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Design optimization for dynamic response of vibration mechanical system with uncertain parameters using convex model

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Abstract

The concept of uncertainty plays an important role in the design of practical mechanical system. The most common methods for solving uncertainty problems are to model the parameters as a random vector. A natural way to handle the randomness is to admit that a given probability density function represents the uncertainty distribution. However, the drawback of this approach is that the probability distribution is difficult to obtain. In this paper, we use the non-probabilistic convex model to deal with the uncertain parameters in which there is no need for probability density functions. Using the convex model theory, a new method to optimize the dynamic response of mechanical system with uncertain parameters is derived. Because the uncertain parameters can be selected as the optimization parameters, the present method can provide more information about the optimization results than those obtained by the deterministic optimization. The present method is implemented for a torsional vibration system. The numerical results show that the method is effective.

1. Introduction

In engineering design it is important to optimize response quantities such as the displacement, stress, vibration frequencies and mode shapes against a given set of design parameters. The deterministic optimization [1–5] of structural behavior has been well developed for specified parameters and loading conditions. However, the design parameters may be uncertain because of complexity of structures, manufacture errors, inaccuracy in measurement, etc. If the design parameters are changed the design will no longer be optimal and may be unstable in response to these changes. Therefore, the concept of uncertainty plays an important role in the investigation of various engineering problems.

The most common approach to study the problems of uncertainty is to model the parameters as random variables or fields. Under the circumstances, all information about the parameters is provided by the joint

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probability density function (or distribution function) of the parameters. But the probabilistic modeling is not the only way to describe the uncertainty, and also uncertainty is not equal to randomness. Indeed, the probabilistic approaches cannot give reliable results unless sufficient experimental data are available to validate the assumptions about the joint probability densities of the random variables or functions involved.

Despite the success of the probabilistic model, one may recognize that uncertainties in engineering can be modeled on the basis of alternative, non-probabilistic conceptual frameworks. These uncertainties in engineering are usually bounded from above and below, and can be considered to be defined within envelope bounds. One of the mathematical models used for uncertainties are interval sets. In the interval models, all parameters with uncertainties are assumed to be bounded in which the magnitude of uncertain parameters are only required, not necessarily knowing the probabilistic distribution densities. Moore [6] and Alefeld and Herzberger [7] have done the pioneering work. The linear interval equations and nonlinear interval equations have been resolved. Because of the complexity of the interval algorithm, it is difficult to deal with practical engineering problems. Recently, Zhang et al. [8,9] have used the interval finite element method to deal with the eigenvalues and dynamic response of uncertain closed-loop system. Hansen in his book [10] discussed the global optimization using interval analysis. Rao and Cao [11] presented the optimum design of uncertain mechanical system using interval analysis combined with truncation procedure for the prediction of system response.

Another non-probabilistic model for uncertain parameters in engineering is convex (ellipsoidal) sets. It was assumed that the parameters fall into the multidimensional ellipsoid or solid ball. Convex models have been used for modeling uncertain phenomena in a wide range of engineering applications. For instance, Ben-Haim and Elishakoff [12] and Lindberg [13] used the convex model to study the dynamic response and failure of structures with pulse loads. Shi and Gao [14] used the convex model to solve the robustness of control system. Recently, convex (ellipsoidal or interval) sets have been used for modeling uncertain phenomena in a wide range of engineering applications by Elishakoff et al. [15,16] and Pantelides and Ganzerli [17]. Based on convex information-gap models of uncertainty, Ben-Haim studied robust reliability, a new non-probabilistic theory of reliability, for mechanical systems [18] and Info-Gap decision theory for decision-making [19]. Hall et al. [20,21] investigated the estimation of the convex set in the problems of robot vision and medical imaging.

The implementation of convex model theory in optimal structural design has not been widely investigated. Natke and Soong [22] considered topological structural optimization in the presence of dynamic loading. Ganzerli and Pantelides [23] discussed the optimal design of structures that are affected by uncertainties present in the loads applied to the structure, and by uncertainties affecting the internal resistance of the structural members.

