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Periodic solution of the generalized Rayleigh equation

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Abstract

The periodic solutions of a strongly cubic nonlinear oscillator whose motion is described with the generalized Rayleigh equation are studied. Approximate analytic solving methods are introduced. A new method based on homotopy and averaging is developed to determine the limit cycle motion. The obtained analytical solutions are compared with those calculated by the elliptic harmonic balance method with generalized Fourier series and Jacobian elliptic functions. Three types of cubic nonlinearity are considered: the coefficients of the linear and cubic terms are positive, the coefficient of the linear term is positive and that of the cubic term is negative and the opposite case. Comparisons of the analytical solution and numerical solution, obtained by using the Runge–Kutta method, are illustrated with examples.

1. Introduction

Throughout the last century many authors have devoted their attention to the study of the Van der Pol oscillator whose motion is described with a differential equation usually called the Rayleigh equation

$$\ddot{x} + c_1 x - \varepsilon(\dot{x} - \dot{x}^3) = 0, \tag{1}$$

where c_1 and ε are constant parameters. For ε a small parameter, approximate techniques (method of harmonic balance (HB) [1,2], Lindstedt–Poincaré (L–P) [3], Krylov–Bogoliubov–Mitropolski (KBM) [3], averaging [4] and multiple scales (MS) [1]) are applied to solve the differential equation (1). The solution of the equation is approximately sinusoidal with a slowly varying amplitude and phase and it approaches a limit cycle at $t \to \infty$, irrespective of the initial conditions. The trajectory in the phase plane tends to a limit cycle (closed curve) and for the steady-state motion it is a circle with a radius a_c which is the limit amplitude independent of the initial state. The analytical solution was enough to explain some of the phenomena which occur in the real systems. For example, in a Van der Pol electrical circuit the existence of a limit cycle was explained by the energy store in the capacitor during the slowly varying part of the motion, while during the

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abrupt changes the energy was being suddenly released. Unfortunately, the quantitative values obtained analytically were not enough accurate. This was the reason why the Rayleigh equation was extended with nonlinear terms. The generalized Rayleigh differential equation is

$$\ddot{x} + c_1 x + c_3 x^3 - \varepsilon(c_0 \dot{x} - c_2 \dot{x}^3) = 0, \tag{2}$$

where ε is a constant which is often assumed to be small ($\varepsilon \ll 1$) and c_i where $i = 0, \ldots, 3$ are constant coefficients. For $c_1 > 0$ and $c_3 > 0$ the generalized Rayleigh equation is studied by using Jacobian elliptic functions with the generalized harmonic balance method [5]. The conditions for existence of a limit cycle are discussed. The method is developed by Bejarano and Sanchez [6].

In this paper an extension to the previous investigation is done. The influence of the type of nonlinearity (hard or soft) on the parameters of limit cycle motion described with the generalized Rayleigh equation is investigated. The new analytical solving procedure based on the homotopy perturbation method [7–10] and averaging procedure is introduced. The case when the coefficients of linear and cubic term are positive ($c_1 > 0$ and $c_3 > 0$) is compared with two other cases: the coefficient of the linear term is positive and that of the cubic term is negative ($c_1 > 0$ and $c_3 < 0$, and the coefficient of the linear term is negative and that of the cubic term is positive ($c_1 < 0$ and $c_3 > 0$). The special case when the linear term is zero ($c_1 = 0$) is also considered. The solution of Eq. (2) is determined using the elliptical harmonic balance method with a generalized Fourier series ([11,12]) too. The suggested method is based on the procedures for solving strong nonlinear differential equations ([13–17]). To prove the accuracy of the analytical procedures three examples are given. The limit cycle amplitude is calculated not only analytically but also numerically. The analytical solutions are compared with the numerical ones.

