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Exact region of stability for a linear orthogonal cutting model

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Abstract

In this work, stability of chatter in orthogonal cutting is investigated in order to identify the critical spindle speeds. The process is modeled by a second-order linear delay-differential equation and the characteristic equation is analyzed in formulating the continuous stability boundaries, explicitly for any given range of the spindle speeds. Moreover, a simple algorithm is introduced for checking whether the system is in the stable zone without determining the whole stability diagram. Numerical simulations and comparison with some other methods are presented to justify the theoretical results.

1. Introduction

The process of cutting in turning exhibits complicated and interesting dynamics, and is best modeled by a system of nonlinear autonomous delay-differential equations [1–6]. The mathematical model capturing the real-world behavior is constructed as simple as possible by taking into account the qualitative features of experimental data. Due to the fact that no analytical solutions are possible for the full nonlinear (infinite dimensional state-space) model and there currently exist no reasonably nonconservative procedures for determining the stability behavior, only qualitative properties may be analyzed based on various numerical and semianalytical techniques. The model exhibits a rich variety of bifurcation phenomena, exhibiting a transition between a stable fixed point and an oscillatory behavior around a periodic or a quasi-periodic orbit, including route to chaos [7–9]. This transition is termed the onset of chatter and presents mostly sub-critical and occasionally super-critical Hopf bifurcation (limit cycle) [10–12]. It involves a jump to a large amplitude of oscillation (a periodic orbit) at the bifurcation point through a change in the stability of the equilibrium point. In close neighborhood to the bifurcation point, use of a local nonlinear mapping, e.g. center manifold reduction and normal forms [13–15], is the most common to deduce the local behavior of the full and weakly nonlinear system. Approximate analytical solutions have also been given for weakly nonlinear systems [16,17] by employing a center manifold reduction.

Even though the actual solution and the post-bifurcation behavior, in general, depend on the strength of the nonlinearity and are obtainable only by numerical means, the exchange of stability and instability (the nature

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Nomenclature

c	damping coefficient ($\mathrm{N\,s\,m^{-1}}$)
F	thrust force (N)
h	instantaneous chip thickness (m)
h_0	nominal chip thickness (m)
k	stiffness coefficient ($\mathrm{N\,m^{-1}}$)
K	material-dependent constant
K_1	design parameter, $Kw\rho h_0^{\rho-1}$
m	mass inertia of the workpiece (kg)
r	radius of the workpiece (m)
v_0	cutting velocity ($\mathrm{m\,s^{-1}}$)
w	chip width (m)
y	horizontal displacement of the workpiece (m)
$y(t–\tau)$	horizontal displacement during the previous pass (m)
α, f	normalized stiffness coefficients ($\mathrm{N\,m^{-1}}$)
β	normalized damping coefficient ($\mathrm{N\,s\,m^{-1}}$)
Θ	Heaviside step function
υ	root tendency
ρ	experimentally determined constant
σ	phase angle (rad)
τ	period between two cuts or time delay (s)
ω	frequency ($\mathrm{rad\,s^{-1}}$)
ω_1, ω_2	chatter frequencies ($\mathrm{rad\,s^{-1}}$)
Ω	constant angular velocity of the workpiece ($\mathrm{rad\,s^{-1}}$)
$(\ll \cdot)$	floor of the related variable

of local bifurcation) at the stability boundary is usually (and well) predicted by a linearized model [18–20]. The main reason in working with the linear model is that the effort to build a nonlinear model is too large and requires a great deal of experimental work. In addition, because the linear model can be profitably used in preventing chatter, there usually is no (immediate) need for investigating such nonlinear phenomena as chaos and bifurcation.