However, the optimization problems described in the aforementioned studies except [19] are limited to the cases in which only the design parameters are uncertain. In some cases, the design variables have some tolerances derived from manufacture errors, inaccuracy in measurement or some other reasons. Thus, it is possible that the optimal points of design variables cannot be obtained precisely and then the values of the objective functions are unreliable. In this paper, the tolerances on the design variables (or unknown parameters) as well as the uncertainties of the pre-assigned parameters (or design data) are to be taken into account for optimization. In other words, we treat the parameters with uncertainties and the design variables with tolerances indiscriminately. Using the convex model theory and Taylor expansion, the optimization problem of a mechanical system with uncertain parameters can be transferred into the approximate deterministic optimization one. Ben-Haim in his book [19] discussed the manufacturing process control. Rigorously to say, the subject the author addressed is the robustness optimization problem. Our present work somewhat relate to that subject. However, in the book [19] the mathematical model of the process is the classical MinMax model and the evaluation of the robustness function is complicated. The innovative idea of the approach presented in this paper is that we reformulate the uncertain optimization problem as an approximated deterministic one, moreover, because the nominal values and the uncertainties of the uncertain parameters are to be selected as the design variables, we can not only get the optimal points but also determine the manufacture errors of design variables in advance to obtain the upper and lower bounds of the objective functions, which the literature addressed little. The present method is applied to a torsional vibration system and the optimization results are compared with those obtained by the deterministic optimization method.

The structure of this paper is that we will start with a brief introduction to convex model, and then discuss the optimization using the convex model. The optimization problems with uncertain parameters can be transformed into the approximate deterministic optimization one. The present method is implemented for a torsional vibration system and the optimization results are compared with those obtained by the deterministic optimization method.

2. A brief introduction to convex model

The method of describing the uncertainties by convex set is called convex model, which does not need precise information and is used broadly. If the uncertain parameter $\boldsymbol{\alpha}$ is confined to a convex set Ω, i.e. $\boldsymbol{\alpha} \in \Omega$, where the elements of the vector $\boldsymbol{\alpha}$ are values or functions, then Ω is defined as the convex model of the uncertain parameter $\boldsymbol{\alpha}$. Here, we follow the approach due to Ben-Haim and Elishakoff [12] and assume that uncertain parameters belong to a bounded quadratic convex set

$$\Omega(W, \theta) = \{\boldsymbol{\alpha} : \boldsymbol{\alpha} \in \mathbb{R}^n, (\boldsymbol{\alpha} - \boldsymbol{\alpha}_0)^{\mathrm{T}}\mathbf{W}(\boldsymbol{\alpha} - \boldsymbol{\alpha}_0) \leqslant \theta^2\} \tag{1}$$

where $\boldsymbol{\alpha}_0$ is the nominal vector of the uncertain parameter vector $\boldsymbol{\alpha}$, $\mathbf{W}$ is the symmetric positive weighted matrix, θ is a given positive real constant and is called the radius of the ellipsoid. Here we consider that the uncertain parameters are on the correlated, so the eigen-structure of $\mathbf{W}$ is chosen so as to reflect the uncertain information. The convex model means that all the uncertain parameters $\boldsymbol{\alpha}$ are constrained into the n-dimension ellipsoid. A convex model depends on its real parameter vector $\boldsymbol{\alpha}$, nominal vector $\boldsymbol{\alpha}_0$ and weight matrix $\mathbf{W}$. Since $\mathbf{W}$ is a positive definite matrix, it is diagonalized by an orthogonal matrix $\mathbf{H}$ whose column vectors are the eigenvectors of $\mathbf{W}$. That is

$$\mathbf{W} = \mathbf{H}\boldsymbol{\Lambda}\mathbf{H}^{\mathrm{T}}, \quad \mathbf{H}^{\mathrm{T}}\mathbf{H} = \mathbf{I} \tag{2}$$

in which, $\boldsymbol{\Lambda} = \mathrm{diag}(\lambda_i)$ is the diagonal matrix of $\lambda_i > 0, i = 1, 2, \ldots, n$, eigenvalues of $\mathbf{W}$. $\mathbf{I}$ is the identity matrix. Then Eq. (1) can be transferred into the following ellipsoidal equation:

$$\Omega(\mathbf{W}, \theta) = \left\{ \boldsymbol{\beta} : \boldsymbol{\beta} \in \mathbb{R}^n, \sum_{i=1}^{n} \frac{(\beta_i - \beta_{i0})^2}{e_i^2} \leqslant \theta^2 \right\} \tag{3}$$

where

$$\begin{gathered} e_i^2 = \frac{1}{\lambda_i}, \quad i = 1, 2, \ldots, n \\ \boldsymbol{\beta} = \mathbf{H}^{\mathrm{T}}\boldsymbol{\alpha}, \quad \boldsymbol{\beta}_0 = \mathbf{H}^{\mathrm{T}}\boldsymbol{\alpha}_0 \end{gathered} \tag{4}$$

In practical engineering numerical analysis, Eq. (3) is far more convenient than Eq. (1).

