0.6 and 0.8) are calculated. For the initial condition $A = 0.6$, the viscous damping and dry friction have approximately the same decrease and their time history diagrams coincide. The two diagrams represent the boundary between two groups of initial conditions: for $A = 0.4$ the dry friction decreases faster than for the viscous damping, and for $A = 0.8$ the decrease is faster for viscous damping than for the dry friction. The obtained results are in good agreement with conclusion 8.
2. In Fig. 2. the solution of the differential equation

$$\ddot{x} + x = 0.05\,\mathrm{sgn}(\dot{x})|\dot{x}|^{2/9},$$

with the initial conditions

$$x(0) = 0.6, \quad \dot{x}(0) = 0,$$

is plotted. Using the suggested analytical procedure, the solution x_A (76) is obtained. Comparing the analytical solution x_A with the 'exact' numerical one x_N, obtained by Runge–Kutta procedure, it is evident that the difference between them is negligible.

4. System with damping force close to dry friction

Let us consider the case when the order of the velocity term in the damping force is a small value ($m/q = \varepsilon$) near zero ($m/q \approx 0$)

$$\ddot{x} + \omega^2 x + (2\delta)\mathrm{sgn}(\dot{x})|\dot{x}|^{\varepsilon} = 0, \tag{81}$$

i.e., between two zero velocities for the motion from right to left

$$\ddot{x} + \omega^2 x = (2\delta)|\dot{x}|^{\varepsilon} \tag{82}$$

and for the motion from left to right

$$\ddot{x} + \omega^2 x = -(2\delta)|\dot{x}|^{\varepsilon}. \tag{83}$$

The damping parameter δ also need to be small. The method of variable amplitude and phase is adopted for solving the differential equation (81).

Remark 4. In the paper the approximate solving method is developed for the differential equation (82) in the time interval between two zero velocities. The same procedure is available for the differential equation (83). By generalization of the obtained solutions the general solution for Eq. (81) is obtained.

For $\varepsilon = 0$, the differential equation (82) transforms to

$$\ddot{x} + \omega^2 x = (2\delta), \tag{84}$$

i.e., the differential equation of dry friction. The general solution of Eq. (84) is

$$x = \frac{2\delta}{\omega^2} + B\sin(\omega t + \alpha), \tag{85}$$

where the arbitrary constants B and α depend on the initial conditions x_0 and $\dot{x}_0 = 0$

$$B = x_0 - \left(\frac{2\delta}{\omega^2}\right), \quad \alpha = \frac{\pi}{2}. \tag{86}$$

Based on the solution (85) of Eq. (84), the trial solution of Eq. (82) is introduced

$$x = \left(\frac{2\delta}{\omega^2}\right) + B(t)\sin\psi(t), \tag{87}$$

where

$$\psi(t) = \omega t + \alpha(t), \tag{88}$$

$B(t) \equiv B$, $\psi(t) \equiv \psi$ and $\alpha(t) \equiv \alpha$ are the unknown time-dependent functions.

Introducing the assumption that the first time derivative has the form of the first time derivative of Eq. (87)

$$\dot{x} = B\omega\cos\psi, \tag{89}$$

the following constraint is to be satisfied:

$$\dot{B}\sin\psi + B\dot{\alpha}\cos\psi = 0. \tag{90}$$

Substituting Eq. (87) and the time derivative of Eq. (89) into Eq. (82), we obtain

$$\dot{B}\omega\cos\psi - B\omega\dot{\alpha}\sin\psi = 2\delta((B\omega\cos\psi)^{\varepsilon} - 1). \tag{91}$$

Using Eqs. (90) and (91) Eq. (81) is expressed as a system of two coupled first-order differential equations

$$\dot{B}\omega = 2\delta((B\omega\cos\psi)^{\varepsilon} - 1)\cos\psi, \tag{92}$$

$$B\dot{\alpha} = \frac{2\delta}{\omega}(1 - (B\omega\cos\psi)^{\varepsilon})\sin\psi. \tag{93}$$

