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Identifying time-dependent damping and stiffness functions by a simple and yet accurate method

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Abstract

The inverse vibration problem is a mathematical process to determine unknown mechanical parameters from measured vibration data. In this study the data of displacement are chosen in order to identify a time-dependent function of damping or stiffness. However, when both functions are to be identified we require both the data of displacement and velocity. This is the first time that a closed-form estimation method for the inverse vibration problems of estimating time-dependent parameters has been constructed. We are able to transform the inverse vibration problem into an identification problem governed by a parabolic-type partial differential equation (PDE). Then, a one-step group-preserving scheme (GPS) for the semi-discretization of PDE is established, which can be used to derive a closed-form solution for estimating parameters. The new Lie-group estimation method has three further advantages: it does not require any prior information on the functional forms of unknown functions; no initial guesses are required; and no iterations are required. Numerical examples were examined to show that the present approach is highly accurate and efficient even for identifying discontinuous and oscillatory parameters. Against the noise is good when only one function is estimated; however, the present approach is slightly weak against the noise when both functions are identified.

1. Introduction

Many researches and efforts have been involved in the science of vibrations. The solution of direct forced vibration problem is concerned with the determination of a system's displacement, velocity and acceleration evolving in time when the initial conditions, external forces, and system parameters are specified. Sometimes we may encounter a problem where some parameters of the system are unknown, and then the resulting problem is an inverse vibration problem. This is concerned with the estimations of these parameters such as the damping coefficient [1–4], stiffness [5,6], as well as external force [7,8] with the aid of measured vibration data, such as frequency, mode shape, displacement, or velocity at different times.

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The parameters' identification problem is known to be highly ill-posed in the sense that a small disturbance of measured data may result in a huge error on the parameters' estimation. In order to overcome this problem, many studies have appeared in this field. Although the system we consider is linear, for the inverse vibration problem we may require to treat a nonlinear problem.

Let us consider a second-order ordinary differential equation (ODE) describing the forced vibration of a linear structure with time-dependent parameters:

$$\ddot{\phi} + c(t)\dot{\phi} + k(t)\phi = F(t) \quad \text{in } 0 < t \leqslant t_f, \tag{1}$$

$$\phi = A_0 \quad \text{at } t = 0, \tag{2}$$

$$\dot{\phi} = B_0 \quad \text{at } t = 0. \tag{3}$$

The direct problem is for the given conditions in Eqs. (2) and (3) and the given functions $c(t)$, $k(t)$ and $F(t)$ in Eq. (1) to find the response of $\phi(t)$ at a time interval of $t \in [0, t_f]$. However, our present inverse vibration problem is to estimate $c(t)$ and $k(t)$ by using some measured data of $\phi(t)$ and $\dot{\phi}(t)$ at a time interval of $t \in [0, t_f]$.

The present approach first transforms the identification problem of ODE in Eq. (1) into an identification problem of a parabolic-type PDE to be introduced in Section 2. This type of approach is appearing in the literature for the first time. When one knows the development of the so-called Lie-group estimation method developed by the author in recent years, one may appreciate that the present approach is very interesting, which results in a closed-form estimating equation without needing any iteration and initial guess of the coefficient functions. More importantly, the novel method does not require to assume a priori a functional form of an unknown coefficient.

Recently, Liu [9–11] extended the group-preserving scheme (GPS) developed by Liu [12] for ODEs to solve the boundary value problems (BVPs), and the numerical results reveal that the GPS is a rather promising method to effectively solve the two-point BVPs. In the construction of the Lie-group method for the calculation of BVPs, Liu [9] has introduced the idea of one-step GPS by utilizing the closure property of the Lie group, and hence, the new shooting method has been named the Lie-group shooting method [13].

On the other hand, in order to effectively solve the backward in time problems of parabolic-type PDEs, a past cone structure and a backward group-preserving scheme have been successfully developed by the author, such that a one-step Lie-group numerical method has been used to solve the backward in time Burgers equation by Liu [14], and the backward in time heat conduction equation by Liu et al. [15].

The Lie-group methods were originally used for the BVPs as designed by Liu [9–11] for direct problems. However, these methods are restricted only to two-dimensional ODEs. In a series of papers by the author and his coworkers, the Lie-group method reveals its excellent behavior for the numerical solutions of different problems, for example, Chang et al. [16] to calculate the sideways heat conduction problem, Chang et al. [17] to treat the boundary layer equation in fluid mechanics, and Liu [18], Liu et al. [15] and Chang et al. [19,20] to treat the backward heat conduction equation, and Liu et al. [21] to treat the Burgers equation.

It should be stressed that the one-step Lie-group property is usually not shared by other numerical methods, because those methods do not belong to the Lie-group types. This important property as first pointed out by Liu [14] was employed to solve the backward in time Burgers equation. After that, Liu [22] used this concept to establish a one-step estimation method to estimate the temperature-dependent heat conductivity, and then extended it to estimate the thermophysical properties of heat conductivity and heat capacity [23–25]. The Lie-group estimation method possesses a great advantage compared to other numerical methods due to its group structure, and it is a powerful technique to solve the inverse problems of parameters' identification as shown by Liu [13] for a heat conduction equation. However, this kind of approach has not yet been applied to the inverse vibration problem.

This paper is organized as follows: we introduce a novel approach of the inverse vibration problem in Section 2 by transforming it into an identification problem of a parabolic-type PDE, and then discretizing the PDE into a system of ODEs at the discretized times. In Section 3 we give a brief sketch of the GPS for a self-content reason. Due to the good property of Lie-group, we will propose an integration technique, such that the one-step GPS can be used to identify the parameters appearing in the PDE. The resulting algebraic equations are derived in Section 4 when we apply the one-step GPS to identify $c(t)$ or $k(t)$. We demonstrate how the Lie-group theory can help us to solve these parameter estimation equations in a closed form. In Section 5 we turn our attention to the estimation of both $c(t)$ and $k(t)$, which again leads to closed-form solutions of $c(t)$ and $k(t)$ simultaneously. In Section 6 several numerical examples are examined to test the Lie-group estimation method (LGEM). In this section we also consider the measurement noise effect on the numerical results obtained from the LGEM. Finally, we conclude with some important results of the new computing method in Section 7.

2. A novel approach

2.1. Transformation into a PDE

In the solutions of a linear PDE, a commonly used technique is the separation of variables, from which the PDE is transformed into some ODEs. In this study we reverse this process by considering

$$u(x,t) = (1+x)\phi(t), \tag{4}$$

such that Eqs. (1)–(3) are changed to a parabolic-type PDE:

$$\frac{\partial u(x,t)}{\partial x} = \frac{\partial^2 u(x,t)}{\partial t^2} + c(t)\frac{\partial u(x,t)}{\partial t} + k(t)u(x,t) + \phi(t) - (1+x)F(t), \tag{5}$$

$$u(0,t) = \phi(t), \tag{6}$$

$$u(x,0) = (1+x)A_0, \tag{7}$$

$$u(x,t_f) = (1+x)\phi(t_f), \tag{8}$$

where $\phi(t_f)$ is a measured displacement at time t_f. In addition to the above data $\phi(t)$ is assumed to be measurable to provide some measured displacement data in the whole time interval of $0<t<t_f$. In Eq. (5) $c(t)$ and $k(t)$ are time-dependent functions to be identified, where the domain we consider is $0\leqslant t\leqslant t_f, 0<x\leqslant x_f$. The coordinate x is a fictitious one; however, from it, together with the independent variable t, we can work in a two-dimensional domain and determine the variation of $c(t)$ and $k(t)$.

2.2. Semi-discretization

Applying a semi-discrete procedure to the above PDE yields a coupled system of ODEs. For Eq. (5) we adopt the following numerical discretizations:

$$\left.\frac{\partial u(x,t)}{\partial t}\right|_{t=i\,\Delta t} = \frac{u_{i+1}(x) - u_i(x)}{\Delta t}, \tag{9}$$

$$\left.\frac{\partial^2 u(x,t)}{\partial t^2}\right|_{t=i\,\Delta t} = \frac{u_{i+1}(x) - 2u_i(x) + u_{i-1}(x)}{(\Delta t)^2}, \tag{10}$$

where $\Delta t = t_f/(n+1)$ is a uniform time increment, and $u_i(x) = u(x, i\,\Delta t)$ for a simple notation, such that Eq. (5) can be approximated by

$$u_i'(x) = \frac{1}{(\Delta t)^2}[u_{i+1}(x) - 2u_i(x) + u_{i-1}(x)] + c_i\frac{u_{i+1}(x) - u_i(x)}{\Delta t} + k_i u_i(x) + h_i(x), \tag{11}$$

where $i = 1,\ldots,n$, $c_i = c(t_i)$, $k_i = k(t_i)$, and $h_i(x) = \phi_i - (1+x)F_i$ with $\phi_i = \phi(t_i)$ and $F_i = F(t_i)$. The superscripted prime on u_i is for the differential of u_i with respect to x.