2. Methods

Both the suggested solving methods, the homotopy averaged method and the generalized harmonic balance method, are based on the generating solution

$$x(t) = a\, ep(\omega t, k^2), \tag{3}$$

which represents the exact analytical solution of the strong nonlinear Duffing equation

$$\ddot{x} + c_1 x + c_3 x^3 = 0 \tag{4}$$

with initial conditions:

$$x(0) = A, \quad \dot{x}(0) = 0, \tag{5}$$

where a, ω and k^2 are constants and $ep(\omega t, k^2)$ denotes a convenient Jacobian elliptic function: $sn(\omega t, k^2)$, $cn(\omega t, k^2)$ or $dn(\omega t, k^2)$ according to the type of equation (4) which depends on the sign of c_1 and c_3 [16]. (A survey of elliptic functions is given in the appendix.) The constants ω and k^2 are the known values which depend on a.

The trial solution of Eq. (2) is assumed in the form (3) but the constants a, ω and k^2 take into consideration the specifics of the damping terms and the influence of c_0 and c_2.

In the paper only the first-order approximation is considered as it gives results with enough technical accuracy.

2.1. Homotopy averaging method

The new homotopy procedure which includes the averaging of the elliptic function is introduced. Based on the homotopy solving method for a differential equation with strong cubic nonlinearity and the averaging of the elliptic functions the method of homotopy averaging is developed.

In the paper of Cveticanin (see Ref. [10]) the homotopy solving method for the differential equation with strong cubic nonlinearity

$$\ddot{x} + c_1 x + c_3 x^3 = \varepsilon N(x, \dot{x}), \tag{6}$$

where $\varepsilon N(x,\dot{x})$ is a small function, is considered. The transformation of the variable $x(t)$ to $X(t,p)$ is done and the embedding parameter $p \in [0,1]$ is introduced. Using the homotopy procedure for Eq. (6) it transforms into

$$(1-p)[(\ddot{X}+c_1X+c_3X^3)-(\ddot{x}_0+c_1x_0+c_3x_0^3)]+p[\ddot{X}+c_1X+c_3X^3-\varepsilon N(X,\dot{X})]=0. \tag{7}$$

Following the suggested method and relation (7) Eq. (2), after homotopy transformation, is

$$(1-p)[(\ddot{X}+c_1X+c_3X^3)-(\ddot{x}_0+c_1x_0+c_3x_0^3)]+p[\ddot{X}+c_1X+c_3X^3-\varepsilon(c_0\dot{X}-c_2\dot{X}^3)]=0 \tag{8}$$

with initial conditions (5)

$$X(0,p)=A, \quad \dot{X}(0,p)=0, \tag{9}$$

where $x_0 \equiv x_0(t)$ is the initial approximate solution which has the form of Eq. (3)

$$x_0 = a\,ep(\omega t, k^2) \equiv a(ep). \tag{10}$$

Using the Maclaurin series expansion

$$X(t,p)=x_0(t)+\sum_{n=1}^{\infty}\left(\frac{x_n}{n!}\right)p^n, \quad n=1,2,3,\ldots, \tag{11}$$

where

$$x_n \equiv x_n(t)=\left(\frac{\partial X(t,p)}{\partial p^n}\right)_{p=0}, \tag{12}$$

the nonlinear differential equation (8) is transformed into the system of n linear differential equations

$$p^0: \quad \ddot{x}_0+c_1x_0+c_3x_0^3=0, \tag{13}$$

$$p^1: \quad \ddot{x}_1+c_1x_1+3c_3x_0^2x_1=-(\ddot{x}_0+c_1x_0+c_3x_0^3)+\varepsilon(c_0\dot{x}_0-c_2\dot{x}_0^3), \tag{14}$$

$$\vdots$$

Substituting the assumed solution (10) into Eq. (13) and using the harmonic balance method we obtain the modulus k and the frequency ω of the elliptic function

$$k=F_k(a,\omega), \quad \omega=F_\omega(a,k). \tag{15}$$

Applying Eq. (10) the differential equation (14) is transformed into the first-order deformation equation

$$\ddot{x}_1+c_1x_1+3c_3a^2(ep)^2x_1=\varepsilon(c_0a\dot{(ep)}-c_2a^3\dot{(ep)}^3), \tag{16}$$

where $\dot{(\bullet)} \equiv \mathrm{d}(\bullet)/\mathrm{d}t$. Relation (16) is a nonlinear nonhomogenous differential equation with a time-variable coefficient. Finding an exact analytical solution for Eq. (16) is not an easy task. Our aim is not to solve the equation but to determine the amplitude of steady-state motion.