The locations of Hopf bifurcations are usually shown by a two-parameter stability diagram [21], as a key element utilizable in preventing the detrimental transient oscillations and thus maximizing chatter-free material removal. The diagram is made of a number of individual lobes separating the stable and the unstable regions. Each lobe span a limited range of spindle speeds, and their convolution results in the overall stability diagram. Using this diagram, one sees that, as the depth-of-cut is closer or further above the stability boundary, the more amplified and unstable the chatter vibration is. In general, the stability boundary is obtained numerically in an iterative manner by either the frequency [22–26] or the time domain simulation [27] techniques. The time domain simulation approach is more reliable than the frequency-based approach, because other effects including the nonlinearity of the process can be easily incorporated into the model. However, time domain simulations are computationally inefficient for the exploration of parameter space and always meet numerical instability (linked to flip bifurcation). In Ref. [23], an iterative procedure employing an online speed selection system is presented in generating the diagram. But, the approach suffers from the requirement of tremendous computational burden (as it is also the case for Refs. [28–30]). This method was later improved by Altintas [25], who employed a simplified linearized model to determine the stability diagram in an iterative manner. Later on, Budak and Altintas [31] described a more general formulation of their method, which was based on a Fourier-series expansion of the time-varying cutting force coefficients. In this method, the closed-loop transfer function of the machine-tool system is first identified, and the real part of the characteristic equation set to zero. Next, at each of the chatter frequencies estimated, the characteristic equation is solved for the eigenvalues and a unique expression for the depth of cut as a function of the spindle speed is obtained. This formulation determines the stability boundaries (lobes), as the chatter frequency varies around all dominant modes of the transfer function. It is obvious that the accuracy of the method depends highly on the number of Fourier terms used and they can be neglected only if variations in the cutting force are small. If this is not the case, the stability lobes diagram cannot be predicted accurately. However, if the approach is supported by an experimentally determined transfer function [32], the stability diagram may be obtained with higher accuracy.

Chattering between the cutting tool and the workpiece can, in general, result from one or more of the following; regenerative effects, mode coupling, loss-of-contact dynamics, friction, structural and other sources of nonlinearities [2,4,7]. Most researches have concentrated in the area of regenerative chatter [21], because unstable chatter vibration is most damaging. Hence, this paper uses the word chatter in addressing regenerative chatter only.

In this paper, a detailed analysis of regenerative chatter is developed with an emphasis on the causes of instabilities (chatter). Our analysis leads to an exact and non-iterative formulation of all of the stability crossing curves explicitly in the parameter space, between the delay and the nominal chip thickness (or the width of cut). This is done by considering the number of purely imaginary characteristic roots and partitioning the delay space into stable and unstable regions. As a byproduct of the proposed approach, a simple algorithm is developed in determining weather the time response of the process is asymptotically stable, without determining the whole stability diagram. The proposed approach provides complete stability characterization of regenerative chatter and differs from the standard direct [21, pp. 55, 131] methods. Moreover, it is numerically more efficient and reliable than the classical iterative methods and the pseudo-delay method [19]. Unlike conventional methods that determine the stable intervals for the spindle speeds in the range of ($N_{\min}$, ∞), our approach identifies the stable intervals for any given range of the spindle speeds.

2. Cutting dynamics

For effective high-speed machining, knowledge and understanding of dynamic characteristics of spindle/tool-holder/tool system is essential, and this study uses the well-known second-order representation [13] shown in Fig. 1 for an orthogonal turning process. The tool is assumed rigid, and the workpiece is allowed to vibrate only in the horizontal y direction as it rotates at a constant angular velocity. The dynamical equation of cutting is [7,13]

$$m\ddot{y} + c\dot{y} + ky = \operatorname{sgn}(v_0 - \dot{y})[F(h) - F(h_0)], \tag{1}$$

where m is the workpiece mass inertia, c the damping, and k the stiffness coefficients in y direction, respectively. A nonlinear cutting force dependence on chip thickness, $F(h) = \Theta(h)Kwh^{\rho}$, is assumed [21]. Here, w is the chip width, h the chip thickness, and K the material-dependent constant. The value of ρ is obtained experimentally, and it is approximately 0.41 for aluminum and 0.75 for steel. $\Theta(h)$ is the Heaviside step function. $F(h_0) = Kwh_0^{\rho}$ with h_0 denoting the nominal chip thickness (or feed). The chip thickness is described by $h = h_0 - y(t) + y(t-\tau)$, where $y(t-\tau)$ corresponds to the position of the workpiece during the previous pass. $\tau = 2\pi/\Omega$ is the period of one revolution of the workpiece, with Ω being the constant angular velocity. Finally, v_0 is the relative velocity between the tool and the workpiece tangent to the workpiece surface ($v_0 = \Omega r$, where r is the radius of the workpiece).