When $i=1$ the term $u_0(x)$ is obtained from the boundary condition (7) with $u_0(x) = A_0(1+x)$. Similarly, when $i=n$ the term $u_{n+1}(x)$ is obtained from the boundary condition (8) with $u_{n+1}(x) = (1+x)\phi_{n+1} = (1+x)\phi(t_f)$. Therefore, Eq. (11) has totally n coupled linear ODEs for the n variables $u_i(x), i = 1,\ldots,n$.

In this section we have transformed the inverse vibration problem of the second-order ODE in Eq. (1) into an inverse problem for the PDE in Eq. (5), and finally we come to an estimation of $2n$ coefficients c_i and k_i in the n-dimensional ODE system (11). In Section 5 we will derive another hyperbolic-type first-order PDE, such that we have enough discretized equations to solve the $2n$ coefficients c_i and k_i.

3. GPS for differential equations system

3.1. Group-preserving scheme

Upon letting $\mathbf{u} = (u_1, \ldots, u_n)^{\mathrm{T}}$ and $\mathbf{f}$ denote a vector, with the ith component being the right-hand side of Eq. (11), we can write it as a vector form:

$$\mathbf{u}' = \mathbf{f}(\mathbf{u}, x), \quad \mathbf{u} \in \mathbb{R}^n, \quad x \in \mathbb{R}. \tag{12}$$

The group-preserving scheme (GPS) can preserve the internal symmetry group of the considered ODE system. Although we do not know previously the symmetry group of a differential equations system, Liu [12] embedded it into an augmented differential system, which is concerned with not only the evolution of state variables themselves but also the evolution of the magnitude of the state variables vector. Let us note that

$$\|\mathbf{u}\| = \sqrt{\mathbf{u}^{\mathrm{T}}\mathbf{u}} = \sqrt{\mathbf{u} \cdot \mathbf{u}}, \tag{13}$$

where the superscript T signifies the transpose, and the dot between two n-dimensional vectors denotes their inner product. Taking the derivatives of both sides of Eq. (13) with respect to x, we have

$$\frac{d\|\mathbf{u}\|}{dx} = \frac{(\mathbf{u}')^{\mathrm{T}}\mathbf{u}}{\sqrt{\mathbf{u}^{\mathrm{T}}\mathbf{u}}}. \tag{14}$$

Then, by using Eqs. (12) and (13) we can derive

$$\frac{d\|\mathbf{u}\|}{dx} = \frac{\mathbf{f}^{\mathrm{T}}\mathbf{u}}{\|\mathbf{u}\|}. \tag{15}$$

It is interesting that Eqs. (12) and (15) can be combined together into a simple matrix equation:

$$\frac{d}{dx}\begin{bmatrix} \mathbf{u} \\ \|\mathbf{u}\| \end{bmatrix} = \begin{bmatrix} \mathbf{0}_{n\times n} & \dfrac{\mathbf{f}(\mathbf{u}, x)}{\|\mathbf{u}\|} \\ \dfrac{\mathbf{f}^{\mathrm{T}}(\mathbf{u}, x)}{\|\mathbf{u}\|} & 0 \end{bmatrix} \begin{bmatrix} \mathbf{u} \\ \|\mathbf{u}\| \end{bmatrix}. \tag{16}$$

It is obvious that the first row in Eq. (16) is the same as the original equation (12), but the inclusion of the second row in Eq. (16) gives us a Minkowskian structure of the augmented state variables of $\mathbf{X}{:=}(\mathbf{u}^{\mathrm{T}}, \|\mathbf{u}\|)^{\mathrm{T}}$, which satisfies the cone condition:

$$\mathbf{X}^{\mathrm{T}}\mathbf{g}\mathbf{X} = 0, \tag{17}$$

where

$$\mathbf{g} = \begin{bmatrix} \mathbf{I}_n & \mathbf{0}_{n\times 1} \\ \mathbf{0}_{1\times n} & -1 \end{bmatrix} \tag{18}$$

is a Minkowski metric and $\mathbf{I}_n$ is the identity matrix of order n. In terms of $(\mathbf{u}, \|\mathbf{u}\|)$, Eq. (17) becomes

$$\mathbf{X}^{\mathrm{T}}\mathbf{g}\mathbf{X} = \mathbf{u} \cdot \mathbf{u} - \|\mathbf{u}\|^2 = \|\mathbf{u}\|^2 - \|\mathbf{u}\|^2 = 0. \tag{19}$$

This follows from the definition given in Eq. (13), and thus Eq. (17) is a natural result.

Consequently, we have an $n+1$-dimensional augmented system:

$$\mathbf{X}' = \mathbf{A}\mathbf{X} \tag{20}$$

with a constraint (17), where

$$\mathbf{A} := \begin{bmatrix} \mathbf{0}_{n\times n} & \dfrac{\mathbf{f}(\mathbf{u},x)}{\|\mathbf{u}\|} \\ \dfrac{\mathbf{f}^{\mathrm{T}}(\mathbf{u},x)}{\|\mathbf{u}\|} & 0 \end{bmatrix} \tag{21}$$

satisfying

$$\mathbf{A}^{\mathrm{T}}\mathbf{g} + \mathbf{g}\mathbf{A} = \mathbf{0} \tag{22}$$

is a Lie algebra $so(n,1)$ of the proper orthochronous Lorentz group $SO_o(n,1)$. This prompts us to devise the group-preserving scheme (GPS), whose discretized mapping $\mathbf{G}$ must exactly preserve the following properties:

$$\mathbf{G}^{\mathrm{T}}\mathbf{g}\mathbf{G} = \mathbf{g}, \tag{23}$$

$$\det \mathbf{G} = 1, \tag{24}$$

$$G_0^0 > 0, \tag{25}$$

where G_0^0 is the 00th component of $\mathbf{G}$.

Although the dimension of the new system is raised once more, it has been shown that the new system permits a GPS given as follows [12]:

$$\mathbf{X}_{\ell+1} = \mathbf{G}(\ell)\mathbf{X}_\ell, \tag{26}$$

where $\mathbf{X}_\ell$ denotes the numerical value of $\mathbf{X}$ at x_ℓ and $\mathbf{G}(\ell) \in SO_o(n,1)$ is the group value of $\mathbf{G}$ at x_ℓ. If $\mathbf{G}(\ell)$ satisfies the properties in Eqs. (23)–(25), then $\mathbf{X}_\ell$ satisfies the cone condition in Eq. (17).