Due to the property of the series expansion (12) and the form of the left-hand side of Eq. (14) the solution of Eq. (16) is assumed in the form of the first time derivative of the elliptic function in Eq. (10)

$$x_1=b\dot{(ep)}, \tag{17}$$

where b is a constant. Substituting the assumed solution (17) into Eq. (16) we obtain

$$b[\dddot{(ep)}+c_1\dot{(ep)}+3c_3a^2(ep)^2\dot{(ep)}]=\varepsilon(c_0a\dot{(ep)}-c_2a^3\dot{(ep)}^3). \tag{18}$$

We are interested only in the limit cycle solution. To obtain the parameters of the limit cycle the averaging of Eq. (18) is introduced. The averaging is done for the period of elliptic function $4K(k^2)$, where $K(k^2) \equiv K$ is the complete elliptic integral of the first kind. The averaged relation (18) is

$$b\{\langle\dddot{(ep)}\dot{(ep)}\rangle+c_1\langle[\dot{(ep)}]^2\rangle+3c_3a^2\langle(ep)^2[\dot{(ep)}]^2\rangle\}=\varepsilon(c_0a\langle[\dot{(ep)}]^2\rangle-c_2a^3\langle[\dot{(ep)}]^4\rangle), \tag{19}$$

where $\langle \bullet \rangle = 1/4K \int_0^{4K} (\bullet)\, d\tau$, $\tau = \omega t$. Then the left-hand side of Eq. (18) is always zero and the right-hand side represents the condition for limit cycle motion

$$c_0 a \langle [(\dot{ep})]^2 \rangle - c_2 a^3 \langle [(\dot{ep})]^4 \rangle = 0. \tag{20}$$

Solving the system of algebraic equations (20) and (15) the limit cycle amplitude a, the frequency of vibration ω and the modulus of the elliptic function k are obtained.

2.2. The generalized harmonic balance method

The generalized harmonic balance method assumes a trial solution in the form (3). Substituting Eq. (3) and the corresponding time derivatives into Eq. (2) and using the Fourier series expansion of the elliptic function [18], we obtain

$$F_1(a, \omega, k^2, \varepsilon, \alpha) \cos \vartheta + F_2(a, \omega, k^2, \varepsilon, \alpha) \sin \vartheta + (\text{higher order harmonics}) = 0, \tag{21}$$

where $\vartheta = am(\omega t, k^2)$ is the generalized circular function which represents the amplitude of the Jacobi elliptic function and gives the relation between the argument of the circular and elliptic function [6], and α collectively denotes any parameter which appears in the nonlinear function. It is worth stating that only the first harmonic of Fourier series is used as it gives enough accurate results. According to the basic assumption that the terms with $\cos\vartheta$ and $\sin\vartheta$ in Eq. (21) are zero

$$F_1(a, \omega, k^2, \varepsilon, \alpha) = 0, \quad F_2(a, \omega, k^2, \varepsilon, \alpha) = 0 \tag{22}$$

and using the harmonic balance method [19] the parameters a, ω and k^2 are obtained.

3. A study of the three types of generalized Rayleigh oscillators

The two suggested methods are applied for solving three types of generalized Rayleigh equations: (a) $c_1 > 0$ and $c_3 > 0$, (b) $c_1 > 0$ and $c_3 < 0$ and (c) $c_1 < 0$ and $c_3 > 0$. All the three types of equations have a physical meaning: case (a) corresponds to the oscillator with a hardening spring [1], case (b) to the oscillator with a softening spring [1] and case (c) is the first modal equation of transversal vibrations of a cantilever beam, for example, Refs. [20–22].