If h^{ρ} is expanded into a power series around the desired chip thickness h_0, and the higher-order nonlinear terms are neglected, Eq. (1) becomes

$$m\ddot{y} + c\dot{y} + (k + k_{\tau})y - k_{\tau}y(t-\tau) = k_0, \tag{2}$$

where $k_{\tau} = \Theta(h)K_1$, $K_1 = Kw\rho h_0^{\rho-1}$, $k_0 = -Kwh_0^{\rho}\Theta(-h)$. Here, $v_0 > \dot{y}$ is assumed during linearization [7]. This is a reasonable approximation since v_0 is generally much greater than $\dot{y}$. For example, consider a tool with a

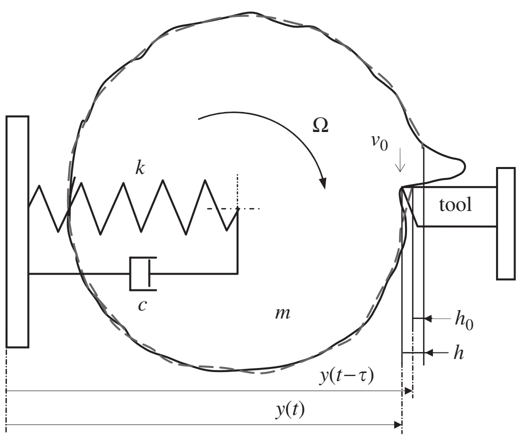

Fig. 1. Orthogonal cutting model [7].

0.1 m workpiece diameter and a spindle speed of 600 rev min^{-1}. Suppose the y-direction natural frequency is 300 Hz with a 10 μm amplitude (A), which is reasonable during chatter oscillations. It is then calculated that $v_0 = \Omega r = 3.14\,\mathrm{m\,s^{-1}}$ is much larger than $\dot{y} = 2\pi f A = 0.0188\,\mathrm{m\,s^{-1}}$.

3. Stability problem

The system stability is in question if $h>0$, or when cutting is in progress. In this case, it follows from Eq. (2) that

$$\ddot{y} + \beta\dot{y} + \alpha y = f y(t-\tau), \tag{3}$$

where $\beta = c/m$, $\alpha = (k+K_1)/m$, $f = K1/m$ are all positive real numbers and

$$\alpha = f + k/m. \tag{4}$$

Making use of the Laplace transform method, the characteristic equation is obtained as

$$s^2 + \beta s + \alpha - f\mathrm{e}^{-s\tau} = 0. \tag{5}$$

For asymptotic stability, this equation must yield roots with negative real parts. Eq. (5) is a transcendental algebraic equation on s, and the stability depends on β, α, f, and the time delay, τ.

Exact analytical solution of Eq. (5) cannot, in general, be obtained in terms of elementary functions. Special cases of Eq. (5) were considered by Chen et al. [33], and exact analytical solutions were given. Asl and Ulsoy [34] presented a closed-form solution to the general problem in matrix form, and stability lobes were computed numerically. Both studies [34,34] are based on a solution of a transcendental equation, expressed in terms of Lambert function, which was first derived by Briggs [35]. Because the solution including a nonelementary (Lambert) function is a series solution extending to infinity, it is not logical to call this solution analytic. Warminski et al. [16] claims to be the first to obtain a weakly analytical solution in the nonlinear cutting process, in terms of the cutting depth and the speed. Perturbation method of multiple scales was used to obtain an analytical solution for the non-resonant case.

The roots of Eq. (5) are the eigenvalues of the equilibrium (origin), and there is infinite number of them. The origin is stable if all the eigenvalues have negative real parts, and unstable if at least one of the eigenvalues has positive real part. Changes of stability may occur whenever an eigenvalue has zero real part. If a complex pair of eigenvalues crosses the imaginary axis, Hopf bifurcation is pronounced. If there are two pairs of complex conjugate roots with zero real parts, two curves of Hopf bifurcation cross (double Hopf bifurcation).

3.1. Mathematical analysis

The stability curves are obtained by setting $s = \mathrm{j}\omega, \mathrm{j} = \sqrt{-1}$, in Eq. (5). This leads to

$$-\omega^2 + \alpha + \omega\beta\mathrm{j} = f\mathrm{e}^{-\mathrm{j}\omega\tau} \tag{6}$$

or

$$\cos(\omega\tau) = \frac{-\omega^2 + \alpha}{f}, \quad \sin(\omega\tau) = \frac{-\omega\beta}{f}. \tag{7}$$

Eq. (7) can be analyzed to yield two positive real roots of ω, namely $\omega_1 > \omega_2 > 0$. Thus, Eq. (5) presents periodic solutions for $s = \pm\mathrm{j}\omega_1$ and $s = \pm\mathrm{j}\omega_2$. Noting that Eq. (6) implies

$$|-\omega^2 + \alpha + \omega\beta\mathrm{j}|^2 - |f|^2 = 0,$$

one can calculate the two positive real roots as

$$\omega_{1,2} = \sqrt{\alpha - 0.5\beta^2 \pm \sqrt{(\alpha - 0.5\beta^2)^2 - \alpha^2 + f^2}}, \tag{8}$$

where α, β, and f are positive real numbers. Also note that α is larger than f by definition, given in Eq. (4). The damping parameter, β, is much smaller than the stiffness parameter, α. Thus, the existence of such solution requires only that

$$f \geqslant \beta\sqrt{\alpha - 0.25\beta^2}. \tag{9}$$

It will be shown in Section 3.2 that, with smaller cutting forces, f, the process is always stable no matter what the delay is. Eq. (9) implies a required amount of minimal damping for having a stable region independent of the delay term.