There is no closed form analytical solution for the systems (92)–(93). As the functions $\cos\psi$ and $\sin\psi$ are periodical, the averaging procedure is introduced in order to find an approximate one. The averaging of the periodic function ψ is done and the averaged differential equations are

$$\dot{B} = \frac{1}{\pi}(2\delta)B^{\varepsilon}\omega^{\varepsilon-1}\int_0^{\pi}\cos^{1+\varepsilon}\psi\,\mathrm{d}\psi, \tag{94}$$

$$\dot{\alpha} = -\frac{1}{\pi}(2\delta)(B\omega)^{\varepsilon-1}\int_0^{\pi}\cos\psi^{\varepsilon}\sin\psi\,\mathrm{d}\psi. \tag{95}$$

Using the series expansion for $\varepsilon = m/q$ [20]

$$\cos^{\varepsilon}\psi = \frac{a_0''}{2} + \sum_{k=1}^{\infty}(a_k''\cos(k\psi) + b_k''\sin(k\psi)), \tag{96}$$

the averaged equations (94) and (95) are obtained

$$\dot{B} = (2\delta)B^{\varepsilon}\omega^{\varepsilon-1}P(\varepsilon), \quad \dot{\alpha} = -(2\delta)(B\omega)^{\varepsilon-1}Q(\varepsilon), \tag{97}$$

where coefficients a_0'', a_k'' and b_k'' depend on ε, and

$$P(\varepsilon) = \frac{1}{\pi}\int_0^{\pi}\sum_{k=1}^{\infty}a_k''\cos(k\psi)\cos\psi\,\mathrm{d}\psi, \quad Q(\varepsilon) = \frac{1}{\pi}\int_0^{\pi}\sum_{k=1}^{\infty}b_k''\sin(k\psi)\sin\psi\,\mathrm{d}\psi. \tag{98}$$

Integrating the differential equations (97), we obtain

$$B = B_0\left[1 + \frac{(2\delta)P(\varepsilon)(1-\varepsilon)}{(B_0\omega)^{1-\varepsilon}}t\right]^{1/(1-\varepsilon)}, \quad \alpha = \alpha_0 - \frac{Q(\varepsilon)}{P(\varepsilon)(1-\varepsilon)}\ln\left|1 + \frac{(2\delta)P(\varepsilon)(1-\varepsilon)}{(B_0\omega)^{1-\varepsilon}}t\right|, \tag{99}$$

where B_0 and α_0 are the initial values. Applying Eq. (99), the solution of Eq. (82) in the first approximation is obtained

$$x = \frac{2\delta}{\omega^2} + B_0\left(1 + \frac{2\delta P(\varepsilon)t}{(B_0\omega)^{1-\varepsilon}}\right)\sin\left[\left(\omega - \frac{(2\delta)Q(\varepsilon)}{(B_0\omega)^{1-\varepsilon}}\right)t + \alpha_0)\right]. \tag{100}$$

The suggested procedure is suitable for solving (83) and after some generalization of Eq. (100) the solution of Eq. (81) in the first approximation is

$$x_n = (-1)^{n-1}\left\{\frac{2\delta}{\omega^2} + B_{n-1}\left(1 + \frac{2\delta P(\varepsilon)t}{(B_{n-1}\omega)^{1-\varepsilon}}\right)\sin\left[\left(\omega - (-1)^{n-1}\frac{(2\delta)Q(\varepsilon)}{(B_{n-1}\omega)^{1-\varepsilon}}\right)t + \alpha_{n-1}\right]\right\}, \tag{101}$$

where B_{n-1} and α_{n-1} are initial conditions for certain time interval of motion between two zero velocities. For the series expansion of functions [21]

$$P(\varepsilon) = \varepsilon P_1 + O(\varepsilon^2), \quad Q(\varepsilon) = \varepsilon Q_1 + O_1(\varepsilon^2), \quad \exp(\varepsilon\delta_{n-1}t) = 1 + \varepsilon\delta_{n-1}t + \cdots, \tag{102}$$

where

$$\delta_{n-1} = \frac{2\delta P_1}{(B_{n-1}\omega)^{1-\varepsilon}}, \tag{103}$$

the solution (101) is

$$x_n = (-1)^{n-1}\left\{\frac{2\delta}{\omega^2} + B_{n-1}\exp(\varepsilon\delta_{n-1}t)\sin\left[\left(\omega - \varepsilon(-1)^{n-1}\frac{(2\delta)Q_1}{(B_{n-1}\omega)^{1-\varepsilon}}\right)t + \alpha_{n-1})\right]\right\}. \tag{104}$$