The Lie group can be generated from $\mathbf{A} \in so(n,1)$ by an exponential mapping,

$$\mathbf{G}(\ell) = \exp[\Delta x\, \mathbf{A}(\ell)] = \begin{bmatrix} \mathbf{I}_n + \dfrac{(a_\ell - 1)}{\|\mathbf{f}_\ell\|^2}\mathbf{f}_\ell\mathbf{f}_\ell^{\mathrm{T}} & \dfrac{b_\ell \mathbf{f}_\ell}{\|\mathbf{f}_\ell\|} \\ \dfrac{b_\ell \mathbf{f}_\ell^{\mathrm{T}}}{\|\mathbf{f}_\ell\|} & a_\ell \end{bmatrix}, \tag{27}$$

where

$$a_\ell := \cosh\left(\frac{\Delta x \|\mathbf{f}_\ell\|}{\|\mathbf{u}_\ell\|}\right), \tag{28}$$

$$b_\ell := \sinh\left(\frac{\Delta x \|\mathbf{f}_\ell\|}{\|\mathbf{u}_\ell\|}\right). \tag{29}$$

Substituting Eq. (27) for $\mathbf{G}(\ell)$ into Eq. (26), we obtain

$$\mathbf{u}_{\ell+1} = \mathbf{u}_\ell + \eta_\ell \mathbf{f}_\ell, \tag{30}$$

$$\|\mathbf{u}_{\ell+1}\| = a_\ell \|\mathbf{u}_\ell\| + \frac{b_\ell}{\|\mathbf{f}_\ell\|}\mathbf{f}_\ell \cdot \mathbf{u}_\ell, \tag{31}$$

where

$$\eta_\ell := \frac{b_\ell \|\mathbf{u}_\ell\| \|\mathbf{f}_\ell\| + (a_\ell - 1)\mathbf{f}_\ell \cdot \mathbf{u}_\ell}{\|\mathbf{f}_\ell\|^2} \tag{32}$$

is an adaptive factor. From $\mathbf{f}_\ell \cdot \mathbf{u}_\ell \geqslant -\|\mathbf{f}_\ell\|\|\mathbf{u}_\ell\|$ we can prove that

$$\eta_\ell \geqslant \left[1-\exp\left(-\frac{\Delta x\|\mathbf{f}_\ell\|}{\|\mathbf{u}_\ell\|}\right)\right]\frac{\|\mathbf{u}_\ell\|}{\|\mathbf{f}_\ell\|}>0, \quad \forall\,\Delta x>0. \tag{33}$$

This scheme has group properties that are preserved for all $\Delta x>0$, and is called the group-preserving scheme.

3.2. One-step GPS

Applying scheme (30) to Eq. (11) we can compute u_i^f by GPS. Throughout this paper the superscript f denotes the value at $x = x_f$, while the superscript 0 denotes the value at $x = 0$. Assume that the total length x_f is divided by K steps, that is, the step size we use in the GPS is $\Delta x = x_f/K$.

Starting from $\mathbf{X}^0 = \mathbf{X}(0)$ we want to calculate the value $\mathbf{X}(x_f)$ at $x = x_f$. From Eq. (26) we can obtain

$$\mathbf{X}^f = \mathbf{G}_K(\Delta x)\ldots\mathbf{G}_1(\Delta x)\mathbf{X}^0, \tag{34}$$

where $\mathbf{X}^f$ approximates the real $\mathbf{X}(x_f)$ within a certain accuracy depending on Δx. However, let us recall that each $\mathbf{G}_i, i = 1,\ldots,K$, is an element of the Lie group $SO_o(n,1)$, and by the closure property of the Lie group, $\mathbf{G}_K(\Delta x)\ldots\mathbf{G}_1(\Delta x)$ is also a Lie group denoted by $\mathbf{G}(x_f)$. Hence, we have

$$\mathbf{X}^f = \mathbf{G}(x_f)\mathbf{X}^0. \tag{35}$$

This is a one-step transformation from $\mathbf{X}^0$ to $\mathbf{X}^f$.

Usually, it is very hard to find the exact solution of $\mathbf{G}(x_f)$; however, a numerical one may be obtained approximately without any difficulty. The most simple method to calculate $\mathbf{G}(x_f)$ is given by

$$\mathbf{G}(x_f) = \begin{bmatrix} \mathbf{I}_n + \dfrac{(a-1)}{\|\mathbf{f}^0\|^2}\mathbf{f}^0(\mathbf{f}^0)^\mathrm{T} & \dfrac{b\mathbf{f}^0}{\|\mathbf{f}^0\|} \\ \dfrac{b(\mathbf{f}^0)^\mathrm{T}}{\|\mathbf{f}^0\|} & a \end{bmatrix}, \tag{36}$$

where

$$a := \cosh\left(\frac{x_f\|\mathbf{f}^0\|}{\|\mathbf{u}^0\|}\right), \tag{37}$$

$$b := \sinh\left(\frac{x_f\|\mathbf{f}^0\|}{\|\mathbf{u}^0\|}\right) \tag{38}$$

and $\mathbf{f}^0 = \mathbf{f}(\mathbf{u}^0, 0)$.

Here, we use the value of $\mathbf{u}(0)$ to calculate $\mathbf{G}(x_f)$. Then from Eqs. (30) and (31) we can obtain a one-step GPS:

$$\mathbf{u}^f = \mathbf{u}^0 + \eta_u\mathbf{f}^0, \tag{39}$$

$$\|\mathbf{u}^f\| = a\|\mathbf{u}^0\| + \frac{b\mathbf{f}^0\cdot\mathbf{u}^0}{\|\mathbf{f}^0\|}, \tag{40}$$

where

$$\eta_u = \frac{(a-1)\mathbf{f}^0\cdot\mathbf{u}^0 + b\|\mathbf{u}^0\|\|\mathbf{f}^0\|}{\|\mathbf{f}^0\|^2}. \tag{41}$$

4. Identifying $c(t)$ or $k(t)$ by the LGEM

In this section we will start to estimate the time-dependent coefficient function $c(t)$ by supposing that $k(t)$ is a given function, or to estimate $k(t)$ by supposing that $c(t)$ is a given function. By using the one-step GPS we also suppose that the value of $u(0,t)=\phi(t)$ is given, which must be nonzero.

By applying the one-step GPS to Eq. (11) from $x=0$ to $x=x_f$ we obtain a nonlinear equation for c_i:

$$u_i^f = u_i^0 + \frac{\eta_u}{(\Delta t)^2}(u_{i+1}^0 - 2u_i^0 + u_{i-1}^0) + \eta_u c_i \frac{u_{i+1}^0 - u_i^0}{\Delta t} + \eta_u k_i u_i^0 + \eta_u h_i(0). \tag{42}$$

At first glance, η_u in the above equation seems to be a nonlinear function of c_i as shown by Eq. (41). However, we will prove below that η_u is fully determined by the data of u_i^0 and u_i^f.

It is not difficult to rewrite Eq. (42) as

$$c_i = \frac{\Delta t}{u_{i+1}^0 - u_i^0}\left[\frac{u_i^f - u_i^0}{\eta_u} - \frac{1}{(\Delta t)^2}(u_{i+1}^0 - 2u_i^0 + u_{i-1}^0) - k_i u_i^0 - h_i(0)\right]. \tag{43}$$

Here we require that $u_{i+1}^0 \neq u_i^0$.

In order to solve c_i, let us return to Eq. (39) obtaining

$$\mathbf{f}^0 = \frac{1}{\eta_u}(\mathbf{u}^f - \mathbf{u}^0), \tag{44}$$

which is substituted into Eq. (40) leading to

$$\frac{\|\mathbf{u}^f\|}{\|\mathbf{u}^0\|} = a + \frac{b(\mathbf{u}^f - \mathbf{u}^0)\cdot\mathbf{u}^0}{\|\mathbf{u}^f - \mathbf{u}^0\|\,\|\mathbf{u}^0\|}, \tag{45}$$

where a and b defined in Eqs. (37) and (38) are simultaneously changed to

$$a = \cosh\left(\frac{x_f\|\mathbf{u}^f - \mathbf{u}^0\|}{\eta_u\|\mathbf{u}^0\|}\right), \tag{46}$$

$$b = \sinh\left(\frac{x_f\|\mathbf{u}^f - \mathbf{u}^0\|}{\eta_u\|\mathbf{u}^0\|}\right). \tag{47}$$

Let

$$\cos\theta := \frac{[\mathbf{u}^f - \mathbf{u}^0]\cdot\mathbf{u}^0}{\|\mathbf{u}^f - \mathbf{u}^0\|\,\|\mathbf{u}^0\|}, \tag{48}$$

$$S := \frac{x_f\|\mathbf{u}^f - \mathbf{u}^0\|}{\|\mathbf{u}^0\|} \tag{49}$$

and from Eqs. (45) to (47) it follows that

$$\frac{\|\mathbf{u}^f\|}{\|\mathbf{u}^0\|} = \cosh\left(\frac{S}{\eta_u}\right) + \cos\theta\sinh\left(\frac{S}{\eta_u}\right). \tag{50}$$

Upon defining

$$Z := \exp\left(\frac{S}{\eta_u}\right) \tag{51}$$

from Eq. (50) we obtain a quadratic equation for Z:

$$(1+\cos\theta)Z^2 - \frac{2\|\mathbf{u}^f\|}{\|\mathbf{u}^0\|}Z + 1 - \cos\theta = 0. \tag{52}$$

The solution is found to be

$$Z = \frac{\frac{\|\mathbf{u}^f\|}{\|\mathbf{u}^0\|} \pm \sqrt{\left(\frac{\|\mathbf{u}^f\|}{\|\mathbf{u}^0\|}\right)^2 - (1-\cos^2\theta)}}{1+\cos\theta} \quad \text{if} \quad \pm\cos\theta > 0 \tag{53}$$

and from Eq. (51) we obtain a closed-form solution of η_u:

$$\eta_u = \frac{x_f\|\mathbf{u}^f - \mathbf{u}^0\|}{\|\mathbf{u}^0\| \ln Z}. \tag{54}$$

Up to here we must point out that for a given x_f, η_u is fully determined by $\mathbf{u}^0$ and $\mathbf{u}^f$, which are supposed to be known. Therefore, the original nonlinear equation (43) becomes a linear equation for c_i.