3. Optimization using convex model

In general, a constrained optimization problem can be stated as follows:

$$\begin{aligned} &\min \quad f(\boldsymbol{\alpha}) \\ &\text{s.t.} \quad \begin{cases} p_i(\boldsymbol{\alpha}) \leqslant 0, & i = 1, 2, \ldots, m \\ q_j(\boldsymbol{\alpha}) = 0, & j = 1, 2, \ldots, l \end{cases} \end{aligned} \tag{5}$$

where $\boldsymbol{\alpha}$ is the set of design variables, $f(\boldsymbol{\alpha})$ is the objective function, and $p_i(\boldsymbol{\alpha})$ and $q_j(\boldsymbol{\alpha})$ are the constraints, respectively.

However, all the design parameters are treated as deterministic quantities in few cases. In some cases, the design variables are assumed to be uncertain or have some tolerances. If the uncertain parameters of mechanical system are studied by convex model, i.e. $\boldsymbol{\alpha}^C \in \Omega$, the objective function we shall minimize is an interval. The constraint functions are intervals, too. Thus, the optimization problem for mechanical system with uncertain parameters using convex model can be expressed as

$$\min \quad f(\boldsymbol{\alpha}^C)$$
$$\text{s.t.} \quad \begin{cases} p_i(\boldsymbol{\alpha}^C) \leqslant 0, & i = 1, 2, \ldots, m \\ q_j(\boldsymbol{\alpha}^C) = 0, & j = 1, 2, \ldots, l \end{cases} \tag{6}$$

in which, $\boldsymbol{\alpha}^C = (\alpha_1^C, \alpha_2^C, \ldots, \alpha_n^C)$ is the uncertain parameter vector of the system constraint into the convex set defined by Eq. (1), $f(\boldsymbol{\alpha}^C)$ is the interval objective function, and $p_i(\boldsymbol{\alpha}^C) \leqslant 0$ and $q_j(\boldsymbol{\alpha}^C) = 0$ are the interval constraint conditions, respectively.

It should be noted that minimizing the $f(\boldsymbol{\alpha}^C)$ means minimize the whole interval, not the length of the interval. However, it is difficult to solve Eq. (6) directly [10]. In order to simplify the uncertain optimization problems, we transform it into equivalent deterministic one. To this end, using the Taylor expansion to expand $f(\boldsymbol{\alpha}^C)$ about the nominal vector $\boldsymbol{\alpha}_0$ of the uncertain vector $\boldsymbol{\alpha}^C$ and neglecting the higher-order terms, one has[1]

$$\begin{aligned} f(\boldsymbol{\alpha}^C) &\approx f(\boldsymbol{\alpha}_0) + \sum_{i=1}^{n} \frac{\partial f(\boldsymbol{\alpha}_0)}{\partial \alpha_i}(\alpha_i - \alpha_{i0}) \\ &= f(\boldsymbol{\alpha}_0) + (\boldsymbol{\alpha}^C - \boldsymbol{\alpha}_0)^{\mathrm{T}} \mathbf{A} \end{aligned} \tag{7}$$

where $(\boldsymbol{\alpha} - \boldsymbol{\alpha}_0)^{\mathrm{T}}$ is a row vector; $\mathbf{A}$ is a column vector

$$\begin{aligned} \mathbf{A} &= \frac{\partial f(\boldsymbol{\alpha}_0)}{\partial \boldsymbol{\alpha}} \\ &= \left[\frac{\partial f(\boldsymbol{\alpha}_0)}{\partial \alpha_1} \; \frac{\partial f(\boldsymbol{\alpha}_0)}{\partial \alpha_2} \cdots \frac{\partial f(\boldsymbol{\alpha}_0)}{\partial \alpha_n} \right]^{\mathrm{T}} \end{aligned} \tag{8}$$

When the uncertain parameters $\boldsymbol{\alpha}^C$ vary within the n-dimension ellipsoid described by Eq. (1), the approximate extremums of the objective function can be determined as

$$\begin{aligned} f(\boldsymbol{\alpha}^C)_{\max} &= \max(f(\boldsymbol{\alpha}_0) + (\boldsymbol{\alpha} - \boldsymbol{\alpha}_0)^{\mathrm{T}} \mathbf{A}) \\ f(\boldsymbol{\alpha}^C)_{\min} &= \min(f(\boldsymbol{\alpha}_0) + (\boldsymbol{\alpha} - \boldsymbol{\alpha}_0)^{\mathrm{T}} \mathbf{A}) \end{aligned} \tag{9}$$

According to the convex model theory, the extremums of Eq. (7) will occur on the boundary of the ellipsoid, $\Omega(\mathbf{W}, \theta)$, described by Eq. (1) [24]. By means of the Lagrange multiplier method, the above equation can be rewritten as

$$\begin{aligned} f(\boldsymbol{\alpha}^C)_{\max} &= f(\boldsymbol{\alpha}_0) + \theta \sqrt{\sum_{i=1}^{n} (e_i f_{,i})^2} \\ f(\boldsymbol{\alpha}^C)_{\min} &= f(\boldsymbol{\alpha}_0) - \theta \sqrt{\sum_{i=1}^{n} (e_i f_{,i})^2} \end{aligned} \tag{10}$$

where $f_{,i} = (\mathbf{H}^{\mathrm{T}}(\partial f(\boldsymbol{\alpha}_0)/\partial \boldsymbol{\alpha}))_i$ is the ith element of the column vector $\mathbf{H}^{\mathrm{T}}(\partial f(\boldsymbol{\alpha}_0)/\partial \boldsymbol{\alpha})$.

