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Nonlinear transverse free vibrations of piles

Chun-Lin Hu[a,b], Chang-Jun Cheng[a,*], Zhong-Xue Chen[b]

[a] Shanghai Institute of Applied Mathematics and Mechanics, Department of Mechanics, Shanghai University, Shanghai 200072, China
[b] School of Civil Engineering and Architecture, Wuhan University of Technology, Wuhan 430070, China

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Abstract

The nonlinear partial differential equation governing the nonlinear transverse vibration of pile was derived under the assumption of that both the materials of pile and soil obey nonlinear elastic and linear viscoelastic constitutive relations. The approximate expressions of the *n*th-order main frequency and the response of the nonlinear vibration of pile with ends hinged have been obtained by the complex mode method and multiple time scales method. Results point out that the main frequency of the nonlinear system is related to not only the natural frequency of linear vibration system, but also the amplitude, damping and nonlinearity of materials. There are high-order harmonic waves with twice the main frequency, and the main frequency of three times as well as the sum and/or difference of 2 or 3 main frequencies besides the harmonic wave with the main frequency in the response of the nonlinear system. Numerical examples were given and the effect of parameters was considered in detail.
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1. Introduction

The pile foundation has been widely used in engineering, such as high-rise building, bridge, offshore platform and nuclear power station and so on, but it is very difficult to perform the analysis of nonlinear mechanical behavior of piles due to complicacy of the interaction between pile and soil, the load transfer as well as deformation and motion. Although there are many papers on linear vibrations and dynamic responses of piles, there are few papers on nonlinear dynamic behaviors and nonlinear vibrations [1], especially, when both the materials of pile and soil are nonlinear elastic and/or visco-elastic ones. Novak [1] gave an overall overview for dynamic analysis of piles and introduced the linear and nonlinear dynamic theory, calculation method, and qualitative conclusion. Li [2] presented a simple and unified approach for the free vibration of a generally supported beam. Surie and Cederbaum [3] and Chen and Cheng [4] studied the stability and chaotic motion of nonlinear visco-elastic columns. Chau and Yang [5] studied the interaction of soil–pile system in nonlinear horizontal vibration from a new model of continuum mechanics. Hu et al. [6] analyzed nonlinear dynamic characteristics of piles under horizontal vibration from a similar model. Many researchers pointed out that the method of multiple time scales is an efficient method for analyzing nonlinear dynamic responses of

*Corresponding author.
E-mail address: chjcheng@mail.shu.edu.cn (C.-J. Cheng).

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structures [7–15], for example, Emam and Nayfeh [8] researched the nonlinear responses of buckled beams subjected to subharmonic-resonance excitations. Chen and Yang [9–11] analyzed the steady-state response of axially moving viscoelastic beam with pulsating speed and the stability in parametric resonance of axially accelerating beams constituted by Boltzmann's superposition principle. However, there are few reports for nonlinear vibration of nonlinear elastic and viscoelastic piles.

In the present paper, it would be assumed that both the materials of a pile and the soil around the pile obey nonlinear elastic and linear viscoelastic constitutive relations. The nonlinear partial differential equation governing the nonlinear transverse vibration of pile is first derived. Under the assumption that both the nonlinear characters of materials of the pile and the soil are weak, the approximate expressions of the nth-order main frequency and the response of the nonlinear vibration system have been obtained by the complex mode method and the method of multiple time scales. The effect of parameters is considered and numerical examples are given.

2. Formulation of the problem

Consider the nonlinear transverse free vibration of a pile with two ends hinged. Assume that the length, the outer diameter, the inner diameter, the cross-sectional area, and density of the pile are denoted by l, D, d, A and ρ, respectively. And suppose that the origin of the X-axis coincides with the geometry center at the top of the pile, the forward direction of the X-axis is downward, and the forward direction of the Y-axis is rightward.

Let $\sigma(X,T)$ and $\varepsilon(X,T)$ denote the stress and strain of the pile at the time T. For the pile composed of a nonlinear elastic and linear viscoelastic material, $\sigma(X,T)$ and $\varepsilon(X,T)$ obey the constitutive equation as follows:

$$\sigma = E_0(\varepsilon + \mathrm{sgn}(-\varepsilon)\beta\varepsilon^2 + \gamma\varepsilon^3) + \eta\frac{\partial\varepsilon}{\partial T} \tag{1}$$

where E_0, β, γ are the elastic coefficients of the material, η the viscosity coefficient, sgn( ) the symbolic function. In the case of small deformation, the relation between the strain $\varepsilon(X,T)$ and the displacement $V(X,T)$ is given as

$$\varepsilon = -Y\frac{\partial^2 V}{\partial X^2} \tag{2}$$

where Y is the distance of the considered point to the neutral axis. Hence, the bending moment is

$$M = -E_0\pi\frac{D^4-d_1^4}{64}\frac{\partial^2 V}{\partial X^2} + E_0\beta\frac{D^5-d_1^5}{60}\left(\frac{\partial^2 V}{\partial X^2}\right)^2 - E_0\gamma\pi\frac{D^6-d_1^6}{512}\left(\frac{\partial^2 V}{\partial X^2}\right)^3 - \eta\pi\frac{D^4-d_1^4}{64}\frac{\partial^3 V}{\partial X^2\,\partial T} \tag{3}$$

If the soil is also one of nonlinear elastic and linear viscoelastic materials, based on the generalized Winkler model, the resistance of the soil to the pile may be expressed as

$$p\left(X, V, \frac{\partial V}{\partial T}\right) = a(k_1V + k_2V^2 + k_3V^3) + c\frac{\partial V}{\partial T} \tag{4}$$

where c is the damping coefficient of the soil, k_1 the linear stiffness coefficient, k_2 and k_3 the nonlinear stiffness coefficients and a the adjustable parameter.

It is not difficult to obtain the nonlinear differential equation governing the transverse motion of the pile as follows:

$$\begin{aligned}&\rho A\frac{\partial^2 V}{\partial T^2} + E_0\pi\frac{D^4-d_1^4}{64}\frac{\partial^4 V}{\partial X^4} - 2E_0\beta\frac{D^5-d_1^5}{60}\left(\frac{\partial^3 V}{\partial X^3}\right)^2 - 2E_0\beta\frac{D^5-d_1^5}{60}\frac{\partial^2 V}{\partial X^2}\frac{\partial^4 V}{\partial X^4}\\&\quad + 6E_0\gamma\pi\frac{D^6-d_1^6}{512}\frac{\partial^2 V}{\partial X^2}\left(\frac{\partial^3 V}{\partial X^3}\right)^2 + 3E_0\gamma\pi\frac{D^6-d_1^6}{512}\left(\frac{\partial^2 V}{\partial X^2}\right)^2\frac{\partial^4 V}{\partial X^4} + \eta\pi\frac{D^4-d_1^4}{64}\frac{\partial^5 V}{\partial X^4\,\partial T}\\&\quad + P_0[1-(1-\alpha)X/l]\frac{\partial^2 V}{\partial X^2} - P_0\frac{1-\alpha}{l}\frac{\partial V}{\partial X} + aD(k_1V + k_2V^2 + k_3V^3) + cD\frac{\partial V}{\partial T} = 0\end{aligned} \tag{5}$$

where P_0 is the static load on the top of the pile and α the constant, commonly $0\leqslant\alpha\leqslant 1$.

For a pile embedded in rock and foundation platform, one may consider that both the top and bottom of the pile are hinged, so the boundary conditions are given as

$$V|_{X=l} = 0;\ M|_{X=l} = 0 \tag{6a}$$

$$V|_{X=0} = 0;\ M|_{X=0} = 0 \tag{6b}$$

Letting $U_0(X)$, $V_0(X)$ be the initial displacement and velocity of the pile, the initial conditions are

$$V(X,T)\big|_{T=0} = U_0(X), \quad \left.\frac{\partial V(X,T)}{\partial T}\right|_{T=0} = V_0(X) \tag{6c}$$

For convenience, assume that the conditions $\partial^2 U_0/\partial X^2|_{X=0,l} = 0$ are valid when $T = 0$.

Let $k = \sqrt{\pi E_0(D^4 - d_1^4)/(64\rho A l^4)}$ and introduce the following dimensionless variables and parameters:

$$\begin{aligned}
&w = V/l,\ t = kT,\ x = X/l,\ \beta_1 = E_0\beta(D^5 - d_1^5)/(30\rho A l^5 k^2)\\
&\gamma_1 = 3\pi E_0\gamma(D^6 - d_1^6)/(512\rho A l^6 k^2),\ \eta_1 = \pi\eta(D^4 - d_1^4)/(64\rho A l^4 k)\\
&p_0 = P_0/(\rho A l^2 k^2),\ k_{11} = aDk_1/(\rho A k^2),\ k_{22} = aDk_2 l/(\rho A k^2)\\
&k_{33} = aDk_3 l^2/(\rho A k^2),\ c_1 = cD/(\rho A k)
\end{aligned}$$

Thus, the equation of motion (5) and boundary conditions (6a) and (6b), initial conditions (6c) in terms of dimensionless displacement can be given as

$$\begin{aligned}
&\frac{\partial^2 w}{\partial t^2} + \frac{\partial^4 w}{\partial x^4} - \beta_1\left(\frac{\partial^3 w}{\partial x^3}\right)^2 - \beta_1\frac{\partial^2 w}{\partial x^2}\frac{\partial^4 w}{\partial x^4} + 2\gamma_1\frac{\partial^2 w}{\partial x^2}\left(\frac{\partial^3 w}{\partial x^3}\right)^2 + \gamma_1\left(\frac{\partial^2 w}{\partial x^2}\right)^2\frac{\partial^4 w}{\partial x^4} + \eta_1\frac{\partial^5 w}{\partial x^4\partial t}\\
&+ p_0[1 - (1-\alpha)x]\frac{\partial^2 w}{\partial x^2} - p_0(1-\alpha)\frac{\partial w}{\partial x} + k_{11}w + k_{22}w^2 + k_{33}w^3 + c_1\frac{\partial w}{\partial t} = 0
\end{aligned} \tag{7a}$$

$$w|_{x=1} = 0, \quad \left.\left[\frac{\partial^2 w}{\partial x^2} + \eta_1\frac{\partial^3 w}{\partial x^2\partial t} - \frac{\beta_1}{2}\left(\frac{\partial^2 w}{\partial x^2}\right)^2 - \frac{\gamma_1}{3}\left(\frac{\partial^2 w}{\partial x^2}\right)^3\right]\right|_{x=1} = 0 \tag{7b}$$

$$w|_{x=0} = 0, \quad \left.\left[\frac{\partial^2 w}{\partial x^2} + \eta_1\frac{\partial^3 w}{\partial x^2\partial t} - \frac{\beta_1}{2}\left(\frac{\partial^2 w}{\partial x^2}\right)^2 - \frac{\gamma_1}{3}\left(\frac{\partial^2 w}{\partial x^2}\right)^3\right]\right|_{x=0} = 0 \tag{7c}$$

$$w(x,t)\big|_{t=0} = u_0(x), \quad \left.\frac{\partial w(x,t)}{\partial t}\right|_{t=0} = v_0(x) \tag{7d}$$

And also assume that the initial displacements at the two ends of the pile satisfy

$$\left.\frac{\partial^2 u_0}{\partial x^2}\right|_{x=0} = 0, \quad \left.\frac{\partial^2 u_0}{\partial x^2}\right|_{x=1} = 0 \tag{7e}$$

3. Multiple time scales method and solution of the problem

Obviously, it is difficult to get the complete solution of the problem, so we try to apply the multiple time scales method to obtain the approximate solution of the problem. Let $\alpha = 1$ in Eq. (5). If both the nonlinear characters of materials of the pile and the soil are weak, we may set $k_{22} = \varepsilon k_{02}$, $k_{33} = \varepsilon k_{03}$, $\beta_1 = \varepsilon\beta_{01}$, $\gamma_1 = \varepsilon\gamma_{01}$, here, ε is a small parameter. Thus, Eq. (7a) may be written as follows:

$$\begin{aligned}
&\frac{\partial^2 w}{\partial t^2} + \frac{\partial^4 w}{\partial x^4} + p_0\frac{\partial^2 w}{\partial x^2} + k_{11}w + \eta_1\frac{\partial^5 w}{\partial x^4\partial t} + c_1\frac{\partial w}{\partial t} = \varepsilon\left[-k_{02}w^2 - k_{03}w^3 + \beta_{01}\left(\frac{\partial^3 w}{\partial x^3}\right)^2\right.\\
&\left.+\beta_{01}\frac{\partial^2 w}{\partial x^2}\frac{\partial^4 w}{\partial x^4} - 2\gamma_{01}\frac{\partial^2 w}{\partial x^2}\left(\frac{\partial^3 w}{\partial x^3}\right)^2 - \gamma_{01}\left(\frac{\partial^2 w}{\partial x^2}\right)^2\frac{\partial^4 w}{\partial x^4}\right]
\end{aligned} \tag{8a}$$