Now, if we substitute the above η_u into Eq. (43) we can immediately find $c_1, \ldots, c_n$. This solution is in a closed form for c_i. Similarly, if $c_1, \ldots, c_n$ are given, from Eq. (42) one can easily find $k_1, \ldots, k_n$ by

$$k_i = \frac{1}{u_i^0}\left[\frac{u_i^f - u_i^0}{\eta_u} - \frac{1}{(\Delta t)^2}(u_{i+1}^0 - 2u_i^0 + u_{i-1}^0) - c_i\frac{u_{i+1}^0 - u_i^0}{\Delta t} - h_i(0)\right]. \tag{55}$$

Obviously, u_i^0 cannot be zero.

In the above we have mentioned that η_u is a nonlinear function of c_i or k_i; however, from Eqs. (48), (53) and (54) it is known that η_u is fully determined by $\mathbf{u}^0$ and $\mathbf{u}^f$. This point is very important for our closed-form solution of the parameter identification. The key points are based in the method by using the one-step GPS for the estimation of a parameter, and the full use of the $n+1$ equations (39) and (40). To distinguish the present method by a combined use of the one-step GPS and the closed-form solution with the aid of Eq. (40), we may call the new method a Lie-group estimation method (LGEM).

5. Identifying both $c(t)$ and $k(t)$ by the LGEM

When both $c(t)$ and $k(t)$ require to be identified, this inverse vibration problem is more difficult. However, if extra data of $\dot{\phi} =: \psi$ are available, we can estimate both $c(t)$ and $k(t)$ as follows.

From Eq. (1) we have

$$\dot{\psi} + c(t)\psi + k(t)\phi = F(t). \tag{56}$$

Furthermore, if we let $v(x,t) = (1+x)\psi(t)$ then we have

$$\frac{\partial v(x,t)}{\partial x} = \frac{\partial v(x,t)}{\partial t} + c(t)v(x,t) + k(t)u(x,t) + \psi(t) - (1+x)F(t), \tag{57}$$

$$v(0,t) = \psi(t), \tag{58}$$

$$v(x,0) = (1+x)B_0, \tag{59}$$

$$v(x,t_f) = (1+x)\psi(t_f), \tag{60}$$

where $\psi(t_f)$ is a measured velocity at time t_f. In addition to the above data $\psi(t)$ is assumed to be measurable to provide some measured velocity data in the whole time interval of $0 < t < t_f$.

As before we can consider a finite difference of Eq. (57) by

$$v_i'(x) = \frac{v_{i+1}(x) - v_{i-1}(x)}{2\Delta t} + c_i v_i(x) + k_i u_i(x) + g_i(x), \tag{61}$$

where $g_i(x) = \psi_i - (1+x)F_i$, and similarly $\psi_i = \psi(t_i)$.

Suppose that we use the one-step GPS to integrate Eq. (61) from $x=0$ to $x=x_f$, then we have other data v_i^f. Therefore, we have

$$A_2^i c_i + B_2^i k_i = C_2^i, \tag{62}$$

where

$$A_2^i = v_i^0, \tag{63}$$

$$B_2^i = u_i^0, \tag{64}$$

$$C_2^i = \frac{v_i^f - v_i^0}{\eta_v} - \frac{1}{2\Delta t}(v_{i+1}^0 - v_{i-1}^0) - g_i(0), \tag{65}$$

in which η_v is defined as in Eq. (54) but replacing $\mathbf{u}$ by $\mathbf{v}$.

On the other hand, Eq. (42) can be written as

$$A_1^i c_i + B_1^i k_i = C_1^i, \tag{66}$$

where

$$A_1^i = \frac{u_{i+1}^0 - u_i^0}{\Delta t}, \tag{67}$$

$$B_1^i = u_i^0, \tag{68}$$

$$C_1^i = \frac{u_i^f - u_i^0}{\eta_u} - \frac{1}{(\Delta t)^2}(u_{i+1}^0 - 2u_i^0 + u_{i-1}^0) - h_i(0). \tag{69}$$

From Eqs. (66) and (62) we can solve

$$c_i = \frac{C_1^i B_2^i - C_2^i B_1^i}{A_1^i B_2^i - A_2^i B_1^i} = \frac{C_1^i - C_2^i}{A_1^i - A_2^i}, \tag{70}$$

$$k_i = \frac{A_1^i C_2^i - A_2^i C_1^i}{A_1^i B_2^i - A_2^i B_1^i} = \frac{C_2^i A_1^i - C_1^i A_2^i}{B_1^i(A_1^i - A_2^i)}. \tag{71}$$

Therefore, we can employ both the displacement and the velocity data to estimate c_i and k_i simultaneously.

6. Numerical examples

6.1. Example 1

Let us consider

$$c(t) = 3 + 2\cos(2\pi t), \tag{72}$$

$$k(t) = 20 + 2\sin(2\pi t), \tag{73}$$

$$F(t) = F_0 + F_1 t. \tag{74}$$

Here we take $F_0 = 5$ and $F_1 = 10$.

In order to obtain the data of $\phi(t)$ and $\psi(t)$ we can apply the fourth-order Runge–Kutta method (RK4) in Eqs. (1)–(3), where $A_0 = 0.1$ and $B_0 = 1$ are fixed. In this calculation we have fixed $\Delta t = 1/300$, i.e., $n = 299$. While the profile of $\phi(t)$ is plotted in Fig. 1(a), the profile of $\psi(t)$ is plotted in Fig. 1(b). Then we apply the one-step GPS to Eq. (11) to obtain the data u_i^f with $x_f = 0.001$. These data are plotted in Fig. 1(c) by the dashed line, which are named the calculated data and are compared with the data $(1 + x_f)\phi_i$ as shown by the solid line, which are calculated from Eq. (4) and are called the measured data. Because these two data are very close, the solid line and dashed line are almost coincident. At the same time, we apply the one-step GPS in Eq. (61) to obtain the data v_i^f. These data are plotted in Fig. 1(d) by the dashed line, and are compared with the measured data $(1 + x_f)\psi_i$ as shown by the solid line.

Applying Eq. (43) using the measured data of $u_i^f = (1 + x_f)\phi_i$, the estimation of c_i shown by the dashed line matches rather well with the exact one as shown in Fig. 2(a) by the solid line, from which the maximum error is about 3.56×10^{-2} as shown in Fig. 2(b) by the dashed line. When we use the one-step GPS calculated data as an input in Eq. (43), the result is much better, with the maximum error in the order of 10^{-12}. In order to give