From the above discussion, we can see that the objective function can be treated as the following interval:

$$f(\boldsymbol{\alpha}^C) = \left[f(\boldsymbol{\alpha}_0) - \theta \sqrt{\sum_{i=1}^{n} (e_i f_{,i})^2}, \quad f(\boldsymbol{\alpha}_0) + \theta \sqrt{\sum_{i=1}^{n} (e_i f_{,i})^2} \right] \tag{11}$$

Similarly, the intervals of the constraints can be obtained as

$$p_i(\boldsymbol{\alpha}^C) = \left[p_i(\boldsymbol{\alpha}_0) - \theta \sqrt{\sum_{k=1}^{n} (e_k p_{i,k})^2}, \quad p_i(\boldsymbol{\alpha}_0) + \theta \sqrt{\sum_{k=1}^{n} (e_k p_{i,k})^2} \right]$$

[1]Here we limit to the case where the uncertainties of the parameters are small.

$$q_j(\boldsymbol{\alpha}^C) = \left[q_j(\boldsymbol{\alpha}_0) - \theta\sqrt{\sum_{k=1}^{n}(e_k q_{j,k})^2}, \quad q_j(\boldsymbol{\alpha}_0) + \theta\sqrt{\sum_{k=1}^{n}(e_k q_{j,k})^2} \right]$$
$$(i = 1,2,\ldots,m, \quad j = 1,2,\ldots,l) \tag{12}$$

where $p_{i,k} = (\mathbf{H}^{\mathrm{T}}(\partial p_i(\boldsymbol{\alpha}_0)/\partial\boldsymbol{\alpha}))_k$ is the kth element of the column vector $\mathbf{H}^{\mathrm{T}}(\partial p_i(\boldsymbol{\alpha}_0)/\partial\boldsymbol{\alpha})$; $q_{j,k} = (\mathbf{H}^{\mathrm{T}}(\partial q_j(\boldsymbol{\alpha}_0)/\partial\boldsymbol{\alpha}))_k$ is the kth element of the column vector $\mathbf{H}^{\mathrm{T}}(\partial q_j(\boldsymbol{\alpha}_0)/\partial\boldsymbol{\alpha})$, respectively.

Then the optimization problem with uncertain parameters (Eq. (6)) can be transformed into the equivalent deterministic optimization one as follows:

$$\min \quad f(\boldsymbol{\alpha}_0) + \theta\sqrt{\sum_{i=1}^{n}(e_i f_{,i})^2}$$
$$\text{s.t.} \quad \begin{cases} p_i(\boldsymbol{\alpha}_0) + \theta\sqrt{\sum_{i=1}^{n}(e_i p_{i,k})^2} \leqslant 0 & (i = 1,2,\ldots,m) \\ q_j(\boldsymbol{\alpha}_0) = 0 & (j = 1,2,\ldots,l) \\ \theta\sqrt{\sum_{i=1}^{n}(e_i q_{j,k})^2} = 0 & (j = 1,2,\ldots,l) \end{cases} \tag{13}$$

4. Numerical examples: design for linear torsional vibration system

Consider the linear torsional vibration system with n degrees of freedom in Fig. 1. Suppose the excitation torque applied to each disk is

$$\mathbf{M}_i^E(t) = \mathbf{M}_{ic}\cos\omega t + \mathbf{M}_{is}\sin\omega t \tag{14}$$