And conditions (7b) and (7c) can be given as

$$w|_{x=1} = 0, \quad \left[\frac{\partial^2 w}{\partial x^2} + \eta_1 \frac{\partial^3 w}{\partial x^2 \partial t} - \frac{\varepsilon \beta_{01}}{2} \left(\frac{\partial^2 w}{\partial x^2} \right)^2 - \frac{\varepsilon \gamma_{01}}{3} \left(\frac{\partial^2 w}{\partial x^2} \right)^3 \right] \Bigg|_{x=1} = 0 \tag{8b}$$

$$w|_{x=0} = 0, \quad \left[\frac{\partial^2 w}{\partial x^2} + \eta_1 \frac{\partial^3 w}{\partial x^2 \partial t} - \frac{\varepsilon \beta_{01}}{2} \left(\frac{\partial^2 w}{\partial x^2} \right)^2 - \frac{\varepsilon \gamma_{01}}{3} \left(\frac{\partial^2 w}{\partial x^2} \right)^3 \right] \Bigg|_{x=0} = 0 \tag{8c}$$

Assume that the approximate solution of Eq. (8a) has the form

$$w(x, t, \varepsilon) = w_0(x, T_0, T_1) + \varepsilon w_1(x, T_0, T_1) + \cdots \tag{9}$$

where $T_0 = t$, $T_1 = \varepsilon t$ are the time variables with different scales. The vibration expressed by Eq. (9) includes the time history with different time scale. Different time scale describes different rhythm of vibration process. For the time scale with lower order, the vibration of the system is slower, and for the time scale with higher order, the vibration of the system is quicker. Specially, $T_0 = t$ corresponds to the quick time scale of the linear vibration system with the natural frequency ω_{dn}, and $T_1 = \varepsilon t$ is a slow time scale arising nonlinear characters of materials of the pile and soil. And also the different time scale is independent of each other. Hence, $w(x, t, \varepsilon)$ is now regarded as a function of spatial variable x and two independent time variables T_0, T_1. Obviously, the following relations are true:

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial T_0} + \varepsilon \frac{\partial}{\partial T_1} + \cdots \tag{10a}$$

$$\frac{\partial^2}{\partial t^2} = \frac{\partial^2}{\partial T_0^2} + 2\varepsilon \frac{\partial^2}{\partial T_0 \partial T_1} + \cdots \tag{10b}$$

Substituting Eq. (9) into Eq. (8a) and observing Eq. (10) as well as comparing the coefficient with ε-same power, the zeroth- and the first-order linear partial differential equations can be obtained as follows:

$$\varepsilon^0 : \frac{\partial^2 w_0}{\partial T_0^2} + \frac{\partial^4 w_0}{\partial x^4} + p_0 \frac{\partial^2 w_0}{\partial x^2} + k_{11} w_0 + \eta_1 \frac{\partial^5 w_0}{\partial x^4 \partial T_0} + c_1 \frac{\partial w_0}{\partial T_0} = 0 \tag{11a}$$

$$\begin{aligned} \varepsilon^1 : \frac{\partial^2 w_1}{\partial T_0^2} + \frac{\partial^4 w_1}{\partial x^4} + p_0 \frac{\partial^2 w_1}{\partial x^2} + k_{11} w_1 + \eta_1 \frac{\partial^5 w_1}{\partial x^4 \partial T_0} + c_1 \frac{\partial w_1}{\partial T_0} &= -2 \frac{\partial^2 w_0}{\partial T_0 \partial T_1} - \eta_1 \frac{\partial^5 w_0}{\partial x^4 \partial T_1} - c_1 \frac{\partial w_0}{\partial T_1} \\ - k_{02} w_0^2 - k_{03} w_0^3 + \beta_{01} \left(\frac{\partial^3 w_0}{\partial x^3} \right)^2 + \beta_{01} \frac{\partial^2 w_0}{\partial x^2} \frac{\partial^4 w_0}{\partial x^4} &- 2\gamma_{01} \frac{\partial^2 w_0}{\partial x^2} \left(\frac{\partial^3 w_0}{\partial x^3} \right)^2 - \gamma_{01} \left(\frac{\partial^2 w_0}{\partial x^2} \right)^2 \frac{\partial^4 w_0}{\partial x^4} \end{aligned} \tag{11b}$$

The boundary conditions corresponding to the variables w_0, w_1 become

$$w_0|_{x=0} = 0, \quad \left[\frac{\partial^2 w_0}{\partial x^2} + \eta_1 \frac{\partial^3 w_0}{\partial x^2 \partial T_0} \right] \Bigg|_{x=0} = 0 \tag{12a}$$

$$w_0|_{x=1} = 0, \quad \left[\frac{\partial^2 w_0}{\partial x^2} + \eta_1 \frac{\partial^3 w_0}{\partial x^2 \partial T_0} \right] \Bigg|_{x=1} = 0 \tag{12b}$$

$$w_1|_{x=0} = 0, \quad \left(\frac{\partial^2 w_1}{\partial x^2} + \eta_1 \frac{\partial^3 w_1}{\partial x^2 \partial T_0} \right) \Bigg|_{x=0} = - \left[\eta_1 \frac{\partial^3 w_0}{\partial x^2 \partial T_1} - \frac{\beta_{01}}{2} \left(\frac{\partial^2 w_0}{\partial x^2} \right)^2 - \frac{\gamma_{01}}{3} \left(\frac{\partial^2 w_0}{\partial x^2} \right)^3 \right] \Bigg|_{x=0} \tag{13a}$$

$$w_1|_{x=1} = 0, \quad \left(\frac{\partial^2 w_1}{\partial x^2} + \eta_1 \frac{\partial^3 w_1}{\partial x^2 \partial T_0} \right) \Bigg|_{x=1} = - \left[\eta_1 \frac{\partial^3 w_0}{\partial x^2 \partial T_1} - \frac{\beta_{01}}{2} \left(\frac{\partial^2 w_0}{\partial x^2} \right)^2 - \frac{\gamma_{01}}{3} \left(\frac{\partial^2 w_0}{\partial x^2} \right)^3 \right] \Bigg|_{x=1} \tag{13b}$$

The corresponding initial conditions are given by

$$w_0|_{T_0=T_1=0} = u_0(x), \quad \left.\frac{\partial w_0}{\partial T_0}\right|_{T_0=T_1=0} = v_0(x), \quad w_1|_{T_0=T_1=0} = 0, \quad \left.\frac{\partial w_1}{\partial T_0}\right|_{T_0=T_1=0} = -\left.\frac{\partial w_0}{\partial T_1}\right|_{T_0=T_1=0} \tag{14}$$

The zeroth- and the first-order approximation solutions can be derived by using the auxiliary condition eliminating the secular terms, boundary conditions and initial conditions. Especially, the displacement can be expressed in terms of complex variable as follows:

$$w_0(x, T_0, T_1) = \sum_{n=1}^{N} \{\phi_n(x) A_n(T_1) \mathrm{e}^{\mathrm{i}\omega_{dn} T_0 - \delta_n T_0} + \bar{\phi}_n(x) \bar{A}_n(T_1) \mathrm{e}^{-\mathrm{i}\omega_{dn} T_0 - \delta_n T_0}\} \tag{15}$$

where $N \geqslant 1$ is integer, $\overline{(\cdot)}$ denotes the conjugate of $(\cdot)$, $\delta_n = c_1 + \eta_1 (n\pi)^4/2$ is the damping coefficient of the derivative linear vibration system, $\beta_{1n} = n\pi$ $(n = 1,2,3,\ldots)$.

Substituting Eq. (15) into Eq. (11b) yields the following equation:

$$\begin{aligned}
&\frac{\partial^2 w_1}{\partial T_0^2} + \frac{\partial^4 w_1}{\partial x^4} + p_0 \frac{\partial^2 w_1}{\partial x^2} + k_{11} w_1 + \eta_1 \frac{\partial^5 w_1}{\partial x^4 \partial T_0} + c_1 \frac{\partial w_1}{\partial T_0} = \sum_{n=1}^{N} \{[(-2\mathrm{i}\phi_n \omega_{dn} + 2\phi_n \delta_n - \eta_1 \phi_n^{(4)} - c_1 \phi_n) \\
&\quad \times \frac{\partial A_n}{\partial T_1} + (-3k_{03}\phi_n^2 \bar{\phi}_n - 4\gamma_{01}\phi_n'' \bar{\phi}_n''' \phi_n''' - 2\gamma_{01}(\phi_n''')^2 \bar{\phi}_n'' - \gamma_{01}(\phi_n'')^2 \bar{\phi}_n^{(4)} \\
&\quad - 2\gamma_{01}\bar{\phi}_n'' \phi_n'' \phi_n^{(4)}) A_n^2 \bar{A}_n \mathrm{e}^{-2\delta_n T_0}] \mathrm{e}^{\mathrm{i}\omega_{dn} T_0 - \delta_n T_0} + (-k_{02}\phi_n^2 + \beta_{01}(\phi_n''')^2 + \beta_{01}\phi_n'' \phi_n^{(4)}) A_n^2 \, \mathrm{e}^{2\mathrm{i}\omega_{dn} T_0 - 2\delta_n T_0} \\
&\quad + (-k_{03}\phi_n^3 - 2\gamma_{01}\phi_n''(\phi_n''')^2 - \gamma_{01}(\phi_n'')^2 \phi_n^{(4)}) A_n^3 \, \mathrm{e}^{3\mathrm{i}\omega_{dn} T_0 - 3\delta_n T_0} + (-k_{02}\phi_n \bar{\phi}_n + \beta_{01}\phi_n''' \bar{\phi}_n''' \\
&\quad + \beta_{01}\bar{\phi}_n'' \phi_n^{(4)}) A_n \bar{A}_n \, \mathrm{e}^{-2\delta_n T_0}\} + \sum_{k=1}^{N-1} \sum_{n=k+1}^{N} [(-2k_{02}\phi_k \phi_n + \beta_{01}\phi_k^{(4)} \phi_n'' + \beta_{01}\phi_k'' \phi_n^{(4)} \\
&\quad + 2\beta_{01}\phi_k''' \phi_n''') A_k A_n \, \mathrm{e}^{\mathrm{i}(\omega_{dk} + \omega_{dn}) T_0 - (\delta_k + \delta_n) T_0} + (-2k_{02}\phi_k \bar{\phi}_n + \beta_{01}\phi_k^{(4)} \bar{\phi}_n'' + \beta_{01}\phi_k'' \bar{\phi}_n^{(4)} \\
&\quad + 2\beta_{01}\phi_k''' \bar{\phi}_n'') A_k \bar{A}_n \, \mathrm{e}^{\mathrm{i}(\omega_{dk} - \omega_{dn}) T_0 - (\delta_k + \delta_n) T_0}] + \sum_{m=1}^{N} \sum_{k=1}^{N-1} \sum_{n=k+1}^{N} [(-2k_{03}\phi_m \phi_k \phi_n - 4\gamma_{01}\phi_m'' \phi_k''' \phi_n''' \\
&\quad - 2\gamma_{01}\phi_m^{(4)} \phi_k'' \phi_n'') A_m A_k A_n \, \mathrm{e}^{\mathrm{i}(\omega_{dm} + \omega_{dk} + \omega_{dn}) T_0 - (\delta_m + \delta_k + \delta_n) T_0} + (-2k_{03}\phi_m \phi_k \bar{\phi}_n - 4\gamma_{01}\phi_m'' \phi_k''' \bar{\phi}_n''' \\
&\quad - 2\gamma_{01}\phi_m^{(4)} \phi_k'' \bar{\phi}_n'') A_m A_k \bar{A}_n \, \mathrm{e}^{\mathrm{i}(\omega_{dm} + \omega_{dk} - \omega_{dn}) T_0 - (\delta_m + \delta_k + \delta_n) T_0} + (-2k_{03}\bar{\phi}_m \phi_k \phi_n - 4\gamma_{01}\bar{\phi}_m''' \phi_k''' \phi_n''' \\
&\quad - 2\gamma_{01}\bar{\phi}_m^{(4)} \phi_k'' \phi_n'') \bar{A}_m A_k A_n \, \mathrm{e}^{\mathrm{i}(\omega_{dk} + \omega_{dn} - \omega_{dm}) T_0 - (\delta_m + \delta_k + \delta_n) T_0} + (-2k_{03}\bar{\phi}_m \phi_k \bar{\phi}_n - 4\gamma_{01}\bar{\phi}_m'' \phi_k''' \bar{\phi}_n''' \\
&\quad - 2\gamma_{01}\bar{\phi}_m^{(4)} \phi_k'' \bar{\phi}_n'') \bar{A}_m A_k \bar{A}_n \, \mathrm{e}^{\mathrm{i}(\omega_{dk} - \omega_{dn} - \omega_{dm}) T_0 - (\delta_m + \delta_k + \delta_n) T_0}] + \sum_{q=1}^{N} \sum_{\substack{n=1 \\ (n \neq q)}}^{N} [(-k_{03}\phi_q \phi_n^2 - 2\gamma_{01}\phi_q''(\phi_n''')^2 \\
&\quad - \gamma_{01}\phi_q^{(4)}(\phi_n'')^2) A_q A_n^2 \, \mathrm{e}^{\mathrm{i}(2\omega_{dn} + \omega_{dq}) T_0 - (2\delta_n + \delta_q) T_0} + (-2k_{03}\phi_q \phi_n \bar{\phi}_n - 4\gamma_{01}\phi_q'' \phi_n''' \bar{\phi}_n''' \\
&\quad - 2\gamma_{01}\phi_q^{(4)} \phi_n'' \bar{\phi}_n'') A_q A_n \bar{A}_n \, \mathrm{e}^{\mathrm{i}\omega_{dq} T_0 - (2\delta_n + \delta_q) T_0} + (-k_{03}\bar{\phi}_q \phi_n^2 - 2\gamma_{01}\bar{\phi}_q''(\phi_n''')^2 \\
&\quad - \gamma_{01}\bar{\phi}_q^{(4)}(\phi_n'')^2) \bar{A}_q A_n^2 \, \mathrm{e}^{\mathrm{i}(2\omega_{dn} - \omega_{dq}) T_0 - (2\delta_n + \delta_q) T_0}] + \mathrm{cc}
\end{aligned} \tag{16}$$

where ϕ_n' and ϕ_n'' denote the derivatives of $\phi_n(x)$ about the variable x, cc is the conjugate of all terms on the right-hand side of the equation. From the homogeneous equation (11a) and the non-homogeneous equation (16), one can see that their left terms have the same expressions, but Eq. (11a) has the nonzero solution $\phi_n(x)$, i.e. the nth-order modal function of free vibration of the linear system, so, the necessary and sufficient condition of existence of solution for Eq. (16) is that the solvability condition must be satisfied, that is to say, the secular term on the right-hand side in Eq. (16) and the solutions of its adjoint Eq. (11a) are orthogonal.