The solutions for ω are independent of the time delay, τ, but τ depends on ω. At the positive real values of ω, one can solve equations in Eq. (7) to obtain positive real values of the time delay, τ, which determine stability limits of cutting. Even with all positive real roots of ω at a certain working-speed, stability switches may still occur because of the sign changes in Eq. (6) as τ increases. Thus, a finite number of stability switches may still occur between $\tau = 0$ and a large value of τ.

Let us look at a simple case first. In Eq. (5), if the damping term, β, is zero, changes of stability may occur when the system has a zero eigenvalue (whenever $\alpha = f$, which defines a steady-state bifurcation but is never the case according to Eq. (4)), or a complex pair of eigenvalues with zero real parts, at $s = \mathrm{j}\omega_{1,2} = \mathrm{j}\sqrt{\alpha \pm f}$, given by Eq. (8). For the steady-state bifurcation, the most critical time delay value is the smallest $\tau > 0$. Solving Eq. (7) in the latter case and using the fact that $\alpha > f$, $\tau = \tau_{1,2} = 2k\pi/\omega_{1,2}$ for $k = 1,2\ldots$ is obtained. $\tau_{1,2}$ and $\omega_{1,2}$ define the two families of surfaces (or curves, if α is fixed). Thus, for $\beta = 0$, the origin is stable in the regions defined by $\tau_1 = 2k\pi/\omega_1 < \tau < \tau_2 = 2k\pi/\omega_2$. For a constant positive f, as τ increases, the origin passes through stable and unstable regions, and stays unstable after $\tau > \tau_2$ (Hopf bifurcation). Whether Hopf is sub or super critical depends on the nonlinearity of the cutting process.

If two pairs of complex roots, $s = \pm\mathrm{j}\omega_1$ and $s = \pm\mathrm{j}\omega_2$, exist for the same τ, Hopf bifurcation curves cross and double Hopf points of intersections are obtained. Since double Hopf bifurcation points are resonant, they greatly influence the system dynamics. As far as the author is aware of, the closed-form solution of these points is not given in literature. However, numerical computation is easy.

Note that with small nonzero damping, the system may still present dynamics of a resonant double Hopf bifurcation, because the system is in a close neighborhood of a double Hopf point. How close depends on the technological parameters and the nonlinearity of the cutting process. Moreover, if certain amount of damping is present, choosing the right values of α and f leads to a stable (Hopf free) system, for all the delay values. In general, more amount of friction helps stabilize the system, and there are various sources of friction during cutting. Chen et al. [33] considered a special case where $\beta = 2\sqrt{\alpha}$. Employing the Lambert function, exact analytical solution was given, and the stability bound was determined as $\beta > 2\sqrt{f}$. This result means working at a low rotational speed and with a small depth of cut. Thus, it is not practical for high-speed machining. But, it shows how sufficient damping globally stabilizes the system. Note also that both α and f increase as the cutting speed or the depth of cut is made larger, and vice versa. Thus, the operator has the tool to accomplish a chatter-free cutting.

Eq. (5) can be written, for the general case, in the form

$$\tau = \frac{1}{s}\ln\left(\frac{f}{s^2 + \beta s + \alpha}\right),$$

which gives the following equation when differentiated with respect to τ,

$$\frac{\mathrm{d}s}{\mathrm{d}\tau} = -s\left(\frac{2s + \beta}{s^2 + \beta s + \alpha} + \tau\right)^{-1},$$

which is called the root sensitivity. The root tendency is then obtained by

$$\vartheta = \mathrm{sgn}\left[\mathrm{Re}\left(\frac{\mathrm{d}s}{\mathrm{d}\tau}\right)\right] = \mathrm{sgn}\left[\mathrm{Re}\left(\frac{s}{K - \tau}\right)\right] = \mathrm{sgn}[\mathrm{Re}(s(\overline{K} - \tau))], \tag{10}$$

where $\overline{K}$ is the complex conjugate of K,

$$K(s) = \frac{-(2s + \beta)}{(s^2 + \beta s + \alpha)}.$$

If we set $s = j\omega$, then it follows from Eq. (10) that

$$\vartheta = \text{sgn}[-(\alpha - 0.5\beta^2) + \omega^2)]. \tag{11}$$

Eq. (11) helps in figuring out how the stability of the system changes as the time delay varies between zero and a large value, τ_f. Thus, further analysis is needed to determine the exact set of time-delay values for which the system is stable in a given delay interval from $\tau = 0$ to τ_f.