3.1. Homotopy averaging method

Oscillator type I: $c_1 > 0, c_3 > 0$

For this type of oscillator the generating solution is as follows [16]:

$$x_0 = a\, cn(\omega t, k^2) \equiv acn, \tag{23}$$

where

$$k^2 = \frac{c_3 a^2}{2\omega^2}, \quad \omega^2 = c_1 + c_3 a^2. \tag{24}$$

According to the aforementioned procedure, the solution of Eq. (14) is assumed to be

$$x_1 = b(\dot{cn}) = -b\omega sndn, \tag{25}$$

where b is a constant. Substituting Eq. (25) into the relation (20) we obtain

$$c_0(M_2 - k^2 M_4) - c_2 a^2 \omega^2 (M_4 - 2k^2 M_6 + k^4 M_8) = 0, \tag{26}$$

where M_{2n}, $n = 1, \ldots, 4$ are the averaged elliptic functions which are given in the Appendix. Using Eqs. (24) and (26) we obtain the relation for k^2

$$\frac{1}{3}\left[(1-k^2) - (1-2k^2)\frac{E}{K}\right] = \left(\frac{c_1^2}{c_3}\frac{c_2}{c_0}\right)\frac{1}{35k^2}\left[(8k^6 - 13k^4 + 3k^2 + 2) - (16k^6 - 24k^4 + 4k^2 + 2)\frac{E}{K}\right], \tag{27}$$

where $E \equiv E(k^2)$ is the complete elliptic integral of the second kind [18]. For the known value of k^2 according to Eq. (24) the limit cycle amplitude and frequency are determined

$$a = \sqrt{\frac{c_1}{c_3}\frac{2k^2}{(1-2k^2)}}, \quad \omega = \sqrt{\frac{c_1}{1-2k^2}}. \tag{28}$$

Then the orbital motion is

$$x = \sqrt{\frac{70k^4[(1-k^2)K-(1-2k^2)E]}{3(1-2k^2)[(8k^6-13k^4+3k^2+2)K-(16k^6-24k^4+4k^2+2)E]}}\sqrt{\frac{1}{c_1}\frac{c_0}{c_2}}cn\left(t\frac{1}{\sqrt{1-2k^2}}\sqrt{c_1}, k^2\right). \tag{29}$$

Two special cases are considered.

1. *Linear case*

For the special case when $c_3 = 0$ the modulus of the elliptic function is zero according to Eq. (24). For $k = 0$ the *cn* elliptic function transforms into a circular cos function, sn into sin function, *dn* is 1 and the period of functions is 2π. The frequency of vibration is $\omega = \sqrt{c_1}$. Using the averaged values of the circular functions $\langle \sin^2 \rangle = 1/2$ and $\langle \sin^4 \rangle = 3/8$ the amplitude of the limit cycle motion is

$$a = \frac{2}{\omega\sqrt{3}}\sqrt{\frac{c_0}{c_2}}. \tag{30}$$

This value was previously obtained by Nayfeh and Mook [1]. The orbital motion is

$$x = 1.1547\sqrt{\frac{1}{c_1}\frac{c_0}{c_2}}\cos(t\sqrt{c_1}). \tag{31}$$

2. *Pure cubic nonlinearity*

According to Eq. (24) the frequency and the modulus of the elliptic function are, respectively,

$$\omega = a\sqrt{c_3}, \quad k^2 = \tfrac{1}{2} = \text{const.} \tag{32}$$

The approximate value of the amplitude of the limit cycle is

$$a = \sqrt[4]{\frac{7}{3}\frac{c_0}{c_2 c_3}} = 1.2359\sqrt[4]{\frac{1}{c_3}\frac{c_0}{c_2}} \tag{33}$$

and the orbital motion is

$$x = 1.2359\sqrt[4]{\frac{1}{c_3}\frac{c_0}{c_2}}cn\left(1.2359t\sqrt[4]{c_3\frac{c_0}{c_2}}, 0.5\right). \tag{34}$$

The period of orbital motion T is

$$T = \frac{4K(0.5)}{1.2359\sqrt[4]{c_3\frac{c_0}{c_2}}} = 6\sqrt[4]{\frac{1}{c_3}\frac{c_2}{c_0}} \tag{35}$$

and does not depend on parameter ε.