For the initial conditions (3), the arbitrary constants in the first interval of motion are

$$B_0\cos\left(\frac{\varepsilon\delta_0}{\omega}\right) = A - \frac{2\delta}{\omega^2}, \quad \alpha_0 = \frac{\pi}{2} + \frac{\varepsilon\delta_0}{\omega}. \tag{105}$$

The motion in this direction stops at

$$x_{1b} = \frac{2\delta}{\omega^2} - B_0\exp\left(\frac{\varepsilon\delta_0\pi}{\omega}\right)\cos\left(\frac{\varepsilon\delta_0}{\omega}\right), \quad \dot{x}_{1b} = 0. \tag{106}$$

These values are the initial values for the motion in the other direction. For $n \geqslant 2$, the arbitrary constants are

$$B_{n-1}\cos\left(\frac{\varepsilon\delta_{n-1}}{\omega}\right) = B_{n-2}\exp\left(\frac{\varepsilon\delta_{n-2}\pi}{\omega}\right)\cos\left(\frac{\varepsilon\delta_{n-2}}{\omega}\right) - 2\left(\frac{2\delta}{\omega^2}\right), \quad \alpha_{n-1} = \frac{\pi}{2} + \frac{\varepsilon\delta_{n-1}}{\omega}. \tag{107}$$

For the sake of simplicity, in Eq. (107) the series expansion of $\cos(\varepsilon\delta_{n-1}/\omega)$ is introduced. Using the first term in the series ($\cos(\varepsilon\delta_{n-1}/\omega) \approx 1$) and the assumption that $\varepsilon p_0/\omega$ is sufficiently small, the arbitrary

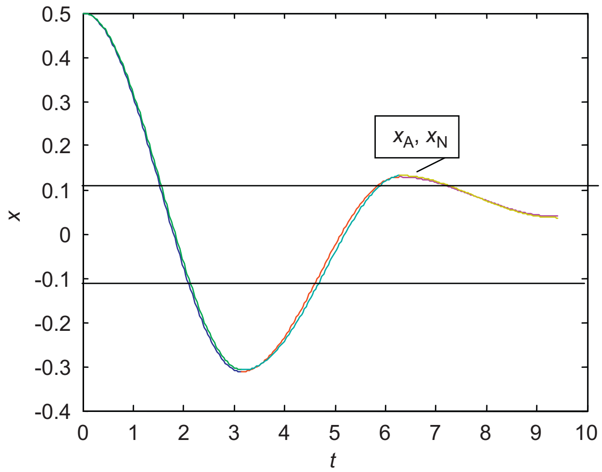

Fig. 3. Limits of motion and time-history diagrams obtained analytically ($x_A - t$) and numerically ($x_N - t$).