Therefore, the solvability condition is given as

$$\langle[(-2\mathrm{i}\phi_n\omega_{dn}+2\phi_n\delta_n-\eta_1\phi_n^{(4)}-c_1\phi_n)\frac{\partial A_n}{\partial T_1}+(-3k_{03}\phi_n^2\bar{\phi}_n-4\gamma_{01}\phi''\bar{\phi}_n'''\phi_n''' \\ -2\gamma_{01}(\phi_n''')^2\bar{\phi}_n''-\gamma_{01}(\phi_n'')^2\bar{\phi}_n^{(4)}-2\gamma_{01}\bar{\phi}_n''\phi_n''\phi_n^{(4)})A_n^2\bar{A}_n\,\mathrm{e}^{-2\delta_nT_0}]\mathrm{e}^{\mathrm{i}\omega_{dn}T_0-\delta_nT_0},\phi_n\rangle=0 \tag{17}$$

where $\langle f, g\rangle$ is the inner product of complex functions in the interval [0,1] defined as

$$\langle f,g\rangle=\int_0^1 f\bar{g}\,\mathrm{d}x \tag{18}$$

In the above Eq. (17), f is taken as the secular term on the right-hand side of Eq. (16) and $g=\phi_n(x)$. From Eqs. (17) and (18), the following relation could be drawn:

$$\left(-2\mathrm{i}\omega_{dn}\int_0^1\phi_n\bar{\phi}_n\,\mathrm{d}x+2\delta_n\int_0^1\phi_n\bar{\phi}_n\,\mathrm{d}x-\eta_1\int_0^1\phi_n^{(4)}\bar{\phi}_n\,\mathrm{d}x-c_1\int_0^1\phi_n\bar{\phi}_n\,\mathrm{d}x\right)\frac{\partial A_n}{\partial T_1}+\left(-3k_{03}\int_0^1\phi_n^2\bar{\phi}_n^2\,\mathrm{d}x\right. \\ \left.-4\gamma_{01}\int_0^1\phi_n''\bar{\phi}_n'''\phi_n'''\bar{\phi}_n\,\mathrm{d}x-2\gamma_{01}\int_0^1(\phi_n''')^2\bar{\phi}_n''\bar{\phi}_n\,\mathrm{d}x-\gamma_{01}\int_0^1(\phi_n'')^2\bar{\phi}_n^{(4)}\bar{\phi}_n\,\mathrm{d}x-2\gamma_{01}\int_0^1\bar{\phi}_n''\phi_n''\phi_n^{(4)}\bar{\phi}_n\,\mathrm{d}x\right) \\ \times A_n^2\bar{A}_n\,\mathrm{e}^{-2\delta_nT_0}=0$$

The equation can be simplified as follows:

$$\frac{\partial A_n}{\partial T_1}-k_{3n}A_n^2\bar{A}_n\,\mathrm{e}^{-2\delta_nT_0}=0 \tag{19}$$

where

$$k_{3n}=\frac{-3k_{03}\int_0^1\phi_n^2\bar{\phi}_n^2\,\mathrm{d}x-4\gamma_{01}\int_0^1\phi_n''\bar{\phi}_n'''\phi_n'''\bar{\phi}_n\,\mathrm{d}x-2\gamma_{01}\int_0^1(\phi_n''')^2\bar{\phi}_n''\bar{\phi}_n\,\mathrm{d}x-\gamma_{01}\int_0^1(\phi_n'')^2\bar{\phi}_n^{(4)}\bar{\phi}_n\,\mathrm{d}x-2\gamma_{01}\int_0^1\bar{\phi}_n''\phi_n''\phi_n^{(4)}\bar{\phi}_n\,\mathrm{d}x}{2\mathrm{i}\omega_{dn}\int_0^1\phi_n\bar{\phi}_n\,\mathrm{d}x-2\delta_n\int_0^1\phi_n\bar{\phi}_n\,\mathrm{d}x+\eta_1\int_0^1\phi_n^{(4)}\bar{\phi}_n\,\mathrm{d}x+c_1\int_0^1\phi_n\bar{\phi}_n\,\mathrm{d}x} \tag{20}$$

From the boundary conditions (12a), (12b) and (7e), it is easy to solve the solution of the zeroth-order system (11a), (12a) and (12b) and get the natural frequency

$$\omega_{dn}=\sqrt{k_{11}+(n\pi)^4-(n\pi)^2p_0}\cdot\sqrt{1-\zeta_n^2}\quad(n=1,2,3,\ldots) \tag{21}$$

where ζ_n is the nth-order relative damping factor given as

$$\zeta_n=\frac{\delta_n}{\omega_n}=\frac{c_1+\eta_1(n\pi)^4}{2\sqrt{k_{11}+(n\pi)^4-(n\pi)^2p_0}}\quad(n=1,2,3,\ldots) \tag{22}$$

The corresponding modal is

$$\phi_n(x)=C_{1n}\sin n\pi x \tag{23}$$

where $C_{1n}=-\mathrm{i}(W_{0n}/2)\mathrm{e}^{\mathrm{i}\theta_{0n}}$ and W_{0n}, θ_{0n} are the vibration amplitude and phase of the derivative linear system, respectively, and they are determined by the initial conditions.

Substituting Eqs. (23) into Eq. (20) yields the following relation:

$$k_{3n}=\frac{-3W_{0n}^2(n^8\pi^8\gamma_{01}+3k_{03})}{16[c_1+n^4\pi^4\eta_1+2(-\delta_n+\mathrm{i}\omega_{dn})]}=\mathrm{i}\frac{3W_{0n}^2(n^8\pi^8\gamma_{01}+3k_{03})}{32\omega_{dn}} \tag{24}$$

Let the solution of Eq. (19) be $A_n=A_n(T_1)=\alpha_n\,\mathrm{e}^{\mathrm{i}\gamma_n}$, and $\alpha_n=\alpha_n(T_1)$, $\gamma_n=\gamma_n(T_1)$ are undetermined functions of the time scale T_1. Substituting it into Eq. (19) leads to the following equation:

$$\frac{\partial\alpha_n}{\partial T_1}+\mathrm{i}\alpha_n\frac{\partial\gamma_n}{\partial T_1}-k_{3n}\alpha_n^3\,\mathrm{e}^{-2\delta_nT_0}=0 \tag{25}$$

Separating the real part and imaginary part yields

$$\frac{\partial \alpha_n}{\partial T_1} = 0 \tag{26}$$

$$\frac{\partial \gamma_n}{\partial T_1} = k_{3n}^I \alpha_n^2 \, \mathrm{e}^{-2\delta_n T_0} \tag{27}$$

From Eq. (26), it is easy to see that $\alpha_n = \alpha_{0n}$ (constant). Inserting it into Eq. (27) and integrating the obtained equation yields $\gamma_n = k_{3n}^{\mathrm{I}} \alpha_{0n}^2 T_1 \, \mathrm{e}^{-2\delta_n T_0} + \gamma_{0n}$ (γ_{0n} is a constant). Therefore, we have

$$A_n = \alpha_n \, \mathrm{e}^{\mathrm{i}\gamma_n} = \alpha_{0n} \, \mathrm{e}^{\mathrm{i}k_{3n}^{\mathrm{I}} \alpha_{0n}^2 T_1 \, \mathrm{e}^{-2\delta_n T_0} + \mathrm{i}\gamma_{0n}} \tag{28}$$

Substituting Eqs. (28) and (23) into Eq. (15) leads to the following expression:

$$\begin{aligned} w_0(x, T_0, T_1) &= \sum_{n=1}^{N} \alpha_{0n} \phi_n(x) \mathrm{e}^{\mathrm{i}(\omega_{dn} T_0 + k_{3n}^{\mathrm{I}} \alpha_{0n}^2 T_1 \, \mathrm{e}^{-2\delta_n T_0}) + \mathrm{i}\gamma_{0n} - \delta_n T_0} + \mathrm{cc} \\ &= \sum_{n=1}^{N} \alpha_{0n} W_{0n} \, \sin[n\pi x] \sin(\omega_{dn} T_0 + k_{3n}^{\mathrm{I}} \alpha_{0n}^2 T_1 \, \mathrm{e}^{-2\delta_n T_0} + \theta_{0n} + \gamma_{0n}) \mathrm{e}^{-\delta_n T_0} \end{aligned} \tag{29}$$

When the nonlinear characters of materials of the pile and the soil are all weak, the nonlinear terms may be ignored, and the above solution ought to coincide with one of the derivative linear system, therefore, $\alpha_{0n} = 1$, $\alpha_{0n} = 0$. Thus, we have

$$A_n = \alpha_n \, \mathrm{e}^{\mathrm{i}\gamma_n} = \mathrm{e}^{\mathrm{i}k_{3n}^I T_1 \, \mathrm{e}^{-2\delta_n T_0}} \tag{30}$$

$$w_0(x, T_0, T_1) = \sum_{n=1}^{N} W_{0n} \, \sin(n\pi x) \sin[\omega_{dn} T_0 + \frac{3W_{0n}^2 (n^8 \pi^8 \gamma_{01} + 3k_{03})}{32\omega_{dn}} T_1 \, \mathrm{e}^{-2\delta_n T_0} + \theta_{0n}] \mathrm{e}^{-\delta_n T_0} \tag{31}$$

Substituting Eq. (31) into the boundary conditions (13a), (13b) and observing Eq. (7e), the boundary conditions may be simplified as

$$w_1|_{x=0} = 0, \quad \left[\frac{\partial^2 w_1}{\partial x^2} + \eta_1 \frac{\partial^3 w_1}{\partial x^2 \, \partial T_0} \right] \Bigg|_{x=0} = 0 \tag{32a}$$

$$w_1|_{x=1} = 0, \quad \left[\frac{\partial^2 w_1}{\partial x^2} + \eta_1 \frac{\partial^3 w_1}{\partial x^2 \, \partial T_0} \right] \Bigg|_{x=1} = 0 \tag{32b}$$