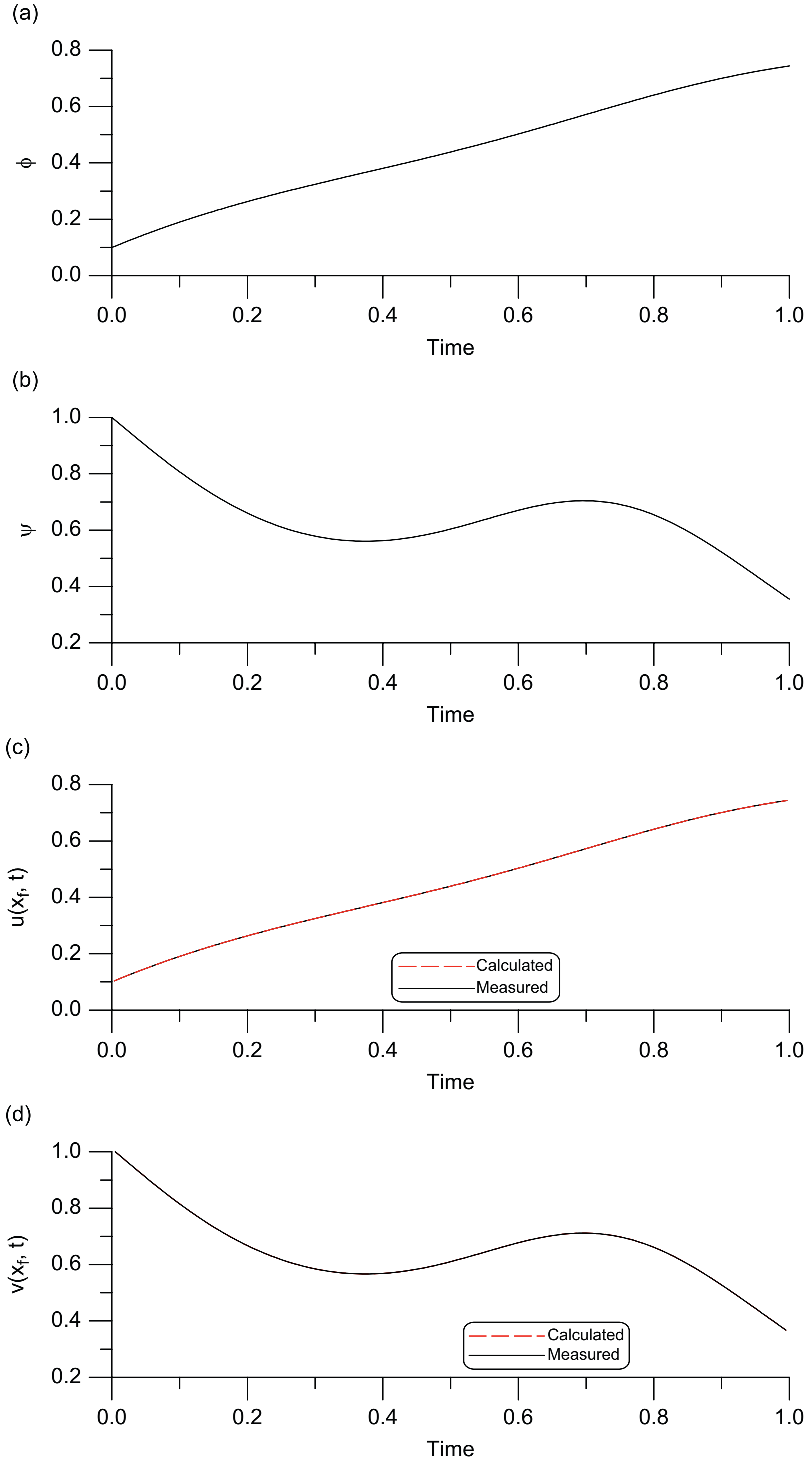

Fig. 1. For Example 1: (a) displaying the displacement, (b) displaying the velocity, and (c) and (d) comparing the calculated and measured data at $x_f = 0.001$.

an impression that the present method is not sensitive to the number of subdivision points, we also show the numerical results in Fig. 2 by employing $n = 99$. It can be seen that the estimated results of $c(t)$ are very accurate no matter which n is used. The result with $n = 99$ is slightly better than that with $n = 299$.

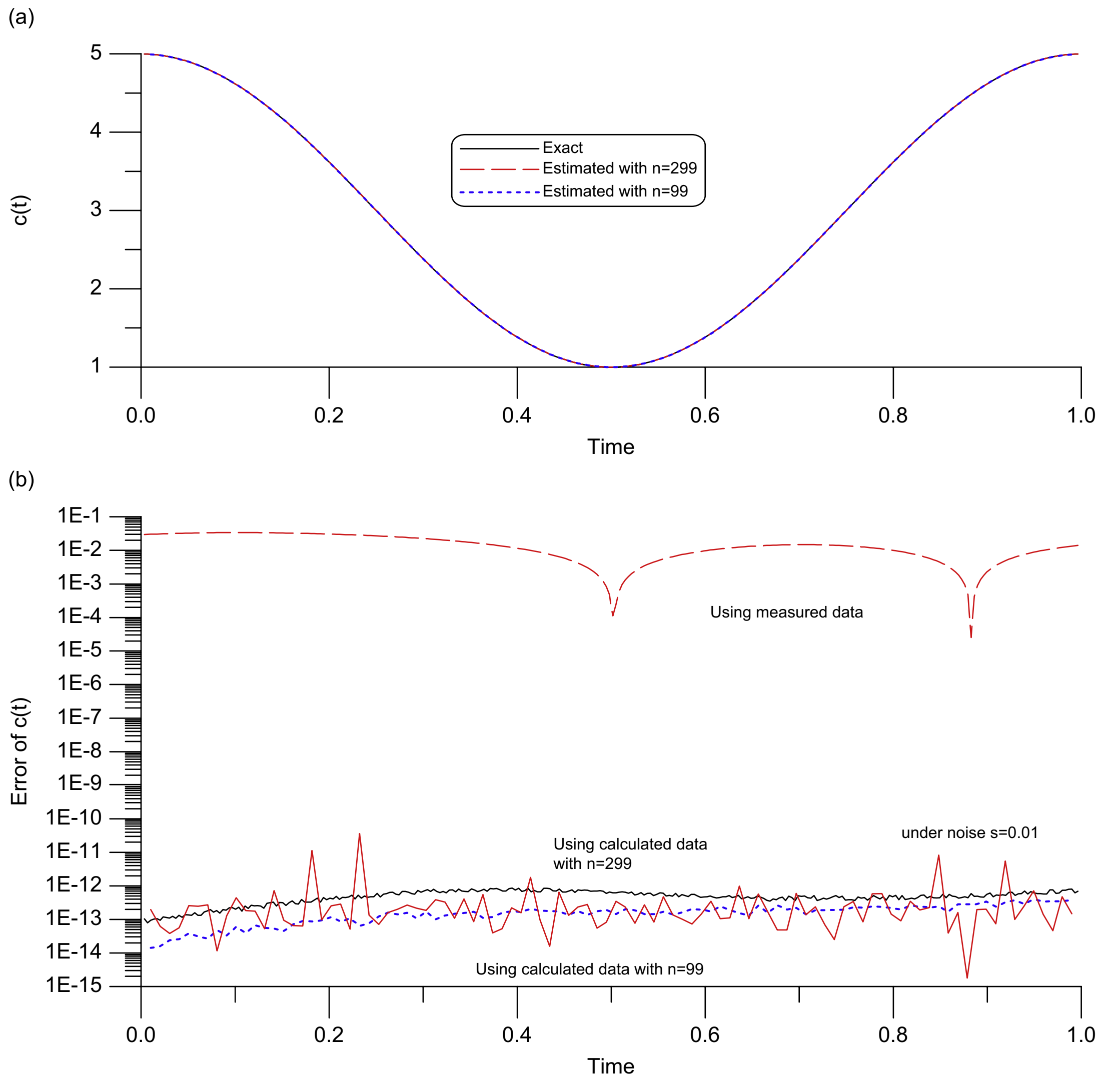

Fig. 2. For Example 1: (a) comparing estimated and exact $c(t)$, and (b) plotting the estimation errors by using the calculated and measured data.

Similarly, from Eq. (55) we can easily estimate k_i when c_i is given. The estimation of k_i shown by the dashed line matches very well with the exact one as shown in Fig. 3(a) by the solid line, where the maximum error is very small using the calculated data. When the data are taken to be $u_i^f = (1 + x_f)\phi_i$ the maximum error is about 1.62×10^{-1} as shown in Fig. 3(b) by the dashed line. It can be seen that the estimated results of $k(t)$ are very accurate whether $n = 299$ or 99 is used.

As mentioned in Section 1 the inverse vibration problem is sensitive to the measurement error. In the case when the measured data are contaminated by random noise, we are concerned with the stability of our estimation method, which is investigated by adding a random noise to the measured data. We use the function RANDOM_NUMBER given in Fortran to generate the noisy data $R(i)$, where $R(i)$ are random numbers in $[-1, 1]$. The noise is obtained by multiplying $R(i)$ by a factor s, and we let $u_i^0 + 2.576sR(i)$ replace u_i^0 in our estimation equations. The factor 2.576 is for considering a 99% confidence bound of the measurement error [5]. In Fig. 2(b) we compare the estimated result of $c(t)$ under a noise of $s = 0.01$ with that of $s = 0$ by using the same number of $n = 99$. Similarly, for the estimation of $k(t)$ we compare two estimations under $s = 0$ and 0.01 in Fig. 3(b). It can be concluded that the present method is robust against the noise.