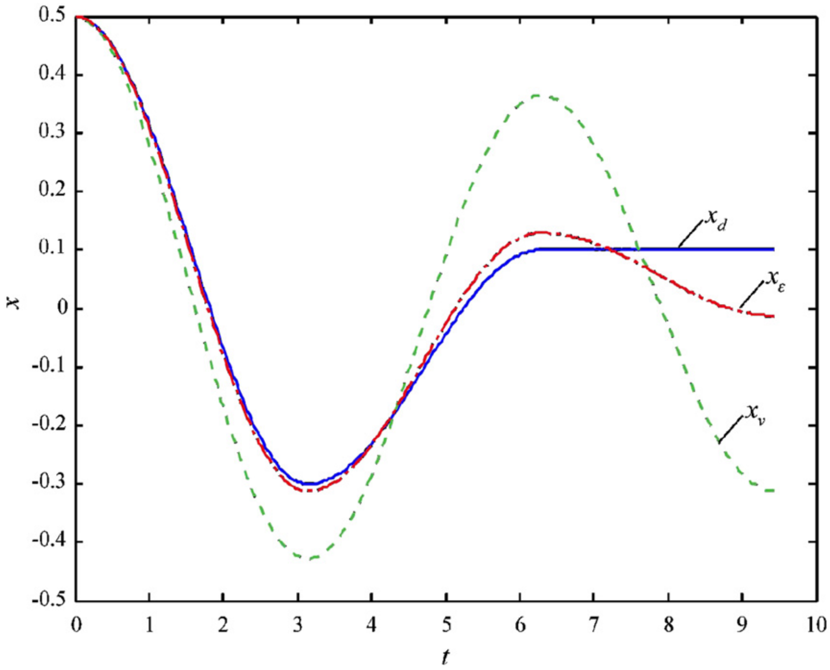

Fig. 4. Time-history diagrams for various values of damping order: viscous damping x_v, dry friction x_d and x_ε when $m/q = 0.05$.

constants are

$$B_{n-1} = A - \frac{2\delta}{\omega^2} \quad \text{for } n = 1, \tag{108}$$

$$B_{n-1} = -2\left(\frac{2\delta}{\omega^2}\right) + B_{n-2}\exp(\varepsilon\delta_{n-2}\pi/\omega) \quad \text{for } n \geqslant 2, \tag{109}$$

and the phase angle

$$\alpha_{n-1} = \frac{\pi}{2}. \tag{110}$$

The motion of the system without an initial velocity is possible only when the elastic force $|\omega^2 x|$ is higher than the damping force $2\delta|\dot{x}|^{\varepsilon}$. For $\varepsilon \ll 1$, the limits of motion are

$$-\frac{2\delta}{\omega^2} < x_n < \frac{2\delta}{\omega^2}. \tag{111}$$

For this interval the motion stops and the velocity is zero.

4.1. Examples

To prove the accuracy of the suggested procedure, some numerical examples are given.

1. The parameters of the system are: $\omega = 1$, $2\delta = 0.1$, $\varepsilon = 0.05$ and the initial conditions $x(0) = 0.5$ and $\dot{x}(0) = 0$. Using the relations (104) and the formula for the limits of motion (111), the approximate time history diagram $x_A - t$ is plotted (Fig. 3). The solution is compared with $x_N - t$ obtained numerically by Runge–Kutta procedure. It is evident that the difference between the curves is negligible.

2. In Fig. 4, the time-history diagrams for various values of the parameter ε are plotted. For $\omega = 1$, $2\delta = 0.1$ and the initial values $x(0) = 0.5$ and $\dot{x}(0) = 0$, the time history diagrams for $m/q = 0$, $m/q = 0.05$ and $\varepsilon = 1$ are

shown. The curve $x_\varepsilon - t$, for $m/q = 0.05$, is between two curves: dry friction $x_d - t$ (when $m/q = 0$), and viscous damping $x_v - t$ (when $m/q = 1$), but very close to the curve of dry friction.

5. Conclusion

The following can be concluded:

1. The oscillations of the system with fraction order damping are between two limits: the motion of the system with dry friction and the motion of the system with linear viscous damping.
2. The vibrations depend on the fraction order of the damping force: for smaller values the motion is closer to dry friction and for higher values to the motion of the system with linear viscous damping.
3. The frequency of vibration and also the quasi-period of vibrations depend on the initial amplitude and fraction order of the damping force. For the case of a small damping coefficient, the influence of the fraction order of the damping force can be omitted.
4. The amplitude decrease decrement depends on the initial amplitude and the fraction order of the damping force. For a small coefficient of the damping force, the amplitude decrease is in arithmetic progression.
5. For the case of a small damping coefficient when $A\omega = 1.6971$, the vibration decrease is independent on the fraction order.
6. The asymptotic results obtained give the possibility of using simple analytical expressions for the vibrations of the system with fraction order damping regardless of the values of the parameter m/q. More specifically:
   - the asymptotic (39) can be used for $0 \leqslant m/q \leqslant 1$ and $2\delta \geqslant 0$,
   - the asymptotic (104) is valid for $0 \leqslant m/q < 0.1$ and $2\delta \geqslant 0$,
   - the asymptotic (76) can be used for $0 \leqslant m/q \leqslant 1$ and $0 \leqslant 2\delta \leqslant 0.1$.