By the fact that $\ln(a + bj) = \ln|a + bj| + \arg(a + bj)j + 2\pi kj$, with real a, b and for $k = \ldots,-2,-1,0,1,2,\ldots$, Eq. (6) implies

$$\tau = \frac{2k\pi}{\omega} - \frac{1}{\omega}\arg(\alpha - \omega^2 + \beta\omega j)$$

or

$$\tau = \frac{2k\pi - \sigma}{\omega}, \tag{12}$$

where

$$\sigma = \arg(\alpha - \omega^2 + \beta\omega j) \tag{13}$$

has the range $[0, \pi]$.

Eq. (12) defines the stable set of positive real time-delay values, for $k = 1,2,\ldots$ The zero time delay must also be added to the set, because it is already known to yield a stable solution.

If Eq. (11) is rewritten as

$$\vartheta_i = \text{sgn}[-(\alpha - 0.5\beta^2) + \omega_i^2)], \quad i = 1, 2, ...,$$

the fact that $\omega_1 > \omega_2 > 0$ implies

$$\vartheta_1 = \text{sgn}[\sqrt{(\alpha - 0.5\beta^2)^2 - \alpha^2 + f^2})] = 1,$$
$$\vartheta_2 = \text{sgn}[-\sqrt{(\alpha - 0.5\beta^2)^2 - \alpha^2 + f^2})] = -1.$$

This result tells us that the system moves out of the stability region at $s = j\omega_1$ and into the stability region at $s = j\omega_2$, with increasing time delay. The stability regions in terms of time delay, τ, may then be defined using Eq. (12), for $k = 1,2\ldots$, as

$$S = [0, \min(R)_{i=1}] \cup [\max(R)_{i=k}, \min(R)_{i=k+1}] \cdots,$$

where

$$R = \left(\frac{2i\pi - \sigma_1}{\omega_1}, \frac{2i\pi - \sigma_2}{\omega_2}\right).$$

Eq. (12) implies that σ, defined in Eq. (13), is a positive real number and $\sigma_1 > \sigma_2 > 0$, due to the assumption that $\omega_1 > \omega_2 > 0$. Thus, the set S can be simplified as

$$S = \left[0, \frac{2\pi - \sigma_1}{\omega_1}\right] \cup \left[\frac{2k\pi - \sigma_2}{\omega_2}, \frac{2(k+1)\pi - \sigma_1}{\omega_1}\right]_{k=1,2,\ldots}. \tag{14}$$

Note that, in the method of Altintas [25], there exist two solutions of the quadratic equation for the eigenvalue, leading to two different stability boundaries. Both stability boundaries are numerically computed, and the lower one is chosen. Eq. (14) eliminates this need.

3.2. Implications

Let us rewrite Eq. (12) as

$$\tau_{k_i} = \frac{2k_i\pi - \sigma_i}{\omega_i}, \tag{15}$$

where $k_i = 1,2,\ldots$ for both $i = 1$ and 2. Noting that away from the stability boundary, k_i is not an integer, let us consider $(\lll k_i)$, the floor of k_i (k_i rounded to the nearest integer towards minus infinity). Because the system steps into the unstable zone at ω_1 and into the stable zone at ω_2, Eq. (15) implies that the system is stable if $(\lll k_1) = (\lll k_2)$ and unstable when $(\lll k_1) \neq (\lll k_2)$. This result is crucial because it makes it possible to tell if the system dynamics presents an asymptotically stable time response, when all the system parameters including the delay are given. In the case that $(\lll k_1) = (\lll k_2) = \mu$ and with a_1, a_2, K, and τ given, the system is stable since

$$\tau_{\min} = \max\left\{0, \frac{2\pi\mu - \sigma_2}{\omega_2}\right\} < \tau < \frac{2\pi(\mu + 1) - \sigma_1}{\omega_1} = \tau_{\max}.$$

This equation describes the amount of change in the spindle speed, $\Omega = 1/\tau$, that causes chattering.