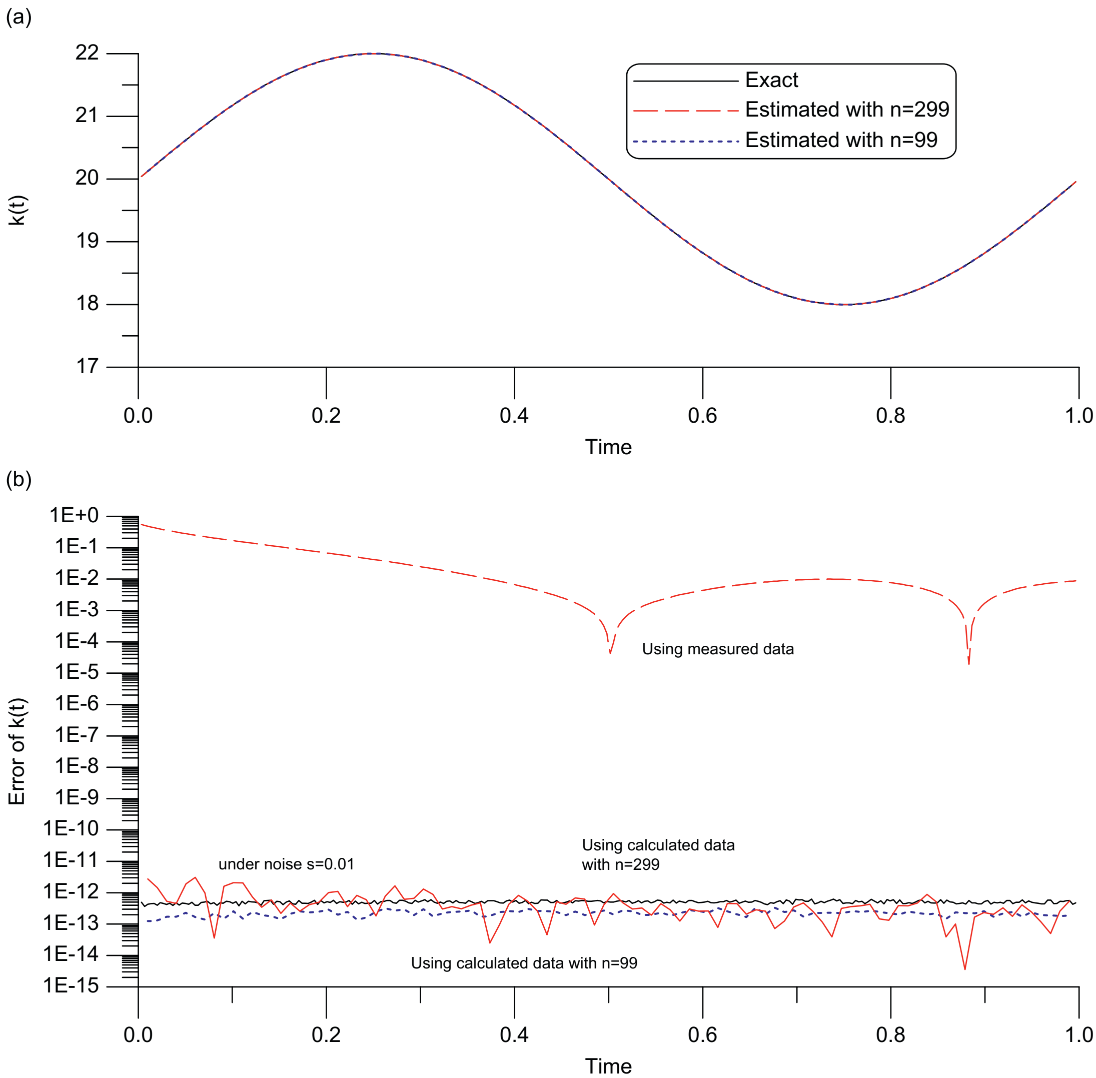

Fig. 3. For Example 1: (a) comparing estimated and exact $k(t)$, and (b) plotting the estimation errors by using the calculated and measured data.

Let us consider Eqs. (72)–(74) again. However, we use the method in Section 5 to estimate both $c(t)$ and $k(t)$. In order to obtain the data of $\phi(t)$ and $\psi(t)$ we apply the RK4 on Eqs. (1)–(3), where $A_0 = 0.1$ and $B_0 = 1$ were fixed. In this calculation we have fixed $\Delta t = 1/200$.

Applying Eqs. (70) and (71), the estimations of c_i and k_i as shown by the dashed lines match very well with the exact ones as shown in Figs. 4(a) and (b) by the solid lines, where the maximum errors are very small as shown in Figs. 4(c) and (d).

6.2. Example 2

Let us consider discontinuous and oscillatory parameters:

$$c(t) = \begin{cases} 2 & t \in [0, 0.1], \\ 10 & t \in (0.1, 0.3), \\ 8 & t \in (0.3, 0.6], \\ 5 + \sin(10\pi t) & t \in (0.6, 1], \end{cases} \tag{75}$$

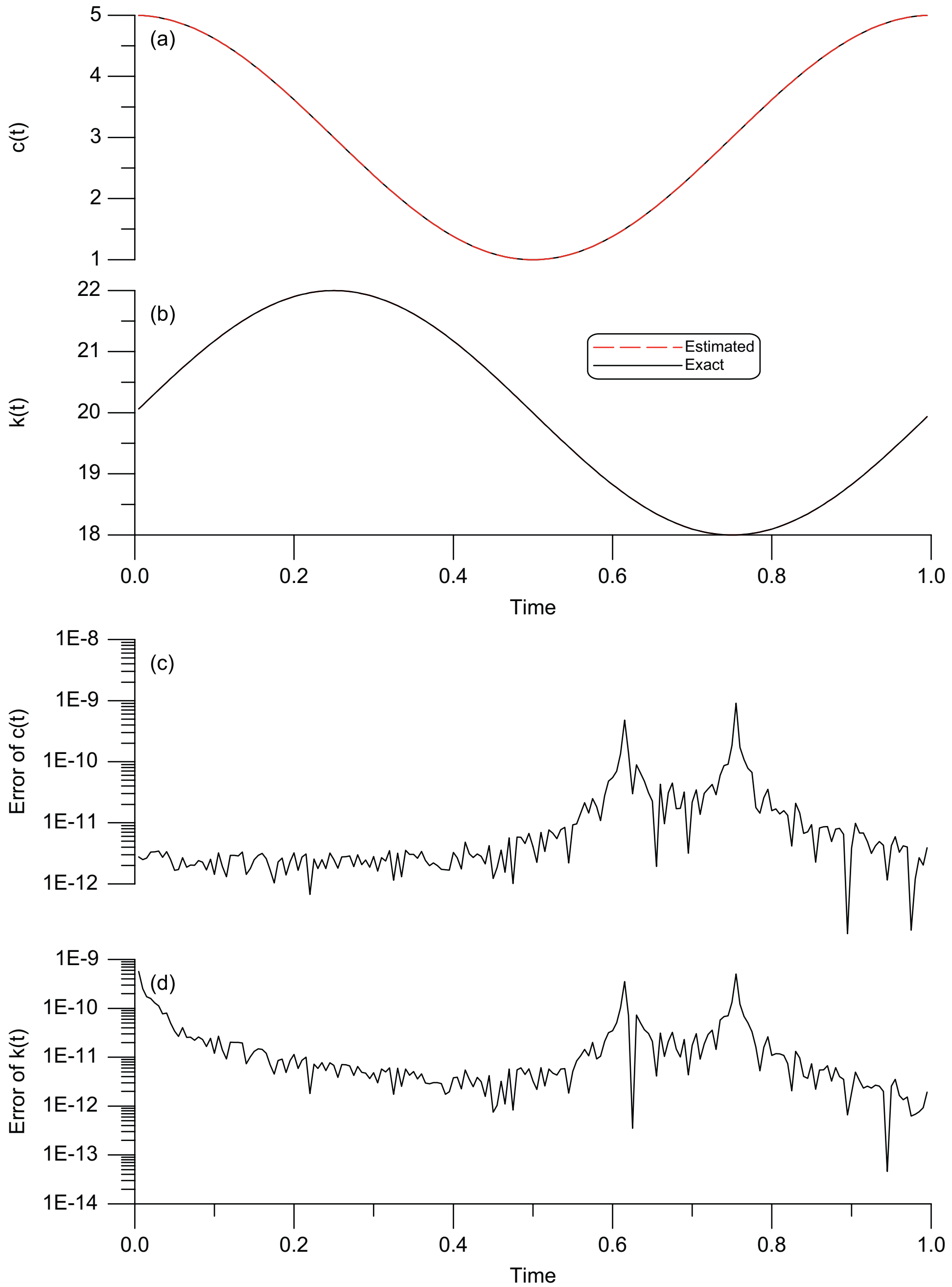

Fig. 4. For Example 1: (a) comparing estimated and exact $c(t)$, (b) comparing estimated and exact $k(t)$, (c) plotting the estimation errors of $c(t)$ by using the calculated data, and (d) plotting the estimation errors of $k(t)$ by using the calculated data.