   Note that there are overlapping domains of asymptotic validity.

All the approximations given in the paper are valid for the time interval for which the length is given by two consecutive zeros of $\dot{x}$. At the end of the interval the displacement x is calculated and it represents the initial condition for the next time interval.

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References

[1] G.W. Hemp, Dynamic analysis of the runaway escapement mechanism, *Shock Vibration Bulletin* 42 (1972) 125–133.
[2] K. Magnus, K. Popp, *Schwingungen (5 Auflage)*, Teubner Verlag.s\sderg, Stuttgart, 1997.
[3] A.A. Andronov, A.A. Vitt, S.E. Khaikin, *Theory of Oscillators*, Dover Publications, New York, 1987.
[4] K. Yabushita, M. Yamashita, K. Tsuboi, An analytic solution of projectile motion with the quadratic resistance law using the homotopy analysis method, *Journal of Physics A: Mathematical and Theoretical* 40 (2007) 8403–8416.
[5] L. Cveticanin, Vibrations of the non-linear oscillator with quadratic non-linearity, *Physica A* 341 (2004) 123–135.
[6] R.E. Mickens, Oscillations in an $x^{4/3}$ potential, *Journal of Sound and Vibration* 246 (2001) 375–378.
[7] K. Cooper, R.E. Mickens, Generalized harmonic balance/numerical method for determining analytical approximations to the periodic solutions of the $x^{4/3}$ potential, *Journal of Sound and Vibration* 250 (2002) 951–954.
[8] R.E. Mickens, Analysis of non-linear oscillators having non-polynomial elastic term, *Journal of Sound and Vibration* 255 (2002) 789–792.
[9] H. Hu, Z.-G. Xiong, Oscillations in an $x^{(2m+2)-(2n+1)}$ potential, *Journal of Sound and Vibration* 259 (2003) 977–980.
[10] J. Awrejcewicz, I.V. Andrianov, Oscillations of non-linear system with restoring force close to sign(x), *Journal of Sound and Vibration* 252 (2002) 962–966.
[11] W.T. van Horssen, On the periods of the periodic solutions of the non-linear oscillator equation $\ddot{x} + x^{1/(2n+1)} = 0$, *Journal of Sound and Vibration* 260 (2003) 961–964.

[12] S.B. Waluya, W.T. van Horssen, On the periodic solutions of a generalized non-linear Van der Pol oscillator, *Journal of Sound and Vibration* 268 (2003) 209–215.
[13] J.-H. He, Homotopy perturbation technique, *Computer Methods in Applied Mechanics and Engineering* 178 (1999) 257–262.
[14] J.-H. He, Coupling method of a homotopy technique and a perturbation technique for non-linear problems, *International Journal of Non-Linear Mechanics* 35 (2000) 37–43.
[15] L. Cveticanin, The homotopy-perturbation method applied for solving complex-valued differential equations with strong cubic nonlinearity, *Journal of Sound and Vibration* 285 (2005) 1171–1179.
[16] L. Cveticanin, Homotopy-perturbation method for pure nonlinear differential equation, *Chaos, Solitons & Fractals* 30 (2006) 1221–1230.
[17] A. Goldbabai, B. Keramati, Modified homotopy perturbation method for solving Fredholm integral equations, *Chaos, Solitons & Fractals* 37 (2008) 1528–1537.
[18] A.H. Nayfeh, D.T. Mook, *Nonlinear Oscillations*, Wiley, New York, 1979.
[19] N.N. Bogolubov, J.A. Mitropolsky, *Asymptotic Methods in the Theory of Non-Linear Oscillations*, Gordon and Breach, New York, 1961.
[20] R.E. Mickens, *The Mathematical Methods for the Natural and Engineering Sciences*, World Scientific, London, 2004.
[21] I.S. Gradshteyn, I.M. Ryzhik, *Table of Integrals, Series, and Products*, sixth ed., Academic Press, New York, 2000.