Eq. (5) can also be used to find the root sensitivity, defined by

$$\left.\frac{\mathrm{d}s}{\mathrm{d}f}\right|_{s=\mathrm{j}\omega} = \frac{\beta(\alpha + \omega^2) + \tau f^2 + (2\omega^2 + \beta^2 - 2\alpha)\mathrm{j}}{f(\tau^2 f^2 + \beta^2 + 4\omega^2 + 2\beta\tau\omega^2 + 2\beta\tau\alpha)}. \tag{16}$$

For this equation, the imaginary part becomes zero if $\omega^2 = \alpha - 0.5\beta^2$ or $\omega = 0$. For $\omega > 0$ and $\alpha > 0.5\beta^2$, $\omega_{1,2} = \sqrt{\alpha - 0.5\beta^2}$. In this case, the root tendency defined by Eq. (11) becomes zero, which implies that the system constantly stabilizes and destabilizes itself, no matter what f is. Moreover, Eq. (8) implies

$$f = \frac{K_1}{m} = \beta\sqrt{\alpha - 0.25\beta^2}, \tag{17}$$

where $\alpha = (k + K_1)/m$, by Eq. (4). This equation defines the minimum value of f to be used in Eq. (14), and the process is stable for all the delay values smaller than this value.

Let us rewrite Eq. (17) in the form of

$$\frac{K_1}{k} = \frac{1}{2}\left(\frac{\beta}{\omega_n}\right)^2 + \frac{\beta}{\omega_n}, \tag{18}$$

where $\omega_n = \sqrt{k/m}$. One can then note that this equation implies that as the damping increases, stable cutting with a larger depth for any delay is possible. However, increase in damping does not always mean a more stable cutting process, at a fixed depth of cut and speed. This is due to the fact that the stability curve stretches as the damping is increased. As a result, in case of cutting near the stability curve, bifurcation may dominate as a result of increase in damping and lead to instability. Because increased damping raises the stability lobes and has a stabilizing effect, if no bifurcation is present, active damping is a good candidate in increasing the machining productivity.

Eq. (18) also implies that increase in k brings down the stability curve and stretches it. Also note that the cutting force, as well as the amplitude of vibrations, gets smaller for the same depth of cut and speed, as k increases. However, in case of cutting in the neighborhood of the stability curve, the bifurcation may take over and lead to an overall increase in amplitude of vibrations. In this case, increase in f may result in further decrease in the vibration amplitude, larger forces (speed) of cutting may then be beneficial and desired.

4. Simulations and discussion

This section presents numerical simulation results, all of which are generated using the realistic values of the technological parameters reported by Litak [7]. These constants are $m = 17.2\,\mathrm{kg}$, $c = 147.92\,\mathrm{N\,s\,m^{-1}}$, and $k = 11452.7232\,\mathrm{kN\,m^{-1}}$.

Eq. (14) is used to obtain the stability diagram, which is illustrated in Fig. 2. The horizontal axis is the angular speed, $\Omega = 1/\tau$, and the vertical axis a dimensionless variable, K_1/k. Stable and unstable regions are marked. The data points represented by circles correspond to K_1/k defined in Eq. (18), and

$$\tau = \frac{2k\pi - \sigma_1}{\omega_1} = \frac{2k\pi - \sigma_2}{\omega_2} \tag{19}$$

holds at these points. These locations, where $\omega_1 = \omega_2$ and $\sigma_1 = \sigma_2$, present resonant double-Hopf characteristics, and the motion is amplified greatly. Note that these double-Hopf points are analytically

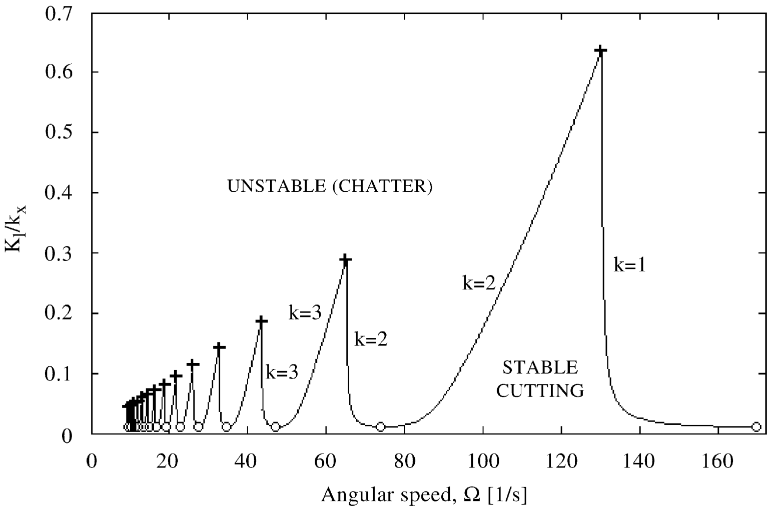

Fig. 2. Stability diagram.