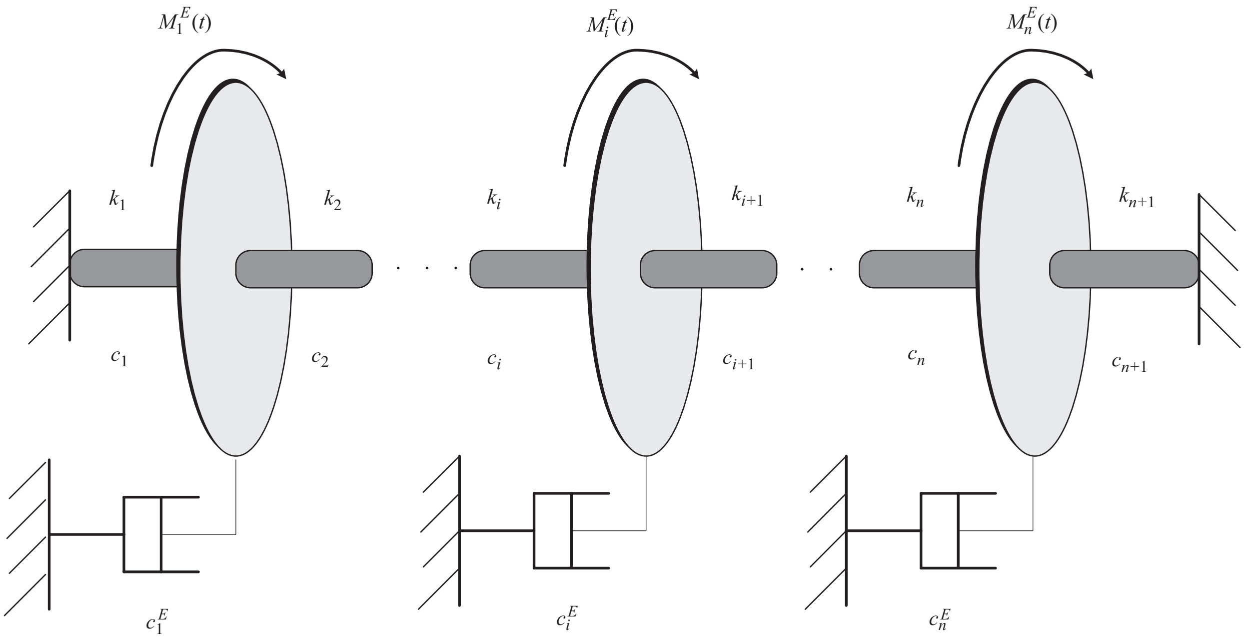

Fig. 1. Linear torsional vibration system with n degrees of freedom.

The goal of the deterministic optimization is to determine design variables I_i, k_i, c_i, c_i^E so that the sum of the amplitude $\sum_{i=1}^{n}|x_i(t)|$ takes minimum. The design variables can be rewritten as an N-dimensional vector

$$\alpha = \{\alpha_1, \alpha_2, \ldots, \alpha_N\}^{\mathrm{T}} = \{I_1, \ldots, I_n, k_1, \ldots, k_{n+1}, c_1, \ldots, c_{n+1}, c_1^E, \ldots, c_n^E\}^{\mathrm{T}}$$

where I_i is the inertia moment, k_i is the torsional stiffness, c_i and c_i^E are the internal and external damping coefficients, respectively.

The deterministic optimization problem is given by

$$\begin{aligned} \min \quad & f(\boldsymbol{\alpha}) = \sum_{i=1}^{n} |x_i(\boldsymbol{\alpha})| \\ \text{s.t.} \quad & \begin{cases} -\theta^2 + \sum\limits_{i=1}^{m} \dfrac{[\mathbf{H}^{\mathrm{T}}(\boldsymbol{\alpha} - \boldsymbol{\alpha}_0)]_i^2}{e_i^2} \leqslant 0 \\ |x_j(\alpha)| - x_j^A \leqslant 0 \quad (j = 1, 2, \ldots, n) \end{cases} \end{aligned} \tag{15}$$

where $|x_i(\boldsymbol{\alpha})|$ denote the displacement amplitudes $x_i(\boldsymbol{\alpha})$, and x_j^A the corresponding permissible deviations.

If at the beginning of design, the design variables are considered to have some uncertainties/tolerances and we use the convex model to deal with the uncertainties/tolerances, then the present optimization for the torsional vibration system is given as follows:

$$\begin{aligned} \min \quad & \sum_{i=1}^{n} \left(|x_i(\boldsymbol{\alpha}_0)| + \theta \sqrt{\sum_{k=1}^{N} (e_k x_{i,k})^2} \right) \\ \text{s.t.} \quad & \begin{cases} -\theta^2 + \sum\limits_{i=1}^{m} \dfrac{[\mathbf{H}^{\mathrm{T}}(\boldsymbol{\alpha} - \boldsymbol{\alpha}_0)]_i^2}{e_i^2} \leqslant 0 & (i = 1, 2, \ldots, m) \\ |x_j(\boldsymbol{\alpha}_0)| + \theta \sqrt{\sum\limits_{k=1}^{N} (e_k x_{i,k})^2} - x_j^A \leqslant 0 & (j = 1, 2, \ldots, n) \end{cases} \end{aligned} \tag{16}$$

The goal of the present optimization is to determine the nominal values and uncertainties of design variables α so that the sum of the amplitude $\sum_i^n |x_i(t)|$ takes minimum and the interval can be derived.