Substituting Eq. (23) into Eq. (16) leads to

$$\begin{aligned} &\frac{\partial^2 w_1}{\partial T_0^2} + \frac{\partial^4 w_1}{\partial x^4} + p_0 \frac{\partial^2 w_1}{\partial x^2} + k_{11} w_1 + \eta_1 \frac{\partial^5 w_1}{\partial x^4 \, \partial T_0} + c_1 \frac{\partial w_1}{\partial T_0} = \sum_{n=1}^{N} [C_{1n}^2 (-k_{02} \sin^2 n\pi x \\ &\quad + \beta_{01} n^6 \pi^6 \cos 2n\pi x) A_n^2 \, \mathrm{e}^{2\mathrm{i}\omega_{dn} T_0 - 2\delta_n T_0} + C_{1n}^3 \sin n\pi x (-k_{03} \sin^2 n\pi x + 2\gamma_{01} n^8 \pi^8 \cos^2 n\pi x \\ &\quad - \gamma_{01} n^8 \pi^8 \sin^2 n\pi x) A_n^3 \, \mathrm{e}^{3\mathrm{i}\omega_{dn} T_0 - 3\delta_n T_0} + C_{1n} \bar{C}_{1n} (-k_{02} \sin^2 n\pi x + \beta_{01} n^6 \pi^6 \cos 2n\pi x) A_n \bar{A}_n \, \mathrm{e}^{-2\delta_n T_0}] \\ &\quad + \sum_{k=1}^{N-1} \sum_{n=k+1}^{N} \{ C_{1k} C_{1n} [(-2k_{02} - \beta_{01} k^4 n^2 \pi^6 - \beta_{01} k^2 n^4 \pi^6) \sin k\pi x \sin n\pi x \\ &\quad + 2\beta_{01} k^3 n^3 \pi^6 \cos k\pi x \cos n\pi x] A_k A_n \, \mathrm{e}^{\mathrm{i}(\omega_{dk} + \omega_{dn}) T_0 - (\delta_k + \delta_n) T_0} + C_{1k} \bar{C}_{1n} [(-2k_{02} - \beta_{01} k^4 n^2 \pi^6 \\ &\quad - \beta_{01} k^2 n^4 \pi^6) \sin k\pi x \sin n\pi x + 2\beta_{01} k^3 n^3 \pi^6 \cos k\pi x \cos n\pi x] A_k \bar{A}_n \, \mathrm{e}^{\mathrm{i}(\omega_{dk} - \omega_{dn}) T_0 - (\delta_k + \delta_n) T_0} \} \\ &\quad + \sum_{m=1}^{N} \sum_{k=1}^{N-1} \sum_{n=k+1}^{N} \{ 2C_{1m} C_{1k} C_{1n} \sin m\pi x [(-k_{03} - \gamma_{01} m^4 k^2 n^2 \pi^8) \sin k\pi x \sin n\pi x \\ &\quad + 2\gamma_{01} m^2 k^3 n^3 \pi^8 \cos k\pi x \cos n\pi x] A_m A_k A_n \, \mathrm{e}^{\mathrm{i}(\omega_{dm} + \omega_{dk} + \omega_{dn}) T_0 - (\delta_m + \delta_k + \delta_n) T_0} + 2C_{1m} C_{1k} \bar{C}_{1n} \sin m\pi x \\ &\quad \times [(-k_{03} - \gamma_{01} m^4 k^2 n^2 \pi^8) \sin k\pi x \sin n\pi x + 2\gamma_{01} m^2 k^3 n^3 \pi^8 \cos k\pi x \cos n\pi x] A_m A_k \bar{A}_n \\ &\quad \times \mathrm{e}^{\mathrm{i}(\omega_{dm} + \omega_{dk} - \omega_{dn}) T_0 - (\delta_m + \delta_k + \delta_n) T_0} + 2\bar{C}_{1m} C_{1k} C_{1n} \sin m\pi x [(-k_{03} - \gamma_{01} m^4 k^2 n^2 \pi^8) \sin k\pi x \sin n\pi x \end{aligned}$$

$$
\begin{aligned}
&+ 2\gamma_{01} m^2 k^3 n^3 \pi^8 \cos k\pi x \cos n\pi x]\bar{A}_m A_k A_n \,\mathrm{e}^{\mathrm{i}(\omega_{dk}+\omega_{dn}-\omega_{dm})T_0-(\delta_m+\delta_k+\delta_n)T_0} \\
&+ 2\bar{C}_{1m} C_{1k} \bar{C}_{1n} \sin m\pi x[(-k_{03} - \gamma_{01} m^4 k^2 n^2 \pi^8) \sin k\pi x \sin n\pi x \\
&+ 2\gamma_{01} m^2 k^3 n^3 \pi^8 \cos k\pi x \cos n\pi x]\bar{A}_m A_k \bar{A}_n \,\mathrm{e}^{\mathrm{i}(\omega_{dk}-\omega_{dn}-\omega_{dm})T_0-(\delta_m+\delta_k+\delta_n)T_0}\} \\
&+ \sum_{q=1}^{N} \sum_{\substack{n=1 \\ (n\neq q)}}^{N} \{C_{1q} C_{1n}^2 \sin q\pi x[(-k_{03} - \gamma_{01} q^4 n^4 \pi^8)\sin^2 n\pi x + 2\gamma_{01} q^2 n^6 \pi^8 \cos^2 n\pi x] \\
&\times A_q A_n^2 \,\mathrm{e}^{\mathrm{i}(2\omega_{dn}+\omega_{dq})T_0-(2\delta_n+\delta_q)T_0} + 2C_{1q} C_{1n} \bar{C}_{1n} \sin q\pi x[(-k_{03} - \gamma_{01} q^4 n^4 \pi^8)\sin^2 n\pi x \\
&+ 2\gamma_{01} q^2 n^6 \pi^8 \cos^2 n\pi x] A_q A_n \bar{A}_n \,\mathrm{e}^{\mathrm{i}\omega_{dq} T_0-(2\delta_n+\delta_q)T_0} + \bar{C}_{1q} C_{1n}^2 \sin q\pi x[(-k_{03} - \gamma_{01} q^4 n^4 \pi^8)\sin^2 n\pi x \\
&+ 2\gamma_{01} q^2 n^6 \pi^8 \cos^2 n\pi x]\bar{A}_q A_n^2 \,\mathrm{e}^{\mathrm{i}(2\omega_{dn}-\omega_{dq})T_0-(2\delta_n+\delta_q)T_0}\} + \mathrm{cc}
\end{aligned} \tag{33}
$$

It can be seen that there are no secular terms in Eq. (33), so its solution is given as

$$
\begin{aligned}
w_1(x, T_0, T_1) &= \sum_{n=1}^{N} [\phi_n(x) A_n^* \,\mathrm{e}^{\mathrm{i}\omega_{dn} T_0 - \delta_n T_0} + a_{1n}(x) A_n^2 \,\mathrm{e}^{2\mathrm{i}\omega_{dn} T_0 - 2\delta_n T_0} + b_{1n}(x) A_n^3 \,\mathrm{e}^{3\mathrm{i}\omega_{dn} T_0 - 3\delta_n T_0} \\
&+ a_{0n}(x) A_n \bar{A}_n \,\mathrm{e}^{-2\delta_n T_0}] + \sum_{k=1}^{N-1} \sum_{n=k+1}^{N} [d_{1kn}(x) A_k A_n \,\mathrm{e}^{\mathrm{i}(\omega_{dk}+\omega_{dn})T_0-(\delta_k+\delta_n)T_0} \\
&+ d_{2kn}(x) A_k \bar{A}_n \,\mathrm{e}^{\mathrm{i}(\omega_{dk}-\omega_{dn})T_0-(\delta_k+\delta_n)T_0}] + \sum_{m=1}^{N} \sum_{k=1}^{N-1} \sum_{n=k+1}^{N} [f_{1mkn}(x) A_m A_k A_n \,\mathrm{e}^{\mathrm{i}(\omega_{dm}+\omega_{dk}+\omega_{dn})T_0-(\delta_m+\delta_k+\delta_n)T_0} \\
&+ f_{2mkn}(x) A_m A_k \bar{A}_n \,\mathrm{e}^{\mathrm{i}(\omega_{dm}+\omega_{dk}-\omega_{dn})T_0-(\delta_m+\delta_k+\delta_n)T_0} + f_{3mkn}(x) \bar{A}_m A_k A_n \,\mathrm{e}^{\mathrm{i}(\omega_{dk}+\omega_{dn}-\omega_{dm})T_0-(\delta_m+\delta_k+\delta_n)T_0} \\
&+ f_{4mkn}(x) \bar{A}_m A_k \bar{A}_n \,\mathrm{e}^{\mathrm{i}(\omega_{dk}-\omega_{dn}-\omega_{dm})T_0-(\delta_m+\delta_k+\delta_n)T_0}] + \sum_{q=1}^{N} \sum_{\substack{n=1 \\ (n\neq q)}}^{N} [g_{1qn}(x) A_q A_n^2 \,\mathrm{e}^{\mathrm{i}(2\omega_{dn}+\omega_{dq})T_0-(2\delta_n+\delta_q)T_0} \\
&+ g_{2qn}(x) A_q A_n \bar{A}_n \,\mathrm{e}^{\mathrm{i}\omega_{dq} T_0-(2\delta_n+\delta_q)T_0} + g_{3qn}(x) \bar{A}_q A_n^2 \,\mathrm{e}^{\mathrm{i}(2\omega_{dn}-\omega_{dq})T_0-(2\delta_n+\delta_q)T_0}] + \mathrm{cc}
\end{aligned} \tag{34}
$$

where A_n^* is a complex number. Substituting Eq. (34) into Eq. (33) yields the following equations:

$$
\begin{aligned}
&[k_{11} + (2\mathrm{i}\omega_{dn} - 2\delta_n)^2 + c_1(2\mathrm{i}\omega_{dn} - 2\delta_n)]a_{1n} + p_0 a''_{1n} + [1 + \eta_1(2\mathrm{i}\omega_{dn} - 2\delta_n)]a_{1n}^{(4)} \\
&\quad = C_{1n}^2(-k_{02} \sin^2 n\pi x + \beta_{01} n^6 \pi^6 \cos 2n\pi x)
\end{aligned} \tag{35a}
$$

$$
\begin{aligned}
&[k_{11} + (3\mathrm{i}\omega_{dn} - 3\delta_n)^2 + c_1(3\mathrm{i}\omega_{dn} - 3\delta_n)]b_{1n} + p_0 b''_{1n} + [1 + \eta_1(3\mathrm{i}\omega_{dn} - 3\delta_n)]b_{1n}^{(4)} \\
&\quad = C_{1n}^3 \sin n\pi x[-k_{03}\sin^2 n\pi x + 2\gamma_{01} n^8 \pi^8(1 - \sin^2 n\pi x) - \gamma_{01} n^8 \pi^8 \sin^2 n\pi x]
\end{aligned} \tag{35b}
$$

$$
\begin{aligned}
&(k_{11} + 4\delta_n^2 - 2\delta_n c_1)a_{0n} + p_0 a''_{0n} + (1 - 2\delta_n \eta_1)a_{0n}^{(4)} \\
&\quad = C_{1n} \bar{C}_{1n}(-k_{02} \sin^2 n\pi x + \beta_{01} n^6 \pi^6 \cos 2n\pi x)
\end{aligned} \tag{35c}
$$

The solutions of the above equations can be obtained as

$$
a_{1n} = \alpha_{1n} \,\mathrm{e}^{-q_1 x} + \alpha_{2n} \,\mathrm{e}^{q_1 x} + \alpha_{3n} \,\mathrm{e}^{-q_2 x} + \alpha_{4n} \,\mathrm{e}^{q_2 x} + A_{1n} \sin^2 n\pi x + B_{1n} \cos 2n\pi x \tag{36a}
$$

$$
b_{1n} = \alpha_{5n} \,\mathrm{e}^{-q_3 x} + \alpha_{6n} \,\mathrm{e}^{q_3 x} + \alpha_{7n} \,\mathrm{e}^{-q_4 x} + \alpha_{8n} \,\mathrm{e}^{q_4 x} + E_{1n} \sin n\pi x + F_{1n} \sin^3 n\pi x \tag{36b}
$$

$$
a_{0n} = \alpha_{01n} \,\mathrm{e}^{-q_5 x} + \alpha_{02n} \,\mathrm{e}^{q_5 x} + \alpha_{03n} \,\mathrm{e}^{-q_6 x} + \alpha_{04n} \,\mathrm{e}^{q_6 x} + A_{00n} \sin^2 n\pi x + B_{00n} \cos 2n\pi x \tag{36c}
$$

where the coefficients are given as

$$
\begin{aligned}
A_{1n} &= \frac{-k_{02}C_{1n}^2}{m_1}, \quad B_{1n} = \frac{C_{1n}^2(2k_{02}n^2\pi^2p_0 - 8k_{02}n^4\pi^4m_2 + \beta_{01}n^6\pi^6m_1)}{m_1(m_1 + 16n^4\pi^4m_2 - 4n^2\pi^2p_0)}\\
E_{1n} &= \frac{2C_{1n}^3[3k_{03}n^2\pi^2p_0 - 30k_{03}n^4\pi^4m_4 + \gamma_{01}n^8\pi^8m_3 - 9\gamma_{01}n^{12}\pi^{12}m_4]}{9n^4\pi^4p_0^2 + m_3^2 - 10n^2\pi^2m_3p_0 - 90n^6\pi^6m_4p_0 + 82n^4\pi^4m_3m_4 + 81n^8\pi^8m_4^2}\\
F_{1n} &= \frac{C_{1n}^3(3\gamma_{01}n^8\pi^8 + k_{03})}{m_3 - 9n^2\pi^2p_0 + 81n^4\pi^4m_4}\\
A_{00n} &= \frac{-C_{1n}\bar{C}_{1n}k_{02}}{m_5}, \quad B_{00n} = \frac{C_{1n}\bar{C}_{1n}(2k_{02}n^2\pi^2p_0 - 8k_{02}n^4\pi^4m_6 + \beta_{01}n^6\pi^6m_5)}{m_5(m_5 + 16n^4\pi^4m_6 - 4n^2\pi^2p_0)}\\
q_1 &= \sqrt{-\frac{p_0 + \sqrt{-4m_1m_2 + p_0^2}}{2m_2}}, \quad q_2 = \sqrt{\frac{-p_0 + \sqrt{-4m_1m_2 + p_0^2}}{2m_2}}\\
q_3 &= \sqrt{-\frac{p_0 + \sqrt{-4m_3m_4 + p_0^2}}{2m_4}}, \quad q_4 = \sqrt{\frac{-p_0 + \sqrt{-4m_3m_4 + p_0^2}}{2m_4}}\\
q_5 &= \sqrt{-\frac{p_0 + \sqrt{-4m_5m_6 + p_0^2}}{2m_6}}, \quad q_6 = \sqrt{\frac{-p_0 + \sqrt{-4m_5m_6 + p_0^2}}{2m_6}}\\
m_1 &= k_{11} + (2\mathrm{i}\omega_{dn} - 2\delta_n)^2 + c_1(2\mathrm{i}\omega_{dn} - 2\delta_n), \quad m_2 = 1 + \eta_1(2\mathrm{i}\omega_{dn} - 2\delta_n)\\
m_3 &= k_{11} + (3\mathrm{i}\omega_{dn} - 3\delta_n)^2 + c_1(3\mathrm{i}\omega_{dn} - 3\delta_n), \quad m_4 = 1 + \eta_1(3\mathrm{i}\omega_{dn} - 3\delta_n)\\
m_5 &= (k_{11} + 4\delta_n^2 - 2\delta_nc_1), \quad m_6 = (1 - 2\delta_n\eta_1)
\end{aligned}
$$