Oscillator type II: $c_1 > 0, c_3 < 0$

For this type of oscillator, the generating function takes the form

$$x_0 = a\,sn(\omega t, k^2), \tag{36}$$

where

$$k^2 = \frac{(-c_3)a^2}{2\omega^2}, \quad \omega^2 = c_1 + \tfrac{1}{2}c_3 a^2. \tag{37}$$

Substituting the assumed solution

$$x_1 = b(\dot{sn}) = b\omega cndn \tag{38}$$

and Eq. (24) into Eq. (20) the following relation is obtained:

$$[k^2 M_4 - (1+k^2)M_2 + 1] - \frac{2}{(1+k^2)^2}\frac{c_1^2}{(-c_3)}\frac{c_2}{c_0}[1 - 2(1+k^2)M_2 + (1+4k^2+k^4)M_4 - 2k^2(1+k^2)M_6 + k^4 M_8 = 0, \tag{39}$$

where the values of M_{2n} are given in the appendix. After some transformation the relation for k^2 is obtained

$$\frac{c_1^2}{(-c_3)}\frac{c_2}{c_0} = -\frac{35k^2}{6}\frac{[(1-k^2)K - (1+k^2)E](1+k^2)^2}{[(2-11k^2+8k^4+k^6)K - (2-10k^2-10k^4+2k^6)E]}. \tag{40}$$

The amplitude and the frequency of the limit cycle motion are

$$a = \sqrt{\frac{2c_1k^2}{(-c_3)(1+k^2)}}, \quad \omega = \sqrt{\frac{c_1}{1+k^2}}. \tag{41}$$

Oscillator type III: $c_1 < 0, c_3 > 0$

For this type of oscillator, we take a generating function

$$x_0 = a\, dn(\omega t, k^2), \tag{42}$$

with

$$\omega^2 = \tfrac{1}{2}c_3 a^2, \quad k^2 = 2 + \frac{c_1}{(1/2)c_3 a^2}. \tag{43}$$

For the limit cycle motion we have

$$c_0(M_2 - M_4) - c_2 a^2 \omega^2 k^4 (M_4 - 2M_6 + M_8) = 0, \tag{44}$$

where the averaged functions M_{2n}, $n = 1, \ldots, 4$ are given in the Appendix. Solving the relations (43) and (44) the parameters of the orbital motion k, a and ω are obtained

$$\frac{c_2}{c_0}\frac{(-c_1)^2}{c_3} = -\frac{35}{6}\frac{[2(1-k^2)K - (2-k^2)E](2-k^2)^2}{K(k^6+15k^4-32k^2+16) - E(2k^6+4k^4-24k^2+16)}, \tag{45}$$

$$a = \sqrt{\frac{(-c_1)}{c_3}\frac{2}{2-k^2}}, \quad \omega = \sqrt{\frac{(-c_1)}{2-k^2}}. \tag{46}$$

3.2. The generalized harmonic balance method

Oscillator type I: $c_1 > 0, c_3 > 0$

For this type of oscillator, one can take a generating function (23). Differentiating Eq. (23) twice with respect to t and substituting into Eq. (2) one obtains

$$[c_3 a^3 - 2a\omega^2 k^2]cn^3 + [c_1 a + 2a\omega^2 k^2 - a\omega^2]cn - c_0 \varepsilon\, \omega\, a\, sn\, dn - c_2 \varepsilon \omega^3 a^3\, sn^3\, dn^3 = 0. \tag{47}$$