The vibration equation of the linear torsional vibration system as shown in Fig. 1 is

$$\mathbf{M}(I_i)\ddot{\mathbf{x}}(t) + \mathbf{C}(c_i, c_i^E)\dot{\mathbf{x}}(t) + \mathbf{K}(k_i)\mathbf{x}(t) = \mathbf{F}_c \cos \omega t + \mathbf{F}_s \sin \omega t \tag{17}$$

where $\mathbf{F}_c = \mathbf{M}_{\{ic\}}^{\mathrm{T}}$, $\mathbf{F}_s = \mathbf{M}_{\{is\}}^{\mathrm{T}}$, $\mathbf{M}(I_i)$ is the mass matrix, $\mathbf{C}(c_i, c_i^E)$ the damping matrix and $\mathbf{K}(k_i)$ the stiffness matrix, respectively.

The solution of the equation is

$$\mathbf{x}(\boldsymbol{\alpha}) = \{\mathbf{x}_i(\boldsymbol{\alpha})\} = \mathbf{G}(\boldsymbol{\alpha})(\mathbf{F}_c - \mathrm{i}\mathbf{F}_s) \tag{18}$$

where the complex frequency response matrix of the system $G(\alpha)$ is given by

$$\mathbf{G}(\boldsymbol{\alpha}) = [-\mathbf{M}(I_i)\omega^2 + \mathrm{i}\omega\mathbf{C}(c_i, c_i^E) + \mathbf{K}(k_i)]^{-1} \tag{19}$$

To simplify the analysis, we only consider the design optimization for the disk I_1. Disk I_1 is excited by $\mathbf{M}_1^E = \mathbf{M}_{1c} \cos \omega t$. The matrices for the system are given in Appendix.

Case 1: The parameters of I_1, k_1 and c_1 are deterministic, the design variables, I_2, k_2 and c_2 have the specified uncertainties/tolerances, Disk I_1 is excited by $\mathbf{M}_1^E = \mathbf{M}_{1c} \cos \omega t$.

Assume that the following parameters can be given in advance: $I_{1,0} = 500\,\mathrm{kg\,m^2}$, $k_{1,0} = 5000\,\mathrm{kg\,m\,rad^{-1}}$, $c_{1,0} = 10$, $\Delta I_1 = 0$, $\Delta k_1 = 0$, $\Delta c_1 = 0$, in which $I_{1,0}$, $k_{1,0}$, $c_{1,0}$, and ΔI_1, Δk_1, Δc_1 are the nominal values and uncertain parts of I_1, c_1 and k_1, respectively. In this case we suppose the uncertainties of the design variables are specified in advance, that is $\Delta I_2 = 0.2\,\mathrm{kg\,m^2}$, $\Delta k_2 = 1.0\,\mathrm{kg\,m\,rad^{-1}}$ and $\Delta c_2 = 0.1$. The present optimization is to minimize the displacement amplitude of the disk I_1 and give the interval in which the value of the objective function lies. There are three optimization parameters

$$I_{2,0}, \quad k_{2,0}, \quad c_{2,0}$$

where $I_{2,0}$, $k_{2,0}$ and $c_{2,0}$ are the nominal values of I_2, k_2 and c_2, respectively. Using Lagrange optimal algorithm, the approximate deterministic optimization can be solved. The results of the present method are given in Table 1. For comparison, the results of the deterministic optimization are also listed in Table 1. From Table 1, it can be seen that the displacement amplitude of the disk I_1 with the deterministic optimization is $2.9877\text{E}-02$, while the corresponding results with the present method is an interval $[2.8932\text{E}-02, 3.4009\text{E}-02]$ and its midpoint is $3.1471\text{E}-02$.

Case 2: The parameters of I_1, k_1 and c_1 are deterministic, and I_2, k_2 and c_2 have unknown uncertainties/tolerances. Disk I_1 is excited by $\mathbf{M}_1^E = \mathbf{M}_{1c}\cos\omega t$.

Assume that the following parameters can be given in advance: $I_{1,0} = 500\,\text{kg}\,\text{m}^2$, $k_{1,0} = 5000\,\text{kg}\,\text{m}\,\text{rad}^{-1}$, $c_{1,0} = 10$, $\Delta I_1 = 0$, $\Delta k_1 = 0$, $\Delta c_1 = 0$, in which $I_{1,0}$, $k_{1,0}$, $c_{1,0}$, and ΔI_1, Δk_1, Δc_1 are the nominal values and uncertain parts of I_1, c_1 and k_1, respectively. The present optimization is to minimize the displacement amplitude of the disk I_1 and give the interval in which the value of the objective function lies. There are six optimization parameters

$$I_{2,0}, \quad k_{2,0}, \quad c_{2,0}, \quad \Delta I_2, \quad \Delta k_2, \quad \Delta c_2$$

where $I_{2,0}$, $k_{2,0}$, $c_{2,0}$ and ΔI_2, Δk_2, Δc_2 are the nominal values and uncertain parts of I_2, k_2 and c_2, respectively.