From the boundary conditions (32a), (32b) and the initial conditions (14), we have

$$
\begin{aligned}
&a_{1n}|_{x=0} = 0, \quad \left.\frac{\partial^2 a_{1n}}{\partial x^2}\right|_{x=1} = 0, \quad b_{1n}|_{x=0} = 0, \quad \left.\frac{\partial^2 b_{1n}}{\partial x^2}\right|_{x=1} = 0, \quad a_{0n}|_{x=0} = 0, \quad \left.\frac{\partial^2 a_{0n}}{\partial x^2}\right|_{x=1} = 0\\
&a_{1n}|_{x=1} = 0, \quad \left.\frac{\partial^2 a_{1n}}{\partial x^2}\right|_{x=0} = 0, \quad b_{1n}|_{x=1} = 0, \quad \left.\frac{\partial^2 b_{1n}}{\partial x^2}\right|_{x=0} = 0, \quad a_{0n}|_{x=1} = 0, \quad \left.\frac{\partial^2 a_{0n}}{\partial x^2}\right|_{x=0} = 0
\end{aligned} \tag{37}
$$

Substituting Eq. (36) into Eq. (37) leads to $\alpha_{1n}, \alpha_{2n}, \alpha_{3n}, \alpha_{4n}, \ldots$ as

$$
\begin{aligned}
\alpha_{1n} &= \frac{\mathrm{e}^{q_1}(-2n^2\pi^2A_{1n} + 4n^2\pi^2B_{1n} + B_{1n}q_2^2)}{(1 + \mathrm{e}^{q_1})(q_1^2 - q_2^2)}, \ \alpha_{2n} = \frac{-2n^2\pi^2A_{1n} + 4n^2\pi^2B_{1n} + B_{1n}q_2^2}{(1 + \mathrm{e}^{q_1})(q_1^2 - q_2^2)}\\
\alpha_{3n} &= \frac{\mathrm{e}^{q_2}(-2n^2\pi^2A_{1n} + 4n^2\pi^2B_{1n} + B_{1n}q_1^2)}{(1 + \mathrm{e}^{q_2})(-q_1^2 + q_2^2)}, \ \alpha_{4n} = \frac{-2n^2\pi^2A_{1n} + 4n^2\pi^2B_{1n} + B_{1n}q_1^2}{(1 + \mathrm{e}^{q_2})(-q_1^2 + q_2^2)}\\
\alpha_{5n} &= 0, \ \alpha_{6n} = 0, \ \alpha_{7n} = 0, \ \alpha_{8n} = 0\\
\alpha_{01n} &= \frac{\mathrm{e}^{q_5}(-2n^2\pi^2A_{00n} + 4n^2\pi^2B_{00n} + B_{00n}q_6^2)}{(1 + e^{q_5})(q_5^2 - q_6^2)}, \ \alpha_{02n} = \frac{-2n^2\pi^2A_{00n} + 4n^2\pi^2B_{00n} + B_{00n}q_6^2}{(1 + \mathrm{e}^{q_5})(q_5^2 - q_6^2)}\\
\alpha_{03n} &= \frac{\mathrm{e}^{q_6}(-2n^2\pi^2A_{00n} + 4n^2\pi^2B_{00n} + B_{00n}q_5^2)}{(1 + \mathrm{e}^{q_6})(-q_5^2 + q_6^2)}, \ \alpha_{04n} = \frac{-2n^2\pi^2A_{00n} + 4n^2\pi^2B_{00n} + B_{00n}q_5^2}{(1 + \mathrm{e}^{q_6})(-q_5^2 + q_6^2)}
\end{aligned}
$$

Similarly, substituting Eq. (30) into Eq. (34) and setting $A_n^* = A_{0n}^* \mathrm{e}^{\mathrm{i}k_{3n}^{\mathrm{I}} A_{0n}^{*2} T_1\,\mathrm{e}^{-2\delta_n T_0}+\mathrm{i}\theta_{0n}^*}$ yields

$$
\begin{aligned}
w_1(x,T_0,T_1) = {} & \sum_{n=1}^{N}[\phi_n(x)A_{0n}^*\,\mathrm{e}^{\mathrm{i}(\omega_{dn}T_0+k_{3n}^{\mathrm{I}}A_{0n}^{*2}T_1\,\mathrm{e}^{-2\delta_nT_0})+\mathrm{i}\theta_{0n}^*-\delta_nT_0} + a_{1n}(x)\mathrm{e}^{2\mathrm{i}(\omega_{dn}T_0+k_{3n}^{\mathrm{I}}T_1\,\mathrm{e}^{-2\delta_nT_0})-2\delta_nT_0} \\
& + a_{0n}(x)\mathrm{e}^{-2\delta_nT_0} + b_{1n}(x)\mathrm{e}^{3\mathrm{i}(\omega_{dn}T_0+k_{3n}^{\mathrm{I}}T_1\,\mathrm{e}^{-2\delta_nT_0})-3\delta_nT_0}] \\
& + \sum_{k=1}^{N-1}\sum_{n=k+1}^{N}\Big[d_{1kn}(x)\mathrm{e}^{\mathrm{i}(\omega_{dk}T_0+\omega_{dn}T_0+k_{3k}^{\mathrm{I}}T_1\,\mathrm{e}^{-2\delta_kT_0}+k_{3n}^{\mathrm{I}}T_1\,\mathrm{e}^{-2\delta_nT_0})-(\delta_k+\delta_n)T_0} \\
& + d_{2kn}(x)\mathrm{e}^{\mathrm{i}(\omega_{dk}T_0-\omega_{dn}T_0+k_{3k}^{\mathrm{I}}T_1\,\mathrm{e}^{-2\delta_kT_0}-k_{3n}^{\mathrm{I}}T_1\,\mathrm{e}^{-2\delta_nT_0})-(\delta_k+\delta_n)T_0}\Big] \\
& + \sum_{m=1}^{N}\sum_{k=1}^{N-1}\sum_{n=k+1}^{N}\Big[f_{1mkn}(x)\mathrm{e}^{\mathrm{i}(\omega_{dm}T_0+\omega_{dk}T_0+\omega_{dn}T_0+k_{3m}^{\mathrm{I}}T_1\,\mathrm{e}^{-2\delta_mT_0}+k_{3k}^{\mathrm{I}}T_1\,\mathrm{e}^{-2\delta_kT_0}+k_{3n}^{\mathrm{I}}T_1\,\mathrm{e}^{-2\delta_nT_0})-(\delta_m+\delta_k+\delta_n)T_0} \\
& + f_{2mkn}(x)\mathrm{e}^{\mathrm{i}(\omega_{dm}T_0+\omega_{dk}T_0-\omega_{dn}T_0+k_{3m}^{\mathrm{I}}T_1\,\mathrm{e}^{-2\delta_mT_0}+k_{3k}^{\mathrm{I}}T_1\,\mathrm{e}^{-2\delta_kT_0}-k_{3n}^{\mathrm{I}}T_1\,\mathrm{e}^{-2\delta_nT_0})-(\delta_m+\delta_k+\delta_n)T_0} \\
& + f_{3mkn}(x)\mathrm{e}^{\mathrm{i}(\omega_{dk}T_0+\omega_{dn}T_0-\omega_{dm}T_0+k_{3k}^{\mathrm{I}}T_1\,\mathrm{e}^{-2\delta_kT_0}+k_{3n}^{\mathrm{I}}T_1\,\mathrm{e}^{-2\delta_nT_0}-k_{3m}^{\mathrm{I}}T_1\,\mathrm{e}^{-2\delta_mT_0})-(\delta_m+\delta_k+\delta_n)T_0} \\
& + f_{4mkn}(x)\mathrm{e}^{\mathrm{i}(\omega_{dk}T_0-\omega_{dn}T_0-\omega_{dm}T_0+k_{3k}^{\mathrm{I}}T_1\,\mathrm{e}^{-2\delta_kT_0}-k_{3m}^{\mathrm{I}}T_1\,\mathrm{e}^{-2\delta_mT_0}-k_{3n}^{\mathrm{I}}T_1\,\mathrm{e}^{-2\delta_nT_0})-(\delta_m+\delta_k+\delta_n)T_0}\Big] \\
& + \sum_{q=1}^{N}\sum_{\substack{n=1\\(n\neq q)}}^{N}[g_{1qn}(x)\mathrm{e}^{\mathrm{i}(2\omega_{dn}T_0+\omega_{dq}T_0+2k_{3n}^{\mathrm{I}}T_1\,\mathrm{e}^{-2\delta_nT_0}+k_{3q}^{\mathrm{I}}T_1\,\mathrm{e}^{-2\delta_qT_0})-(2\delta_n+\delta_q)T_0} \\
& + g_{2qn}(x)\mathrm{e}^{\mathrm{i}(\omega_{dq}T_0+k_{3q}^{\mathrm{I}}T_1\,\mathrm{e}^{-2\delta_qT_0})-(2\delta_n+\delta_q)T_0} + g_{3qn}(x)\mathrm{e}^{\mathrm{i}(2\omega_{dn}T_0-\omega_{dq}T_0+2k_{3n}^{\mathrm{I}}T_1\,\mathrm{e}^{-2\delta_nT_0}-k_{3q}^{\mathrm{I}}T_1\,\mathrm{e}^{-2\delta_qT_0})} \\
& - (2\delta_n+\delta_q)T_0] + \mathrm{cc} \\
= {} & \sum_{n=1}^{N}\{A_{0n}^*W_{0n}\,\sin(n\pi x)\sin[\omega_{dn}T_0 + \frac{3W_{0n}^2A_{0n}^{*2}(n^8\pi^8\gamma_{01}+3k_{03})}{32\omega_{dn}}T_1\,\mathrm{e}^{-2\delta_nT_0} + \theta_{0n} + \theta_{0n}^*]\mathrm{e}^{-\delta_nT_0} \\
& - 2\sqrt{(a_{1n}^R)^2+(a_{1n}^{\mathrm{I}})^2}\,\sin[2(\omega_{dn}T_0 + \frac{3W_{0n}^2(n^8\pi^8\gamma_{01}+3k_{03})}{32\omega_{dn}}T_1\,\mathrm{e}^{-2\delta_nT_0}) - \theta_{1n}]\mathrm{e}^{-2\delta_nT_0} \\
& - 2\sqrt{(b_{1n}^R)^2+(b_{1n}^{\mathrm{I}})^2}\,\sin[3(\omega_{dn}T_0 + \frac{3W_{0n}^2(n^8\pi^8\gamma_{01}+3k_{03})}{32\omega_{dn}}T_1\,\mathrm{e}^{-2\delta_nT_0}) - \theta_{2n}]\mathrm{e}^{-3\delta_nT_0} \\
& + 2a_{0n}^R\,\mathrm{e}^{-2\delta_nT_0}\} + \sum_{k=1}^{N-1}\sum_{n=k+1}^{N}\{d_{1kn}^*\,\sin[(\omega_{dk}T_0+\omega_{dn}T_0+k_{3k}^{\mathrm{I}}T_1\,\mathrm{e}^{-2\delta_kT_0}+k_{3n}^{\mathrm{I}}T_1\,\mathrm{e}^{-2\delta_nT_0}) \\
& - \theta_{1kn}]\mathrm{e}^{-(\delta_k+\delta_n)T_0} + d_{2kn}^*\sin[(\omega_{dk}T_0-\omega_{dn}T_0+k_{3k}^{\mathrm{I}}T_1\,\mathrm{e}^{-2\delta_kT_0}-k_{3n}^{\mathrm{I}}T_1\,\mathrm{e}^{-2\delta_nT_0}) \\
& - \theta_{2kn}]\mathrm{e}^{-(\delta_k+\delta_n)T_0}\} + \sum_{m=1}^{N}\sum_{k=1}^{N-1}\sum_{n=k+1}^{N}\{f_{1mkn}^*\,\sin[(\omega_{dm}T_0+\omega_{dk}T_0+\omega_{dn}T_0+k_{3m}^{\mathrm{I}}T_1\,\mathrm{e}^{-2\delta_mT_0} \\
& + k_{3k}^{\mathrm{I}}T_1\,\mathrm{e}^{-2\delta_kT_0}+k_{3n}^{\mathrm{I}}T_1\,\mathrm{e}^{-2\delta_nT_0}) - \theta_{1mkn}]\mathrm{e}^{-(\delta_m+\delta_k+\delta_n)T_0} + f_{2mkn}^*\,\sin[(\omega_{dm}T_0+\omega_{dk}T_0-\omega_{dn}T_0 \\
& + k_{3m}^{\mathrm{I}}T_1\,\mathrm{e}^{-2\delta_mT_0}+k_{3k}^{\mathrm{I}}T_1\,\mathrm{e}^{-2\delta_kT_0}-k_{3n}^{\mathrm{I}}T_1\,\mathrm{e}^{-2\delta_nT_0}) - \theta_{2mkn}]\mathrm{e}^{-(\delta_m+\delta_k+\delta_n)T_0} \\
& + f_{3mkn}^*\,\sin[(\omega_{dk}T_0+\omega_{dn}T_0-\omega_{dm}T_0+k_{3k}^{\mathrm{I}}T_1\,\mathrm{e}^{-2\delta_kT_0}+k_{3n}^{\mathrm{I}}T_1\,\mathrm{e}^{-2\delta_nT_0}-k_{3m}^{\mathrm{I}}T_1\,\mathrm{e}^{-2\delta_mT_0}) \\
& - \theta_{3mkn}]\mathrm{e}^{-(\delta_m+\delta_k+\delta_n)T_0} + f_{4mkn}^*\,\sin[(\omega_{dk}T_0-\omega_{dn}T_0-\omega_{dm}T_0 \\
& + k_{3k}^{\mathrm{I}}T_1\,\mathrm{e}^{-2\delta_kT_0}-k_{3m}^{\mathrm{I}}T_1\,\mathrm{e}^{-2\delta_mT_0}-k_{3n}^{\mathrm{I}}T_1\,\mathrm{e}^{-2\delta_nT_0}) - \theta_{4mkn}]\mathrm{e}^{-(\delta_m+\delta_k+\delta_n)T_0}\} \\
& + \sum_{q=1}^{N}\sum_{\substack{n=1\\(n\neq q)}}^{N}\{g_{1qn}^*\,\sin[(2\omega_{dn}T_0+\omega_{dq}T_0+2k_{3n}^{\mathrm{I}}T_1\,\mathrm{e}^{-2\delta_nT_0}+k_{3q}^{\mathrm{I}}T_1\,\mathrm{e}^{-2\delta_qT_0}) - \theta_{1qn}]\mathrm{e}^{-(2\delta_n+\delta_q)T_0} \\
& + g_{2qn}^*\,\sin[(\omega_{dq}T_0+k_{3q}^{\mathrm{I}}T_1\,\mathrm{e}^{-2\delta_qT_0}) - \theta_{2qn}]\mathrm{e}^{-(2\delta_n+\delta_q)T_0} \\
& + g_{3qn}^*\,\sin[(2\omega_{dn}T_0-\omega_{dq}T_0+2k_{3n}^{E}T_1\,\mathrm{e}^{-2\delta_nT_0}-k_{3q}^{\mathrm{I}}T_1\,\mathrm{e}^{-2\delta_qT_0}) - \theta_{3qn}]\mathrm{e}^{-(2\delta_n+\delta_q)T_0}\}
\end{aligned} \tag{38}
$$

