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Rapid Communication

Reduction of thermo-elastic damping with a secondary elastic field

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Abstract

Thermo-elastic damping is the dominant mode of energy loss due to the coupling of thermal and elastic fields in a body vibrating at or near resonant frequency. While the literature contains both exact and numerical schemes to quantify it, no technique is available yet to reduce thermo-elastic damping. We address this issue by introducing a secondary elastic field to derive an exact expression that predicts linear reduction in thermo-elastic damping with respect to frequency. Contrary to the current understanding, introduction of a static axial stress in addition to the flexural stresses is shown to increase quality factor and resonant frequency simultaneously.

1. Introduction

Deformation in a body changes its heat content, giving rise to the coupling of elastic and thermal fields. At or near resonant frequency, the coupling results in appreciable temperature gradient in the body, causing energy loss through irreversible heat flow. Known as thermo-elastic damping, it is the dominant energy loss mode that lowers the quality factor. Due to its significance in micro- and nano-resonator applications, attempts have been made to quantify it in terms of physical properties. Zener's [1–4] approximate expression for thermo-elastic damping was first known attempt to quantify the thermo-elastic damping using anelastic solid model. Recently, Lifshitz and Roukes [5] developed an exact expression for thermo-elastic damping that also predicts increase in thermo-elastic damping with increase in natural frequency of vibration. Other schemes for quantification of thermo-elastic damping involve non-Fourier heat diffusion equation [6], solution of the coupled thermo-elasticity equations [7] and using eigenvalues and eigenvectors of uncoupled thermal and elastic equations [7,8]. For complicated geometry, these different approaches can be implemented using commercially available Finite Element solver [7]. It is important to note that the widely accepted Lifshitz–Roukes model does not have dimension or size effects explicitly, which makes it inadequate for device design purposes [9,10].

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While the above-mentioned techniques can be used to quantify thermo-elastic damping, there is no technique available in the literature that reduces the losses. Miniaturization makes it worse because as the size of the beam is reduced its natural frequency increases, and thermo-elastic damping also increases in the process. To the best of our knowledge, there is no model available in the literature that can reduce thermo-elastic damping, a reason why it is dubbed as a fundamental loss mechanism. In this brief communication, we solve this problem by modeling, for the first time, a secondary elastic field to modify the thermo-elastic coupling to reduce the thermo-elastic damping. A simple implementation of this scheme is to apply a static stress field in the resonator. It is known that for flexural resonator, such axial stress increases natural frequency of beams [11–13], which in turn, is expected to increase the thermo-elastic damping. We analytically show that to the contrary, Q-factor and resonance frequency can be increased simultaneously. The premise for our analytical model is recent experimental studies on nano-mechanical resonators that suggest an increase in the quality factor with application of tensile stress [14,15].

2. Mathematical formulation

We modify the Lifshitz–Roukes thermo-elastic damping model by introducing static axial stress in a flexural resonator with length l, width b, depth d and moment of inertia I. Deformation due to applied axial force F will result in change in temperature which negligible at isothermal or adiabatic conditions. If T_0 is the equilibrium temperature of the beam, flexural displacement $w(x, y, z, t)$ will result in temperature field $T = T_0+\theta$, which is introduced to the constitutive relationship as

$$\sigma_{xx} = \sigma_0 - Ey\frac{\partial^2 w}{\partial x^2} - E\alpha\theta, \tag{1}$$

where σ_0 is stress due to applied axial force and α is coefficient of thermal expansion and E is the Young's modulus. The equation of motion for flexural vibration is then

$$\frac{\partial^2}{\partial x^2}\left(EI\frac{\partial^2 w}{\partial x^2} + E\alpha I_{\mathrm{T}} - Fw\right) + \rho A\frac{\partial^2 w}{\partial t^2} = 0, \tag{2}$$

where I_{T} is thermal moment of inertia. The heat diffusion equation with thermo-elastic damping is given as

$$\left(1 + 2\Delta_{\mathrm{E}}\frac{1+v}{1-2v}\right)\frac{\partial\theta}{\partial t} = \chi\frac{\partial^2\theta}{\partial y^2} + y\frac{\Delta_{\mathrm{E}}}{\alpha}\frac{\partial}{\partial t}\frac{\partial^2 w}{\partial x^2}, \tag{3}$$

where $\Delta_{\mathrm{E}} = E\alpha^2 T/C$, χ is solid's thermal diffusivity and C is volumetric heat capacity. Now neglecting the Δ_{E} term on the left side and assuming the solution for coupled thermo-elastic equations, we get, the temperature profile to be