The generalized Fourier expansion, if the expression is limited to the first harmonic [18], [?], gives

$$[(\tfrac{3}{4})(c_3 a^3 - 2a\omega^2 k^2) + (c_1 a + 2a\omega^2 k^2 - a\omega^2)]\cos\vartheta - [c_0 \varepsilon a \omega H_1 + c_2 \varepsilon a^3 \omega^3 H_2]\sin\vartheta + (\text{higher harmonics}) = 0, \tag{48}$$

where

$$H_1 = \frac{1}{\pi}\int_0^{2\pi} snu\, dnu \sin\vartheta\, \mathrm{d}\vartheta = \frac{1}{\pi}\int_0^{4K} sn^2 u\, dn^2 u\, \mathrm{d}u = \frac{4}{3\pi k^2}[(2k^2-1)E + (1-k^2)K], \tag{49}$$

$$H_2 = \frac{1}{\pi}\int_0^{2\pi} sn^3 u\, dn^3 u \sin\vartheta\, \mathrm{d}\vartheta = \frac{1}{\pi}\int_0^{4K} sn^4 u\, dn^4 u\, \mathrm{d}u = \frac{4}{35\pi k^4}[(8k^6 - 13k^4 + 3k^2 + 2)K - (16k^6 - 24k^4 + 4k^2 + 2)E] \tag{50}$$

and the argument of $\sin\vartheta$ and $\cos\vartheta$ in the Fourier expansion is the amplitude function $\vartheta = am(\omega t, k^2) = am\, u$, so that $\cos\vartheta = cn(\omega t, k^2) = cn\, u = cn$, $\sin\vartheta = sn(\omega t, k^2) = sn\, u = sn$. If we introduce these transformations into equations (48), we obtain

$$\begin{aligned}&[\tfrac{3}{4}(c_3 a^3 - 2a\,\omega^2 k^2) + (c_1 a + 2a\,\omega^2 k^2 - a\,\omega^2)]\cos\vartheta - [c_0 \varepsilon a\omega([4(2k^2-1)E + 4(1-k^2)K]/3\pi k^2)\\ &\quad + c_2\varepsilon a^3\omega^3([4(8k^6 - 13k^4 + 3k^2 + 2)K - 4(16k^6 - 24k^4 + 4k^2 + 2)E]/35\pi k^4)]\sin\vartheta\\ &\quad + (\text{higher harmonics}) = 0.\end{aligned} \tag{51}$$

Following the method of harmonic balance and equating the coefficient of $\sin\vartheta$ with zero in Eq. (51), one obtains

$$a^2 = \frac{-35 c_0 k^2[(2k^2-1)E + (1-k^2)K]}{3c_2\omega^2[(8k^6 - 13k^4 + 3k^2 + 2)K - (16k^6 - 24k^4 + 4k^2 + 2)E]}. \tag{52}$$

Equating the first and the second terms of coefficient $\cos\vartheta$ with zero we obtain relations (24).

This result has been previously shown by Margallo and Bejarano [5]. Substituting Eq. (24) into Eq. (52) the relation for the modulus of elliptic function is obtained

$$\frac{c_1^2 c_2}{c_0 c_3} = \frac{35[(1-2k^2)E - (1-k^2)K](1-2k^2)^2}{6[(8k^6 - 13k^4 + 3k^2 + 2)K - (16k^6 - 24k^4 + 4k^2 + 2)E]}. \tag{53}$$

Comparing Eq. (53) with Eq. (27) it is obvious that they are equal. The amplitude and the modulus of the elliptic function depend on k^2 and Eq. (28) are the corresponding relations.

Oscillator type II: $c_1 > 0, c_3 < 0$

For this type of oscillator, the generating function takes the form

$$x = a\, sn(\omega t, k^2). \tag{54}$$

Following the suggested procedure, assumption (54) and the Fourier expansion of the function Eq. (2) transforms into

$$[(\tfrac{3}{4})(2a\omega^2 k^2 + c_3 a^3) + (c_1 a - a\omega^2 k^2 - a\omega^2)]\sin\vartheta + [c_0\varepsilon a\omega H_3 + c_2\varepsilon a^3\omega^3 H_4]\cos\vartheta + (\text{higher harmonics}) = 0, \tag{55}$$

where

$$H_3 = \frac{1}{\pi}\int_0^{2\pi} cn\, u\, dn\, u \cos\vartheta\, \mathrm{d}\vartheta = \frac{4}{3\pi k^2}[(1+k^2)E - k'^2 K], \tag{56}$$