where $\sin\theta_{1n} = a_{1n}^R / \sqrt{(a_{1n}^R)^2 + (a_{1n}^{\mathrm{I}})^2}$, $\sin\theta_{2n} = b_{1n}^R / \sqrt{(b_{1n}^R)^2 + (b_{1n}^{\mathrm{I}})^2}$ and also A_{0n}^*, θ_{0n}^* are the real numbers determined by the initial conditions. Substituting Eqs. (38) and (31) into the initial conditions (14) leads to

$$A_{0n}^* W_{0n} \sin(n\pi x)\sin(\theta_{0n} + \theta_{0n}^*) + 2a_{1n}^R + 2b_{1n}^R + 2a_{0n}^R + \cdots = 0 \tag{39a}$$

$$\begin{aligned} &A_{0n}^* W_{0n} \sin(n\pi x)[\omega_{dn} \cos(\theta_{0n} + \theta_{0n}^*) - \delta_n \sin(\theta_{0n} + \theta_{0n}^*)] - 4a_{1n}^{\mathrm{I}}\omega_{dn} - 4\delta_n a_{1n}^R - 6b_{1n}^{\mathrm{I}}\omega_{dn} - 6\delta_n b_{1n}^R \\ &- 4a_{0n}^R\delta_n + \sin(n\pi x)\frac{3W_{0n}^3(n^8\pi^8\gamma_{01} + 3k_{03})}{32\omega_{dn}}\cos\theta_{0n} + \cdots = 0 \end{aligned} \tag{39b}$$

When $N = 1$, $n = 1$, the omitted part in Eq. (39) is zero. And A_{0n}^*, θ_{0n}^* in Eq. (39) can be numerically obtained. Substituting Eqs. (31) and (38) into Eq. (9) yields

$$\begin{aligned} w(x,t,\varepsilon) = &\sum_{n=1}^{N}\{W_{0n}\sin(n\pi x)\sin[\omega_{dn}T_0 + \frac{3W_{0n}^2(n^8\pi^8\gamma_{01} + 3k_{03})}{32\omega_{dn}}T_1\,\mathrm{e}^{-2\delta_n T_0} + \theta_{0n}]\mathrm{e}^{-\delta_n T_0} \\ &+ \varepsilon A_{0n}^* W_{0n}\sin(n\pi x)\sin[\omega_{dn}T_0 + \frac{3W_{0n}^2 A_{0n}^{*2}(n^8\pi^8\gamma_{01} + 3k_{03})}{32\omega_{dn}}T_1\,\mathrm{e}^{-2\delta_n T_0} + \theta_{0n} + \theta_{0n}^*]\mathrm{e}^{-\delta_n T_0} \\ &- 2\varepsilon\sqrt{(a_{1n}^{\mathrm{R}})^2 + (a_{1n}^{\mathrm{I}})^2}\,\sin[2(\omega_{dn}T_0 + \frac{3W_{0n}^2(n^8\pi^8\gamma_{01} + 3k_{03})}{32\omega_{dn}}T_1\,\mathrm{e}^{-2\delta_n T_0}) - \theta_{1n}]\mathrm{e}^{-2\delta_n T_0} \\ &- 2\varepsilon\sqrt{(b_{1n}^{\mathrm{R}})^2 + (b_{1n}^{\mathrm{I}})^2}\,\sin[3(\omega_{dn}T_0 + \frac{3W_{0n}^2(n^8\pi^8\gamma_{01} + 3k_{03})}{32\omega_{dn}}T_1\,\mathrm{e}^{-2\delta_n T_0}) - \theta_{2n}]\mathrm{e}^{-3\delta_n T_0} \\ &+ 2\varepsilon a_{0n}^{\mathrm{R}}\,\mathrm{e}^{-2\delta_n T_0}\} + \sum_{k=1}^{N-1}\sum_{n=k+1}^{N}\{d_{1kn}^*\sin[(\omega_{dk}T_0 + \omega_{dn}T_0 + k_{3k}^{\mathrm{I}}T_1\,\mathrm{e}^{-2\delta_k T_0} + k_{3n}^{\mathrm{I}}T_1\,\mathrm{e}^{-2\delta_n T_0}) \\ &- \theta_{1kn}]\mathrm{e}^{-(\delta_k+\delta_n)T_0} + d_{2kn}^*\sin[(\omega_{dk}T_0 - \omega_{dn}T_0 + k_{3k}^{\mathrm{I}}T_1\,\mathrm{e}^{-2\delta_k T_0} - k_{3n}^{\mathrm{I}}T_1\,\mathrm{e}^{-2\delta_n T_0}) - \theta_{2kn}]\mathrm{e}^{-(\delta_k+\delta_n)T_0}\} \\ &+ \sum_{m=1}^{N}\sum_{k=1}^{N-1}\sum_{n=k+1}^{N}\{f_{1mkn}^*\sin[(\omega_{dm}T_0 + \omega_{dk}T_0 + \omega_{dn}T_0 + k_{3m}^{\mathrm{I}}T_1\,\mathrm{e}^{-2\delta_m T_0} + k_{3k}^{\mathrm{I}}T_1\,\mathrm{e}^{-2\delta_k T_0} + k_{3n}^{\mathrm{I}}T_1\,\mathrm{e}^{-2\delta_n T_0}) \\ &- \theta_{1mkn}]\mathrm{e}^{-(\delta_m+\delta_k+\delta_n)T_0} + f_{2mkn}^*\sin[(\omega_{dm}T_0 + \omega_{dk}T_0 - \omega_{dn}T_0 + k_{3m}^{\mathrm{I}}T_1\,\mathrm{e}^{-2\delta_m T_0} + k_{3k}^{\mathrm{I}}T_1\,\mathrm{e}^{-2\delta_k T_0} \\ &- k_{3n}^{\mathrm{I}}T_1\,\mathrm{e}^{-2\delta_n T_0}) - \theta_{2mkn}]\mathrm{e}^{-(\delta_m+\delta_k+\delta_n)T_0} + f_{3mkn}^*\sin[(\omega_{dk}T_0 + \omega_{dn}T_0 - \omega_{dm}T_0 + k_{3k}^{\mathrm{I}}T_1\,\mathrm{e}^{-2\delta_k T_0} \\ &+ k_{3n}^{\mathrm{I}}T_1\,\mathrm{e}^{-2\delta_n T_0} - k_{3m}^{\mathrm{I}}T_1\,\mathrm{e}^{-2\delta_m T_0}) - \theta_{3mkn}]\mathrm{e}^{-(\delta_m+\delta_k+\delta_n)T_0} + f_{4mkn}^*\sin[(\omega_{dk}T_0 - \omega_{dn}T_0 - \omega_{dm}T_0 \\ &+ k_{3k}^{\mathrm{I}}T_1\,\mathrm{e}^{-2\delta_k T_0} - k_{3m}^{\mathrm{I}}T_1\,\mathrm{e}^{-2\delta_m T_0} - k_{3n}^{\mathrm{I}}T_1\,\mathrm{e}^{-2\delta_n T_0}) - \theta_{4mkn}]\mathrm{e}^{-(\delta_m+\delta_k+\delta_n)T_0}\} \\ &+ \sum_{q=1}^{N}\sum_{\substack{n=1\\(n\neq q)}}^{N}[g_{1qn}^*\sin[(2\omega_{dn}T_0 + \omega_{dq}T_0 + 2k_{3n}^{\mathrm{I}}T_1\,\mathrm{e}^{-2\delta_n T_0} + k_{3q}^{\mathrm{I}}T_1\,\mathrm{e}^{-2\delta_q T_0}) - \theta_{1qn}]\mathrm{e}^{-(2\delta_n+\delta_q)T_0} \\ &+ g_{2qn}^*\sin[(\omega_{dq}T_0 + k_{3q}^{\mathrm{I}}T_1\,\mathrm{e}^{-2\delta_q T_0}) - \theta_{2qn}]\mathrm{e}^{-(2\delta_n+\delta_q)T_0} + g_{3qn}^*\sin[(2\omega_{dn}T_0 - \omega_{dq}T_0 \\ &+ 2k_{3n}^{\mathrm{I}}T_1\,\mathrm{e}^{-2\delta_n T_0} - k_{3q}^{\mathrm{I}}T_1\,\mathrm{e}^{-2\delta_q T_0}) - \theta_{3qn}]\mathrm{e}^{-(2\delta_n+\delta_q)T_0}\} \end{aligned} \tag{40}$$