determined by Eqs. (18) and (19). On the other hand, the locations depicted by plus signs are defined by

$$\tau = \frac{2(k+1)\pi - \sigma_1}{\omega_1} = \frac{2k\pi - \sigma_2}{\omega_2}.$$

The curves from a circle marked data point to a plus one in the direction of increasing Ω, are represented (and obtained) by $\tau = 2k\pi - \sigma_1/\omega_1$. Similarly, the curves from a plus marked data point to a circle one are given by $\tau = 2k\pi - \sigma_2/\omega_2$.

Let $(\Omega, K_1/k)$ be the representation of any points in Fig. 2, and consider two different points in this figure (100, 0.1) and (100, 0.3). For the first data location, we get $(\ll k_1) = (\ll k_2) = 1$, and this result indicates a stable system, by our analysis in Section 3.1. If the second data point is used for the same purpose $(\ll k_1) = 2$ and $(\ll k_2) = 1$ are obtained. Because $(\ll k_1) \neq (\ll k_2)$, the system is unstable.

Although the governing delay-differential equation used to model regenerative chatter dynamics in turning has a standard structure, there exist various linear and nonlinear cutting force models [36]. Each of these models yields a slightly different force and causes small variations in the stability diagram. This is due to the fact that chatter is characterized as a limit cycle of the nonlinear process and the stability limit is (well) approximated by a linear model of the system. Because our approach is applicable to any type of linear delay differential equations with a single delay term, it presents an exact match to the stability diagram associated with the specific cutting force model. Moreover, compared with the traditional frequency domain or numerical-based methods, our algorithm determines the critical values of delay at the stability limit of the system with very simple calculations.

5. Generalization and comparison

The method demonstrated here for machine-tool chatter is applicable to a wide variety of systems represented by systems of linear delay differential equations with a single delay. A demonstration of the approach in teleoperation with a fourth-order delay-differential equation is presented in Ref. [37]. The example shows how our algorithm may be used for determining stability characteristics and the critical delays by anyone utilizing such linear delay-differential equation models.

The algorithm may be extended for use in cases with multiple delays (that appears in multiple regenerative effect in chatter [21]) by applying a special procedure based on the Floquet theory. The reasoning for this the underlying dynamics of the multidelay equation are equivalent to those of a single delay equation [38]. However, it must be noted that no exact analytical procedure is possible for determining the exact stability limit of the general linear [39] (and the full nonlinear) delay-differential equation system. One has to resort to either numerical or conservative analytic asymptotic techniques in such cases.

In order to show how our method can also be applied to linear delay-differential equations with commensurate delays, consider

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{x}(t-\tau) + \mathbf{C}\dot{\mathbf{x}}(t-\tau), \tag{20}$$

which has the characteristic equation

$$\det[s\mathbf{I} - \mathbf{A} - (\mathbf{B} + \mathbf{C}s)\mathrm{e}^{-\tau s}] = 0.$$

Here, the matrices are $n \times n$ and $\mathbf{x}$ is $n \times 1$. Specifically, let us consider the example studied by Olgac and Sipahi in Ref. [29], where

$$\mathbf{A} = \begin{bmatrix} 2 & 1 \\ -137.52 & -116.41 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} -3.8 & -2.2 \\ 142.45 & 117.68 \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} 1 & 0.3 \\ -1.208 & -0.2253 \end{bmatrix}.$$

With $s = \mathrm{j}\omega$, $\tau = T/\omega$, $\mathrm{e}^{\mathrm{Tj}} = x$, the characteristic equation may be shown to be

$$f(x) = x^2 + \frac{p_1}{p_2}x + \frac{p_0}{p_2} = 0, \tag{21}$$

where

$$P_2 = -\omega^2 - 95.3 + 114.41\omega\mathrm{j},$$
$$P_1 = 0.7747\omega^2 + 32.724 - 188.2766\omega\mathrm{j},$$
$$P_0 = -0.1371\omega^2 - 133.794 + 73.14354\omega\mathrm{j}.$$