$$k(t) = \begin{cases} 20 & t \in [0, 0.3], \\ 30 & t \in (0.3, 0.6], \\ 20 + \sin(10\pi t) & t \in (0.6, 1]. \end{cases} \tag{76}$$

For this case while the profile of $\phi(t)$ is plotted in Fig. 5(a), the profile of $\psi(t)$ is plotted in Fig. 5(b). Then we apply the one-step GPS to Eq. (11) to obtain the data u_i^f with $x_f = 0.001$. These data are plotted in Fig. 5(c) by

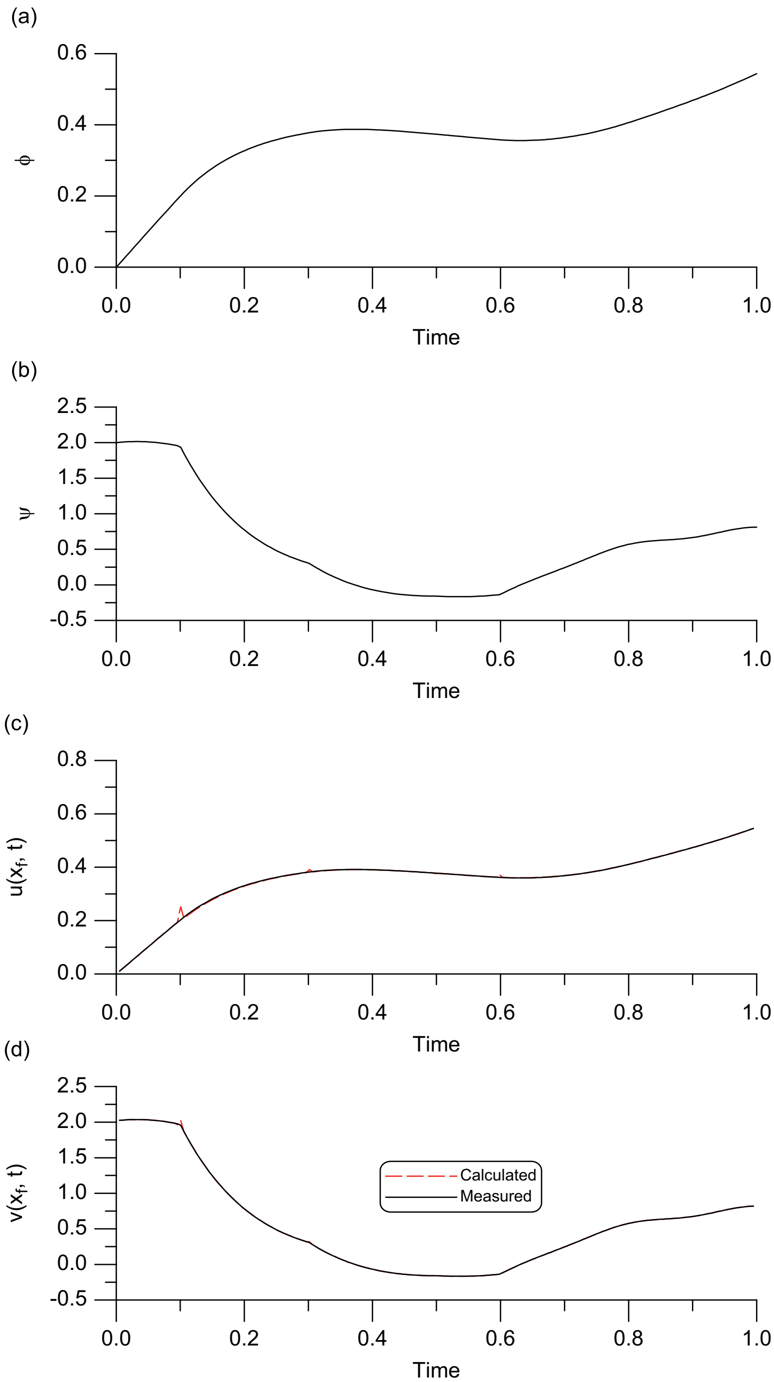

Fig. 5. For Example 2: (a) displaying the displacement, (b) displaying the velocity, and (c) and (d) comparing the calculated and measured data at $x_f = 0.001$.

the dashed line, and are compared with the measured data $(1 + x_f)\phi_i$ as shown by the solid line. Because these two data are very close, the solid line and the dashed line are almost coincident. At the same time, we apply the one-step GPS on Eq. (61) to obtain the data v_i^f. These data are plotted in Fig. 5(d) by the dashed line, and are compared with the measured data $(1 + x_f)\psi_i$ as shown by the solid line.