Substituting $T_0 = t$, $T_1 = \varepsilon t$ into the above expression yields the displacement response as follows:

$$\begin{aligned} w(x,t,\varepsilon) = &\sum_{n=1}^{N}\{W_{0n}\sin(n\pi x)\sin(\omega_{dn}^{\mathrm{NL}}t + \theta_{0n})\mathrm{e}^{-\delta_n t} - 2\varepsilon\sqrt{(a_{1n}^{\mathrm{R}})^2 + (a_{1n}^{\mathrm{I}})^2}\sin(2\omega_{dn}^{\mathrm{NL}}t - \theta_{1n})\mathrm{e}^{-2\delta_n t} \\ &- 2\varepsilon\sqrt{(b_{1n}^{\mathrm{R}})^2 + (b_{1n}^{\mathrm{I}})^2}\sin(3\omega_{dn}^{\mathrm{NL}}t - \theta_{2n})\mathrm{e}^{-3\delta_n t} + 2\varepsilon a_{0n}^{\mathrm{R}}\,\mathrm{e}^{-2\delta_n t} \\ &+ \varepsilon A_{0n}^* W_{0n}\sin(n\pi x)\sin[\omega_{dn}^{\mathrm{NL}}(A_{0n}^{*2} + (1 - A_{0n}^{*2})\omega_{dn}/\omega_{dn}^{\mathrm{NL}})t + \theta_{0n} + \theta_{0n}^*]\mathrm{e}^{-\delta_n t}\} \\ &+ \varepsilon\sum_{k=1}^{N-1}\sum_{n=k+1}^{N}\{d_{1kn}^*\sin[(\omega_{dk}^{\mathrm{NL}} + \omega_{dn}^{\mathrm{NL}})t - \theta_{1kn}]\mathrm{e}^{-(\delta_k+\delta_n)t} + d_{2kn}^*\sin[(\omega_{dk}^{\mathrm{NL}} - \omega_{dn}^{\mathrm{NL}})t \end{aligned}$$

$$- \theta_{2kn}]e^{-(\delta_k+\delta_n)t}\} + \varepsilon \sum_{m=1}^{N}\sum_{k=1}^{N-1}\sum_{n=k+1}^{N}\{f^*_{1mkn}\sin[(\omega^{NL}_{dm}+\omega^{NL}_{dk}+\omega^{NL}_{dn})t-\theta_{1mkn}]e^{-(\delta_m+\delta_k+\delta_n)t}$$
$$+ f^*_{2mkn}\sin[(\omega^{NL}_{dm}+\omega^{NL}_{dk}-\omega^{NL}_{dn})t-\theta_{2mkn}]e^{-(\delta_m+\delta_k+\delta_n)t} + f^*_{3mkn}\sin[(\omega^{NL}_{dk}+\omega^{NL}_{dn}-\omega^{NL}_{dm})t$$
$$- \theta_{3mkn}]e^{-(\delta_m+\delta_k+\delta_n)t} + f^*_{4mkn}\sin[(\omega^{NL}_{dk}-\omega^{NL}_{dn}-\omega^{NL}_{dm})t-\theta_{4mkn}]e^{-(\delta_m+\delta_k+\delta_n)t}\}$$
$$+ \varepsilon\sum_{q=1}^{N}\sum_{\substack{n=1\\(n\neq q)}}^{N}[g^*_{1qn}\sin[(2\omega^{NL}_{dn}+\omega^{NL}_{dq})t-\theta_{1qn}]e^{-(2\delta_n+\delta_q)t}$$
$$+ g^*_{2qn}\sin[\omega^{NL}_{dq}t-\theta_{2qn}]e^{-(2\delta_n+\delta_q)t} + g^*_{3qn}\sin[(2\omega^{NL}_{dn}-\omega^{NL}_{dq})t-\theta_{3qn}]e^{-(2\delta_n+\delta_q)t}\} \tag{41}$$

The nth-order main frequency ω^{NL}_{dn} of the nonlinear system is given as

$$\omega^{NL}_{dn} = \omega_{dn} + \frac{3Q_3W^2_{0n}}{32\omega_{dn}}e^{-2\delta_n t} \tag{42}$$

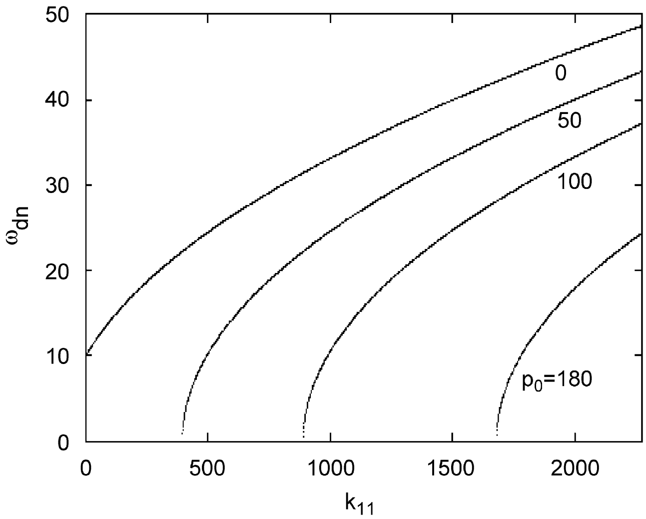

Fig. 1. Curves of frequency vs. stiffness of soil for different values of load p_0 ($n = 1$).

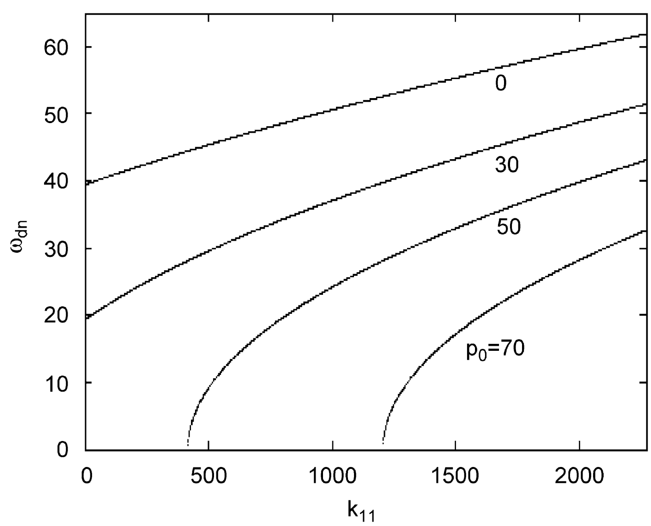

Fig. 2. Curves of frequency vs. stiffness of soil for different values of load p_0 ($n = 2$).

In Eq. (42), Q_3 is a quantity describing the nonlinear characteristics of pile and soil defined as

$$Q_3 = \varepsilon(n^8\pi^8\gamma_{01} + 3k_{03}) = n^8\pi^8\gamma_1 + 3k_{33} \tag{43}$$

From Eq. (42), one can see that the nth-order main frequency $\omega_{dn}^{\rm NL}$ is related to not only the natural frequency ω_{dn} of derivative linear vibration system, but also the amplitude W_{0n}, damping coefficient δ_n and nonlinear characteristic quantity Q_3, and also it decreases with increase of time.

It can be seen that, from Eq. (41), there are high-order harmonic waves with frequencies $2\omega_{dn}^{\rm NL}$ and $3\omega_{dn}^{\rm NL}$ as well as the $\omega_{dk}^{\rm NL}+\omega_{dn}^{\rm NL}$, $\omega_{dk}-\omega_{dn}^{\rm NL}$, $\omega_{dm}^{\rm NL}+\omega_{dk}^{\rm NL}+\omega_{dn}^{\rm NL}$, $\omega_{dm}^{\rm NL}+\omega_{dk}^{\rm NL}-\omega_{dn}^{\rm NL}$, $\omega_{dk}^{\rm NL}-\omega_{dn}^{\rm NL}-\omega_{dm}^{\rm NL}$, $2\omega_{dn}^{\rm NL}+\omega_{dq}^{\rm NL}$, $2\omega_{dn}^{\rm NL}-\omega_{dq}^{\rm NL}$, $\omega_{dq}^{\rm NL}$ ($n\neq q$) besides the harmonic wave with the main frequency ω_{dn} in the response of the nonlinear system. The phase angle of the nonlinear system is also different to θ_{0n} of the derivative linear

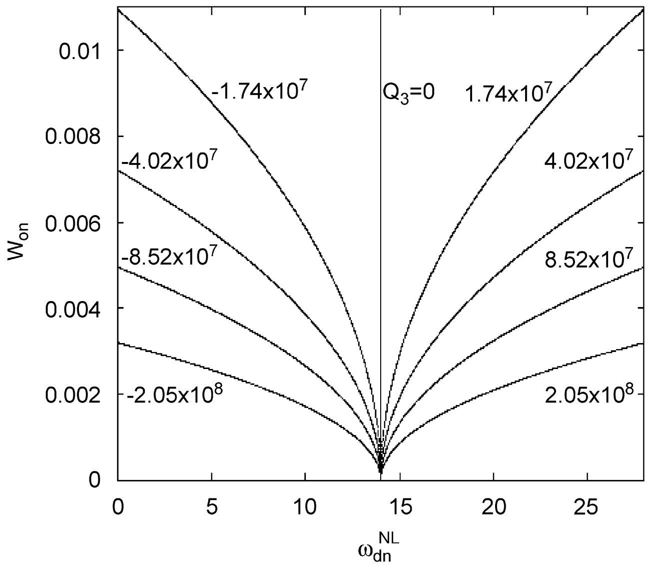

Fig. 3. Curves of amplitude vs. frequency for different values of Q_3 ($n = 1$).

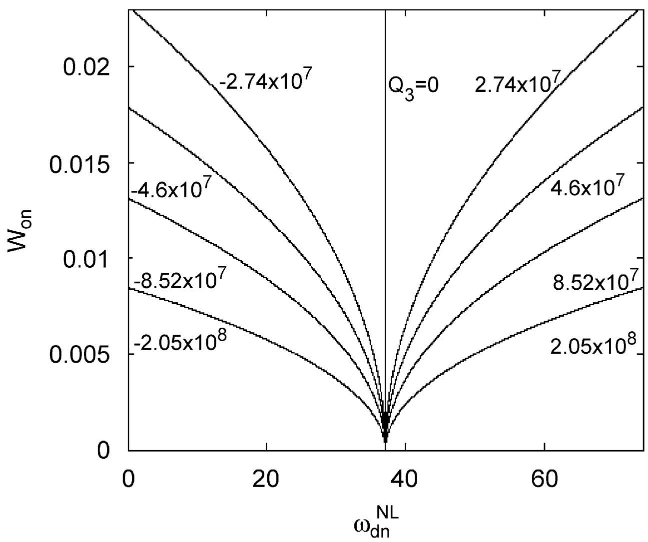

Fig. 4. Curves of amplitude vs. frequency for different values of Q_3 ($n = 2$).

system. The system's response decreases with increase of time due to the effect of viscosity, but the attenuation velocity of every term is different. The vibration zero-point is changed and the symmetry axis occurs the offset due to the effect of nonlinearity of materials of the pile and the soil. From the fifth term on the right-hand side of Eq. (41), one can also see that the effect of nonlinearity of materials on the frequency is very complex.

4. Numerical examples and results analysis

In this section, we shall give numerical results of solution (41) and (42). From the experimental results in Ref. [16], the parameters are given as $l = 15\,\mathrm{m}$, $D = 0.35\,\mathrm{m}$, $d = 0$, $\rho = 2.4 \times 10^3\,\mathrm{kg/m^3}$, $E_0 = 2.1 \times 10^{10}\,\mathrm{Pa}$, $A = 0.0962\,\mathrm{m^2}$, $a = 0.2$, $P_0 = 6.5 \times 10^5\,\mathrm{N}$, $\alpha = 1$; $W_{0n} = 0.002$, $\theta_{0n} = \pi/2$, $n = 1$(or $n = 2$), $\varepsilon = 0.01$, $x = 0.5$, $\eta = 1.02 \times 10^5\,\mathrm{N\,s/m^2}$ ($\eta = 0$ for the pile without viscosity), $\gamma = 1.19 \times 10^4$, $\beta = 2.62 \times 10^2$, $k_1 = 4.2 \times 10^6\,\mathrm{N/m^3}$, $k_2 = -1.3 \times 10^8\,\mathrm{N/m^4}$, $k_3 = 1.3 \times 10^9\,\mathrm{N/m^5}$ ($\gamma = 0$, $\beta = 0$, $k_2 = 0$, $k_3 = 0$ for the linear system), $c = 0.8 \times 10^3\,\mathrm{N\,s/m^3}$ ($c = 0$ for the soil

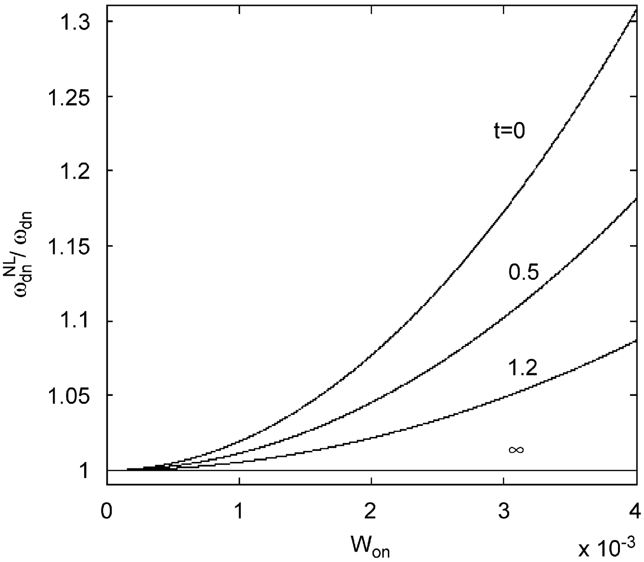

Fig. 5. Curves of frequency ratio vs. amplitude for different time ($n = 1$).