Using Lagrange optimal algorithm, the approximate deterministic optimization can be solved. The results of the present method are given in Table 2. For comparison, the results of the deterministic optimization are also listed in Table 2. From Table 2, it can be seen that the displacement amplitude of the disk I_1 with the deterministic optimization is $2.9877\text{E}-02$, while the corresponding results with the present method is an interval $[2.9536\text{E}-02, 3.3245\text{E}-02]$ and its midpoint is $3.1391\text{E}-02$.

Case 3: The parameters of I_1, I_2, c_1, c_2, k_1 and k_2 and uncertain.

There are 12 optimization parameters

$$I_{1,0}, \quad k_{1,0}, \quad c_{1,0}, \quad I_{2,0}, \quad k_{2,0}, \quad c_{2,0}, \quad \Delta I_1, \quad \Delta k_1, \quad \Delta c_1, \quad \Delta I_2, \quad \Delta k_2, \quad \Delta c_2$$

where $I_{1,0}$, $k_{1,0}$, $c_{1,0}$ and ΔI_1, Δk_1, Δc_1 are the nominal values and uncertain parts of I_1, k_1 and c_1, respectively.

Using Lagrange optimal algorithm, the approximate deterministic optimization can be solved. The results of the present method are given in Table 3. For comparison, the results of the deterministic optimization are also listed in Table 3.

From Tables 1 and 2, it can be seen that in Case 2 we take the uncertainties/tolerances of the design variables as the optimization parameters, so the interval value of the objective function obtained by Case 2 is sharper than that obtained by Case 1. At first glance, maybe this is contradict to the intuition that the more uncertainty there is in the input space, the more spread of possible output values in the output space. But in our study, in Case 2 the uncertain parts of the parameters are selected as the optimization objects, which is pre-assigned in Case 1.

Through the above three cases, we can see that if we control the manufacturing precision of the design variables within their uncertainties, we can obtain the range of the objective function, while it cannot be

Table 1
Comparison of results of deterministic optimization and the present method with three parameters, for Case 1 ($\mathbf{M}_{1c} = 100\,\text{kg}\,\text{m}$, $\omega = 4\,\text{rad}\,\text{s}^{-1}$)

	Initial value	Deterministic optimization values	The present method
$I_{2,0}$	10.0000	10.3751	10.7690
$k_{2,0}$	300.0000	301.6022	300.5018
$c_{2,0}$	10.0000	12.3730	14.7180
ΔI_2 (specified)	0.2000	0	0.2000
Δk_2 (specified)	1.0000	0	1.0000
Δc_2 (specified)	0.1000	0	0.1000
$\min\lvert x_1(\alpha)\rvert$		$2.9877\text{E}-02$	$[2.8932\text{E}-02,\ 3.4009\text{E}-02]$

Table 2
Comparison of results of deterministic optimization and the present method with six parameters, for Case 2 ($\mathbf{M}_{1c} = 100\,\mathrm{kg\,m}$, $\omega = 4\,\mathrm{rad\,s^{-1}}$)

	Initial value	Deterministic optimization values	The present method
$I_{2,0}$	10.0000	10.3751	10.7872
$k_{2,0}$	300.0000	301.6022	305.7537
$c_{2,0}$	10.0000	12.3730	13.1139
ΔI_2	0.2000	0	0.1880
Δk_2	1.0000	0	0.9021
Δc_2	0.1000	0	0.0900
$\min\lvert x_1(\alpha)\rvert$		2.9877E − 02	[2.9536E − 02, 3.3245E − 02]

Table 3
Comparison of results of deterministic optimization and the present method with 12 parameters, for Case 3 ($\mathbf{M}_{1c} = 100\,\mathrm{kg\,m}$, $\omega = 4\,\mathrm{rad\,s^{-1}}$)

	Initial values	Deterministic optimization values	The present method
$I_{1,0}$	300.0000	450.0000	450.0000
$I_{2,0}$	15.0000	21.3760	25.8263
$k_{1,0}$	5000.0000	5000.0000	5075.6255
$k_{2,0}$	1000.0000	1001.1087	1015.0007
$c_{1,0}$	100.0000	100.0658	124.9472
$c_{2,0}$	100.0000	52.2269	59.0210
ΔI_1	0.2000	0	0.7054
ΔI_2	0.2000	0	1.0235
Δk_1	1.0000	0	2.7366
Δk_2	1.0000	0	6.0913
Δc_1	0.1000	0	0.1578
Δc_2	0.1000	0	0.1297
$\min\lvert x_1(\alpha)\rvert$		3.6487E − 02	[3.6421E − 02, 3.6572E − 02]

obtained in the deterministic optimization. Using the present optimization method, we can find not only the optimum points but also the interval in which the objective function value lies.