$$\theta_0(x, y) = \frac{\Delta_{\mathrm{E}}}{\alpha}\frac{\mathrm{d}^2 W}{\mathrm{d}x^2}\left(y - \frac{\sin(ky)}{k\cos(\mathrm{d}k/2)}\right), \tag{4a}$$

$$k = (1 + \mathrm{i})\sqrt{\frac{\omega_n}{2\chi}}. \tag{4b}$$

Utilizing the temperature profile, thermal moment of inertia is calculated and the beam vibration equation can be expressed as

$$E_\omega I\frac{\mathrm{d}^4 W}{\mathrm{d}x^4} - F\frac{\mathrm{d}^2 W}{\mathrm{d}x^2} + \rho A\omega^2 W = 0, \tag{5}$$

where

$$E_\omega = E[1 + \Delta_{\mathrm{E}}(1 + f(\omega_n))]$$

and

$$f(\omega_n) = \frac{24}{d^3 k^3}\left[\frac{\mathrm{d}k}{2} - \tan\left(\frac{\mathrm{d}k}{2}\right)\right]. \tag{6}$$

Solving equation of motion by assuming boundary conditions to be simply supported, real and imaginary part of first natural frequency are given as

$$\mathrm{Re}(\omega) = \omega_n\left[1 + \frac{\Delta_\mathrm{E}/2}{1+(F/F_\mathrm{cr})}\left(1 + \frac{\sin(\xi) - \sinh(\xi)}{\cos(\xi)+\cosh(\xi)}\right)\right], \tag{7a}$$

$$\mathrm{Im}(\omega) = \omega_n \frac{\Delta_\mathrm{E}/2}{1+(F/F_\mathrm{cr})}\left[\frac{6}{\xi^3}\frac{\sin(\xi)+\sinh(\xi)}{\cos(\xi)+\cosh(\xi)} - \frac{6}{\xi^2}\right], \tag{7b}$$

where $\xi = d\sqrt{\omega_n/2\chi}$.

Now quality factor (Q) is defined as

$$Q^{-1} = 2\left|\frac{\mathrm{Im}(\omega)}{\mathrm{Re}(\omega)}\right|. \tag{8}$$

Neglecting Δ_E term in real part results in damping to be expressed as

$$Q^{-1} = \frac{(E\alpha^2 T/C)\{(6/\xi^2) - (6/\xi^3)[\sinh(\xi) + \sin(\xi)/\cosh(\xi) + \cos(\xi)]\}}{1+(F/F_\mathrm{cr})}. \tag{9}$$

The above expression has been derived for a simply supported beam but it can be extended for any boundary condition. For clamped–clamped beam, the solution for natural frequency [12] is given as

$$\omega_n = \omega_0\sqrt{1 + 0.97\frac{F}{F_\mathrm{cr}}}. \tag{10}$$

Similarly, natural frequency for other boundary conditions also can be expressed as

$$\omega_n = \omega_0\sqrt{1 + a\left(\frac{F}{F_\mathrm{cr}}\right)}, \tag{11}$$

where a is factor which depends on boundary condition. Using this, expression for thermo-elastic damping can also be extended for other boundary conditions using factor a:

$$Q^{-1} = \frac{E\alpha^2 T/C\{(6/\xi^2) - (6/\xi^3)[\sinh(\xi) + \sin(\xi)/\cosh(\xi) + \cos(\xi)]\}}{1+(F/F_\mathrm{cr})}, \tag{12}$$

3. Results and discussion

The effect of a second elastic field due to the axial force F is clearly seen in Eq. (12), where the Q factor increases for any positive value of F. It also indicates the presence of a discontinuity in case of compressive critical buckling force F_cr. This expression in the limiting case of $F = 0$ turns out to be same as Lifshitz–Roukes expression for thermo-elastic damping. Value of 'a' is 1 for simply supported beams, 0.925 for cantilever beams and 0.97 [12] for clamped–clamped beams. Fig. 1 shows the normalized thermo-elastic damping against the factor ξ, which is a function of natural frequency, thermal diffusivity and thickness of the resonator as given in Eq. (7). It is observed from Fig. 1 that for same value of ξ, stressed resonator will have significantly less thermo-elastic damping compared to its unstressed counterpart. Several orders of magnitude improvement is expected for a modest tensile stress (10 times the critical buckling stress in a flexural resonator).