Eq. (21) is a second-order polynomial in x:

$$x = -\frac{p_1}{2p_2} \pm \sqrt{\left(\frac{p_1}{2p_2}\right)^2 - \frac{p_0}{p_2}}. \tag{22}$$

Using $|x| = 1$, we obtain, by Eq. (22), the ω values corresponding to the roots of identity as $\omega_1 = 31.84828949092779$ and $\omega_2 = 1.67208372674041$. Then, at each ω, the angle of the right-hand side of Eq. (22) are $\sigma_1 = -0.24741341597768$ and $\sigma_2 = 0.04072634008986$. By a similar analysis given in Section 3.1, the stability zone is obtained:

$$S = \left[\frac{\sigma_2}{\omega_2}, \frac{2\pi + \sigma_1}{\omega_1}\right] \cup \left[\frac{2k\pi + \sigma_2}{\omega_2}, \frac{2(k+1)\pi + \sigma_1}{\omega_1}\right]_{k=1,2,\ldots}$$

$$S = [0.02435663922718,\ 0.18951635983217] \cup [3.782,\ 0.387]\ldots$$

Here, $3.782 > 0.387$ implies that the system is asymptotically stable at only the first interval. Hence,

$$S = [0.02435663922718,\ 0.18951635983217].$$

One can easily verify that Eq. (21) is satisfied for the pair $(\tau,\ \omega)$ at both $(0.02435663922718, 1.67208372674041)$ and $(0.18951635983217, 31.84828949092779)$. Moreover, we observe that these values are slightly different from those obtained by the "cluster treatment of characteristic roots" method used in Ref. [29] and the values given in Ref. [29] do not satisfy Eq. (21). This may be due to the error caused by the (1,1) Pade approximation (obtained by expanding the exponential in a McLauren series) used in Ref. [29].

Note that, for n larger than 2 in Eq. (20), a general circulant matrix representation of $f(x)$ may be formulated. Then, the roots, $x = \mathrm{e}^{\tau\omega\mathrm{j}}$, become equal to the unitary eigenvalues of the circulant matrix. This information is then used to calculate n possible values of ω satisfying the characteristic equation. However, as n gets larger, it is obvious that the root finding process becomes numerically cumbersome. Nevertheless, as it is shown by this example, analytic solution of the stability boundary is made possible by our approach in the case of commensurate delays.

For comparison, we have also used the closed-loop solution derived by Asl and Ulsoy [34] in obtaining the stability diagram. The stability lobes obtained by the method described in Ref. [34] did not coincide with the ones reported here. Although the authors of Ref. [34] claimed to have found a closed-form solution to the general problem in matrix form, their approach is not mathematically sound [40]. The authors [34] express Eq. (3) in the state-space form, and derive the characteristic equation in matrix form. The main problem with their approach is that they treat the characteristic equation as a matrix identity, and the eigenvalues computed by their formula do not satisfy Eq. (6).

6. Conclusion

A mathematical perspective is presented for the regenerative chatter phenomenon using a second-order linear delay-differential equation. The system is shown to enter into the unstable region at the higher chatter frequency, and into the stable region at the lower one. This information is used to arrive at an explicit formulation (14) of the stability boundaries for certain intervals of the spindle speeds.

Formulation (14) defines each stability lobe for certain intervals of the spindle speeds, and the stability diagram is generated for each set of stable spindle speeds for the design parameter, $K_1/k = Kw\rho h_0^{\rho-1}/k$. Thus, the analysis provides insight into the stability not just for the width-of-cut, but also for h_0, k, ρ, and K. For certain combinations of these parameters, it may then be possible to maximize chatter-free material removal rated. The convolution of the stability lobes determines the overall stability diagram, which is otherwise obtained by numerical means in an iterative manner. In literature, no analytical method of bringing together the adjacent lobes in one formula is reported, and this is provided by Eq. (14).

By our approach, the number of stability switches as the spindle speed varies between two values can also be easily formulated or calculated by counting the number of stable intervals, defined in Eq. (14). This information implies that it can be used in an actual system to automate the variation of the spindle speeds in achieving stable maximum depths-of-cuts or chip-widths in the Mechatronics sense.

We contribute to the literature by an analytical representation of the continuous stability boundary for a general second-order linear delay-differential equation. We also present a generic tool that detects weather the time response of the linear delay-differential equation is asymptotically stable. This tool also enables one to easily determine the delay variations that cause instability. The proposed approach can be used in determining the stability and the stable delay intervals of higher-order delay-differential equations with one delay or commensurate delays.

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