5. Conclusion

In this paper, a new optimization method is proposed for dynamic response of mechanical system with uncertain parameters using convex method. The optimization problem for the uncertain system is transformed into the approximate deterministic optimization one, so we can use the standard algorithm for nonlinear optimization to solve the optimization problem for the uncertain system. Using the present method, more information for the optimal mechanical system can be obtained, such as how the optimization results change if the uncertainties of parameters are imposed on the system. Because the present method is based on the first-order Taylor expansion, the application of the method is limited to the cases where the uncertainties of the parameters are small. If the uncertainties of the parameters are fairly large, in order to obtain higher computing accuracy, the second-order Taylor expansion should be considered.

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Appendix

$$\mathbf{F}_c = \{\mathbf{M}_{ic}\}^{\mathrm{T}} = \left\{ \begin{array}{c} M_{1c} \\ 0 \end{array} \right\} \tag{A.20}$$

$$\mathbf{F}_s = \{\mathbf{M}_{is}\}^{\mathrm{T}} = \left\{ \begin{array}{c} 0 \\ 0 \end{array} \right\} \tag{A.21}$$

$$\mathbf{M}(I_i) = \begin{bmatrix} I_1 & 0 \\ 0 & I_2 \end{bmatrix} \tag{A.22}$$

$$\mathbf{C}(c_i, c_i^E) = \begin{bmatrix} c_1 + c_2 & -c_2 \\ -c_2 & c_2 \end{bmatrix} \tag{A.23}$$

$$\mathbf{K}(k_i) = \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} \tag{A.24}$$

$$\begin{aligned} \mathbf{G}(\alpha) &= [-\mathbf{M}(I_i)\omega^2 + \mathrm{i}\omega\mathbf{C}(c_i, c_i^E) + \mathbf{K}(k_i)]^{-1} \\ &= \begin{bmatrix} -I_1\omega^2 + k_1 + k_2 + \mathrm{i}\omega(c_1 + c_2) & -k_2 - \mathrm{i}\omega c_2 \\ -k_2 - \mathrm{i}\omega c_2 & k_2 - I_2\omega^2 + \mathrm{i}\omega c_2 \end{bmatrix}^{-1} \end{aligned} \tag{A.25}$$

$$\{\mathbf{x}(\boldsymbol{\alpha})\} = \left\{ \begin{array}{c} \mathbf{x}_1(\boldsymbol{\alpha}) \\ \mathbf{x}_2(\boldsymbol{\alpha}) \end{array} \right\} = G(\alpha) \left\{ \begin{array}{c} M_{1c}(\alpha) \\ 0 \end{array} \right\} \tag{A.26}$$

and the displacement response for the first disk is

$$x_1(\boldsymbol{\alpha}) = \frac{A + \mathrm{i}B}{C^2 + D^2} \tag{A.27}$$

where

$$\begin{aligned} A = M_{1c}\big(&k_1k_2^2 - 2\omega^2 I_2 k_1 k_2 - \omega^2 I_1 k_2^2 - \omega^2 I_2 k_2^2 + 2\omega^4 I_1 I_2 k_2 \\ &+ \omega^4 I_2^2 k_1 + \omega^4 I_2^2 k_2 - \omega^6 I_1 I_2^2 + \omega^2 k_1 c_2^2 - \omega^4 I_1 c_2^2 - \omega^4 I_2 c_2^2\big) \end{aligned} \tag{A.28}$$

$$B = M_{1c}(-\omega k_2^2 c_1 + 2\omega^3 I_2 k_2 c_1 - \omega^3 c_1 c_2^2 - \omega^5 I_2^2 c_1 - \omega^5 I_2^2 c_2) \tag{A.29}$$

$$C = k_1k_2 - \omega^2 I_2 k_1 - \omega^2 I_2 k_2 - \omega^2 I_1 k_2 + \omega^4 I_1 I_2 - \omega^2 c_1 c_2 \tag{A.30}$$

$$D = \omega k_1 c_2 - \omega^3 I_1 c_2 + \omega k_2 c_1 - \omega^3 I_2 c_1 - \omega^3 I_2 c_2 \tag{A.31}$$

The norm of $x_1(\boldsymbol{\alpha})$ is

$$|x_1(\boldsymbol{\alpha})| = \frac{\sqrt{A^2 + B^2}}{C^2 + D^2} \tag{A.32}$$

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