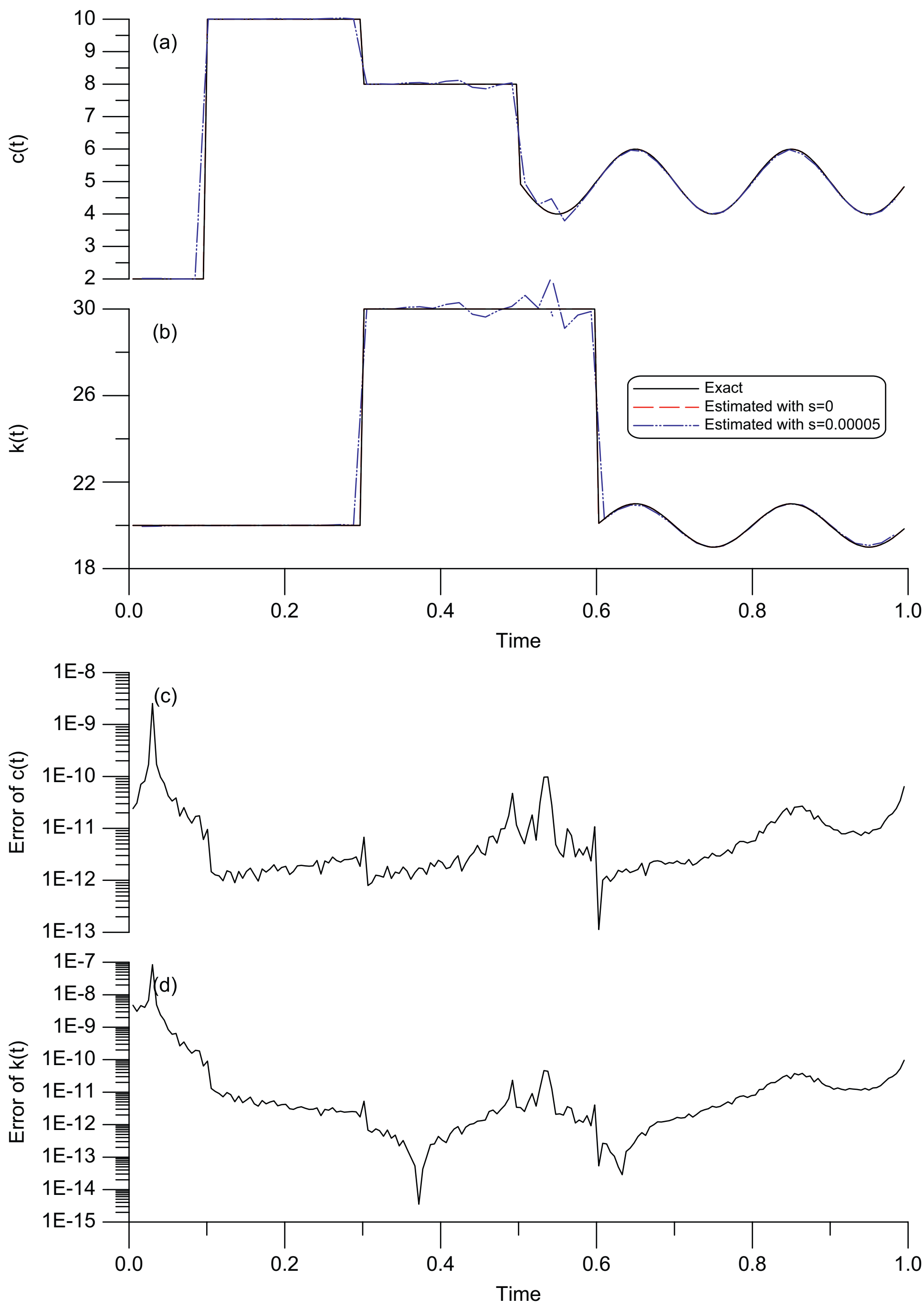

Fig. 6. For Example 2: (a) comparing estimated and exact $c(t)$, (b) comparing estimated and exact $k(t)$, (c) plotting the estimation errors of $c(t)$ by using the calculated data, and (d) plotting the estimation errors of $k(t)$ by using the calculated data.

Applying Eqs. (70) and (71), the estimations of c_i and k_i as shown by the dashed lines match very well with the exact ones as shown in Figs. 6(a) and (b) by the solid lines, where the maximum errors are very small as shown in Figs. 6(c) and (d). We also consider a noise with $s = 0.00005$ on the input data, whose results are shown in Figs. 6(a) and (b) by the dashed–dotted lines. The against the noise is weak for this computation.

6.3. Example 3

Huang [5] has used the conjugate gradient method to identify $k(t)$ under the following conditions:

$$c = 2, F(t) = 100 + t, \tag{77}$$

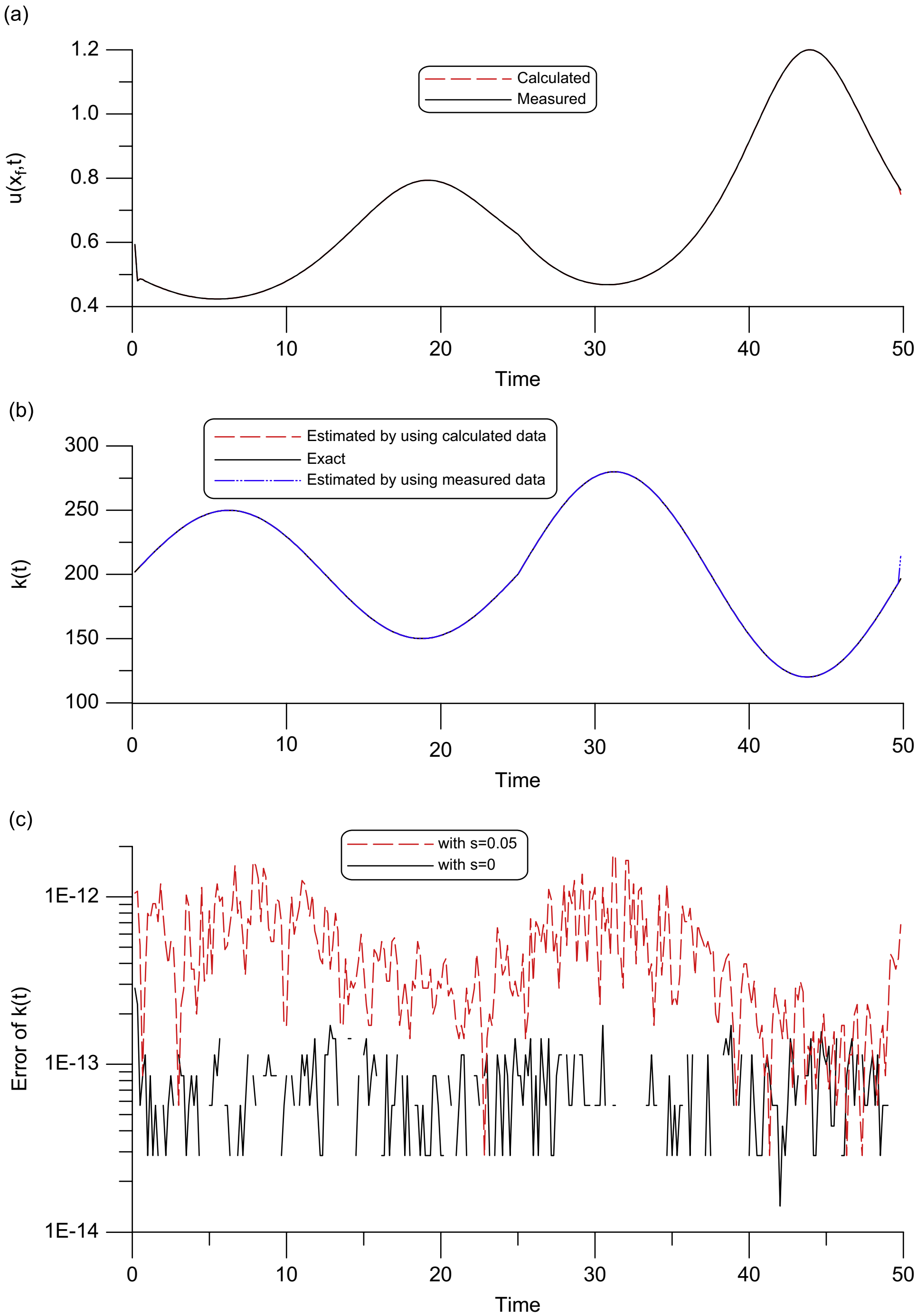

Fig. 7. For Example 3: (a) comparing the calculated and measured data, (b) comparing estimated and exact $k(t)$, and (c) plotting the estimation errors by using the calculated data under noises $s = 0$ and 0.05.

$$k(t) = \begin{cases} 200 + 50\sin\left(\dfrac{2\pi t}{25}\right), & t \in [0, 25], \\ 200 + 80\sin\left(\dfrac{2\pi t}{25}\right), & t \in (25, 50], \end{cases} \tag{78}$$

where $t_f = 50$ was fixed.

The numerical results with noises $s = 0$ and 0.05 were compared with the exact solutions in Fig. 7, where we use $\Delta t = 50/300$. The data of displacement at x_f are compared for measured and calculated ones in Fig. 7(a). From Fig. 7(c) it can be seen that the noise level with $s = 0.05$ disturbs the numerical solution deviating from the exact solution very little with an estimating error in the order of 10^{-12}. It appears that the measurement noise has no obvious effect on our estimation even if the level of noise is large up to five times.

In comparing our estimation results with that obtained by Huang [5] as shown in Figs. 2 and 4, one may highly appreciate that the present method is much more accurate and stable than the conjugate gradient method. To the best of our knowledge, no report appears in the open literature that in the estimations of c and k one can obtain closed-form estimating solutions. It is clear that the accuracy and efficiency of our LGEM are much better than other methods.

7. Conclusions

In order to estimate the time-dependent damping and stiffness functions under the measured data of displacement and velocity, we have employed the LGEM to derive algebraic equations and solved them in a closed form. The key points were that we have transformed the inverse vibration problem into an identification problem for a parabolic-type PDE and a hyperbolic-type first-order PDE and then established a one-step GPS for the semi-discretizations of these two PDEs.

Numerical examples were worked out, which show that the new LGEM is applicable for both the estimations of damping and stiffness functions. Through this study, it can be concluded that the new estimation method is accurate, effective, and stable. Its numerical implementation is very simple and the computational speed is very fast.

In contrast to other parameter estimation methods, the advantages of the present method are that it does not need any prior information of the functional forms of damping and stiffness coefficients, no initial guesses are required, no iterations are required, and closed-form solutions are available. This is the first time that a closed-form estimation method has been constructed for the inverse vibration problems.

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