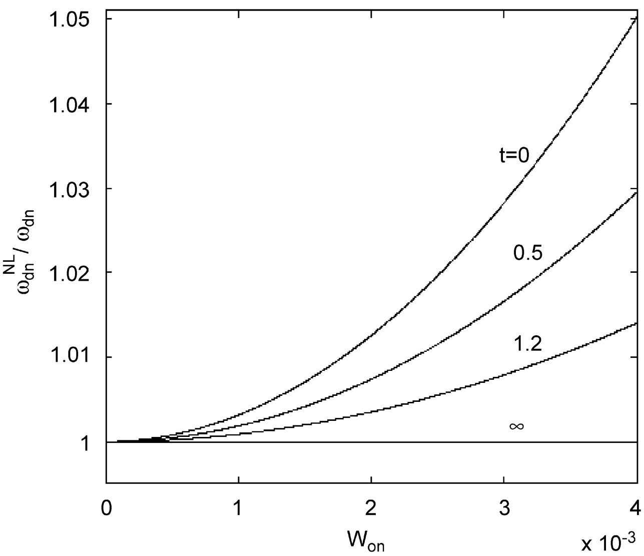

Fig. 6. Curves of frequency ratio vs. amplitude for different time ($n = 2$).

without viscosity). From the given values, it is easy to get $Q_3 = 4.02 \times 10^7$ (when $n = 1$) and $Q_3 = 4.61 \times 10^7$ (when $n = 2$). The numerical results are shown in the following figures.

Figs. 1 and 2 show the curves of the natural frequency vs. the linear stiffness of the soil for the linear system with the viscosity and different values of load p_0 when $n = 1$ and 2, respectively. One can see that the natural frequency of the linear system with the viscosity increases rapidly with increase of the linear stiffness of the soil and decreases with increase of p_0.

For different values of Q_3, Figs. 3 and 4 show the curves of the amplitude vs. the natural frequency for the nonlinear system without the viscosity when $n = 1$ and 2, respectively. One can see that the effect of Q_3 is very obvious.

For different time, Figs. 5 and 6 show the curves of the frequency ratio $\omega_{dn}^{\mathrm{NL}}/\omega_{dn}$ vs. the amplitude of the nonlinear system with viscosity when $n = 1$ and 2, respectively. One can see that the frequency ratio increases rapidly with the initial amplitude. For different amplitude, Figs. 7 and 8 show the curves of frequency ratio vs. time when $n = 1$ and 2, respectively. It is obvious that the frequency ratio decreases rapidly with the increase

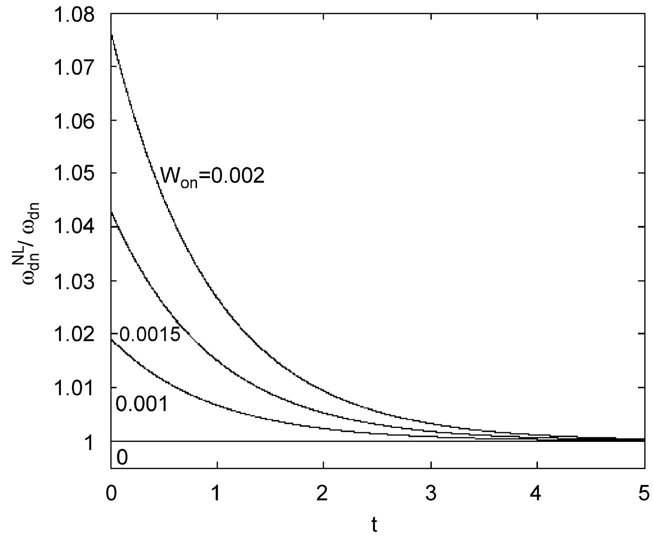

Fig. 7. Curves of frequency ratio vs. time for different amplitude vs. ($n = 1$).

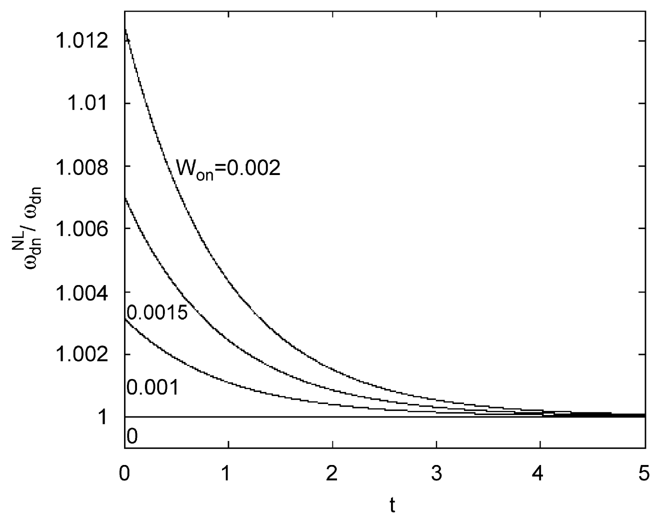

Fig. 8. Curves of frequency ratio vs. time for different amplitude ($n = 2$).

of time. For different time, Figs. 9 and 10 show the curves of the frequency ratio vs. the characteristic quantity Q_3 of the nonlinear system with viscosity when $n = 1$ and 2, respectively. It can be seen that the frequency ratio increases with the increase of Q_3 linearly.

Figs. 11 and 12 show the time–displacement curves for the linear and nonlinear systems with and without viscosity when $n = N = 1$. It is obvious that the vibration of the linear system without viscosity is periodic, its frequency is also invariable, but the vibration of the nonlinear system without viscosity is approximately periodic, the amplitude changes weakly, and also the vibration zero-point changes sometimes. The amplitude of the linear system with viscosity decreases rapidly in terms of a fixed frequency, but the frequency and the amplitude as well as the vibration zero-point of the nonlinear system with viscosity all change, specially, the amplitude decreases rapidly.

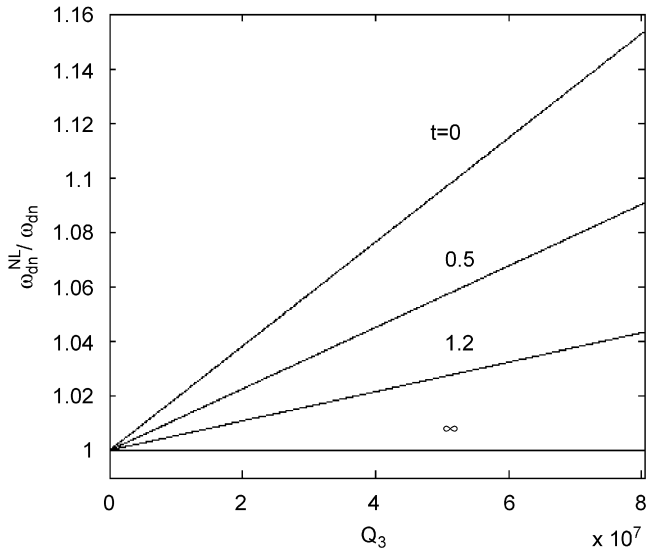

Fig. 9. Curves of frequency ratio vs. Q_3 for different time ($n = 1$).

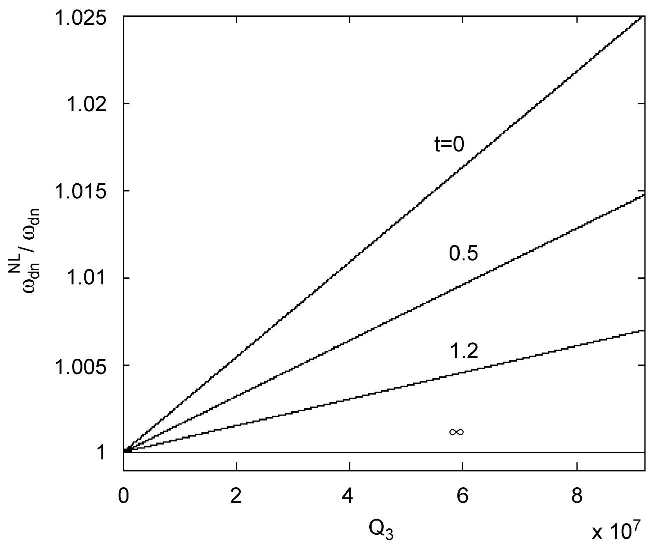

Fig. 10. Curves of frequency ratio vs. Q_3 for different time ($n = 2$).

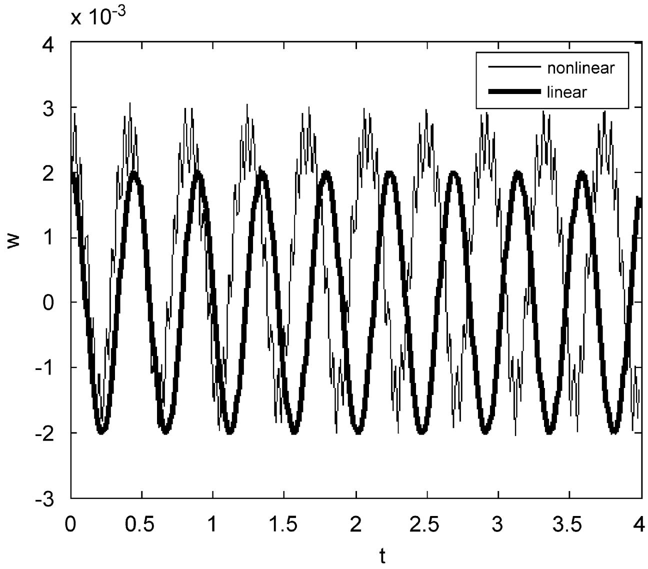

Fig. 11. Curves of time–displacement of linear and nonlinear system without viscosity ($n = N = 1$).

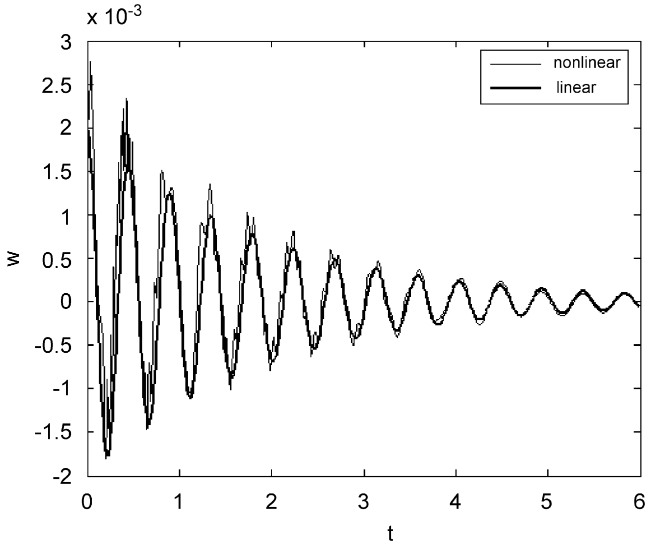

Fig. 12. Curves of time–displacement of linear and nonlinear system with viscosity ($n = N = 1$).

5. Conclusions

In this paper, in the case of assuming that both the materials of the pile and the soil are nonlinear elastic and linear viscoelastic ones, the partial differential equation governing the nonlinear vibration of piles is first derived. The nonlinear transverse free vibration of piles with two ends are hinged is analyzed by using the method of multiple time scales, and the nth-order main frequency and the approximate expression of the displacement response are obtained. Research results point out that the main frequency of the nonlinear system is related to not only the natural frequency of derivative linear vibration system, but also the amplitude, damping coefficient and nonlinearity of materials. There are high order harmonic waves with frequencies $2\omega_{dn}^{\mathrm{NL}}$ and $3\omega_{dn}^{\mathrm{NL}}$ as well as the $\omega_{dk}^{\mathrm{NL}} + \omega_{dn}^{\mathrm{NL}}$, $\omega_{dk}^{\mathrm{NL}} - \omega_{dn}^{\mathrm{NL}}$, $\omega_{dm}^{\mathrm{NL}} + \omega_{dk}^{\mathrm{NL}} + \omega_{dn}^{\mathrm{NL}}$, $\omega_{dm}^{\mathrm{NL}} + \omega_{dk}^{\mathrm{NL}} - \omega_{dn}^{\mathrm{NL}}$,

$\omega_{dk}^{\mathrm{NL}} - \omega_{dn}^{\mathrm{NL}} - \omega_{dm}^{\mathrm{NL}}$, $2\omega_{dn}^{\mathrm{NL}} + \omega_{dq}^{\mathrm{NL}}$, $2\omega_{dn}^{\mathrm{NL}} - \omega_{dq}^{\mathrm{NL}}$, $\omega_{dq}^{\mathrm{NL}}$ $(n \neq q)$ besides the harmonic wave with the main frequency ω_{dn} in the response of the nonlinear system. The vibration zero-point of the system changes and the symmetry axis offsets. The phase angle of the nonlinear system is also different to θ_{0n} of the derivative linear system. Due to the effect of viscosity, the response of the system attenuates with the increase of time, and the attenuating velocity is different to that of derivative linear system, too.

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