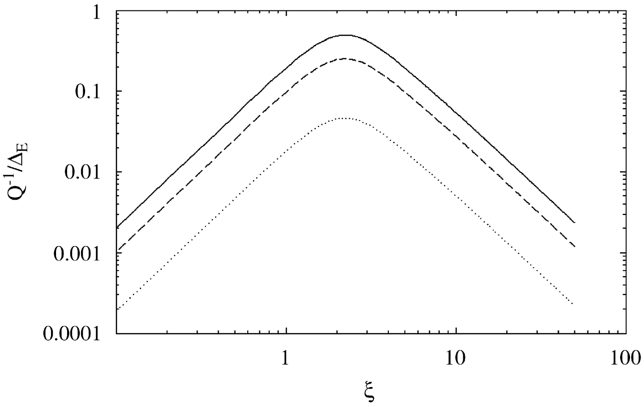

Fig. 1. Variation of normalized thermo-elastic damping due to a secondary axial stress, solid line indicates unstressed beam, dashed-stress equal to critical stress and dotted line indicates stress 10 times critical stress.

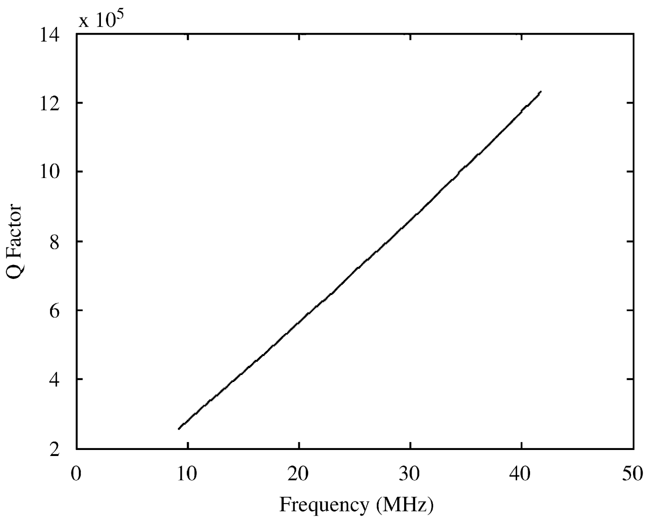

Fig. 2. Variation of the quality factor with respect to applied axial stress (expressed in form of the natural frequency).

The new model for thermo-elastic damping also allows the quality factor to be expressed in terms of dimensional properties. This is given by the following equations:

$$\omega_n = \beta_1 \frac{d}{l^2}\sqrt{\frac{E}{\rho}}\sqrt{1 + a\frac{\sigma}{a_1\pi^2 E}\left(\frac{l}{d}\right)^2}, \tag{13}$$

$$Q^{-1} = \frac{E\alpha^2 T/C\{(6/\xi^2) - (6/\xi^3)[\sinh(\xi) + \sin(\xi)/\cosh(\xi) + \cos(\xi)]\}}{1 + a(\sigma/a_1\pi^2 E)(l/d)^2}, \tag{14}$$

where a_1 and a are factors that represent the boundary conditions. It is important to note that the relationship between beam geometry and the quality factor is derived in an exact form. Eq. (14) is therefore expected to be a useful design tool for micro- or nano-resonators, giving the designer choice over the resonator geometry.

In order to compare the model performance with experimental results available in literature, the quality factor of a silicon nitride beam resonator with dimensions $l = 10\,\mu m$, $b = 1\,\mu m$ and $d = 110\,nm$ [15] was computed. Silicon nitride beams are commonly fabricated by LPCVD [16] type deposition and their properties are well known in the literature [16–18]. The results are shown in Fig. 2, where the theoretical quality factor considering thermo-elastic damping alone is plotted as a function of the resonant frequency, which itself is a function of the applied axial stress. Here, the resonant frequency increases from 10 MHz for an unstressed beam to 40 MHz for applied axial stress of 20 times the critical buckling stress. Interestingly, the exact formulation for thermo-elastic damping [5] predicts decrease in quality factor, which apparently is not the case shown in Fig. 2. Here, the quality factor increases in a linear fashion with respect to the applied stress, as predicted by Eq. (12).

4. Conclusion

A new concept for reducing thermo-elastic damping in vibrating beams by introducing a secondary elastic field is proposed. The concept is implemented for a flexural resonator undergoing a static axial stress and an exact expression for thermo-elastic damping is developed. The new model suggests simultaneous increase in quality factor and resonant frequency under the applied axial stress for all aspect ratios. It also quantifies the quality factor corresponding to thermo-elastic damping in terms of the vibrating beam geometry and therefore will be useful in design of micro-/nano-resonators for reduced thermo-elastic damping and quantitative prediction of the increase in quality factor as an axial stress is applied.

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