$$H_4 = \frac{1}{\pi}\int_0^{2\pi} cn^3 u\, dn^3 u \cos\vartheta\, \mathrm{d}\vartheta = \frac{4}{35\pi k^4}[(k^6 + 8k^4 - 11k^2 + 2)K - (2k^6 - 10k^4 - 10k^2 + 2)E], \tag{57}$$

and $k' = \sqrt{1-k^2}$ is the complementary modulus. Equating the terms with $\sin\vartheta$ and $\cos\vartheta$ to zero, one obtains relations (37) and

$$a^2 = \frac{-35c_0k^2[(1+k^2)E - k'^2K]}{3c_2\omega^2[(k^6+8k^4-11k^2+2)K - (2k^6-10k^4-10k^2+2)E]}. \tag{58}$$

Comparing formulas (37) and (58) with Eqs. (40) and (41) it is evident that they are the same.

Oscillator type III: $c_1 < 0, c_3 > 0$

For this type of oscillator, we take a trial function

$$x = a\,dn(\omega t, k^2). \tag{59}$$

The application of Fourier series and Eq. (59) in Eq. (2) yields

$$[(\tfrac{3}{4})(c_3a^3 - 2a\omega^2) + [c_1a + a\omega^2(2-k^2)]]\cos\vartheta - [c_0\varepsilon a\omega k^2H_5 + c_2\varepsilon a^3\omega^3k^6H_6]\sin\vartheta + (\text{higher harmonics}) = 0, \tag{60}$$

where

$$H_5 = \frac{1}{\pi}\int_0^{2\pi} sn\,u\,cn\,u\,\sin\vartheta\,\mathrm{d}\vartheta = \frac{4}{3\pi k^2}[(2-k^2)E - 2k'^2K], \tag{61}$$

$$H_6 = \frac{1}{\pi}\int_0^{2\pi} sn^3\,u\,cn^3u\,\sin\vartheta\,\mathrm{d}\vartheta = \frac{4}{35\pi k^6}[(k^6+15k^4-32k^2+16)K - (2k^6+4k^4-24k^2+16)E]. \tag{62}$$

The Fourier expansion in terms of $\sin\vartheta$ and $\cos\vartheta$ is calculated with the approximate value $\vartheta = k\int cn\,u\,\mathrm{d}u$, which gives $\cos\vartheta = dn\,u = dn$ and $\sin\vartheta = k\,sn\,u = k\,sn$. Putting the coefficients of $\sin\vartheta$ and $\cos\vartheta$, respectively, equal to zero in Eq. (60) we obtain Eq. (43) and

$$a^2 = \frac{-35c_0[(2-k^2)E - 2k'^2K]}{3c_2\omega^2[(k^6+15k^4-32k^2+16)K - (2k^6+4k^4-24k^2+16)E]}. \tag{63}$$

Solving relations (63) and (43) we obtain that the modulus of elliptic function k depends on the values of the coefficients (c_2/c_0) and (c_1^2/c_3) and is given with Eq. (45). The amplitude and frequency of orbital motion satisfy relations (46).

4. Numerical examples

It is important to compare the analytical approximate results with the ‘exact’ solutions obtained by numerical integration of the equations of motion. The numerical solutions are obtained by using the symbolic language Mathematica 5.2. Three numerical examples are considered. The coefficients in the examples are arbitrary and have no special physical meaning. In all the examples $\varepsilon = 0.1$.

Example 1. Consider the equation

$$\ddot{x} + x^3 - \varepsilon(1-\dot{x}^2)\dot{x} = 0, \tag{64}$$

which is a special case of the oscillator of type I. Using the approximate analytical solution

$$x = 1.2359cn(1.2359t, 0.5) \tag{65}$$

and its first time derivative

$$\dot{x} = -1.5274sn(1.2359t, 0.5)dn(1.2359t, 0.5) \tag{66}$$

by eliminating the parameter t the limit cycle in $x-\dot{x}$ is obtained (Fig. 1)

$$\dot{x}^2 = 1.1666 - 0.5x^4. \tag{67}$$

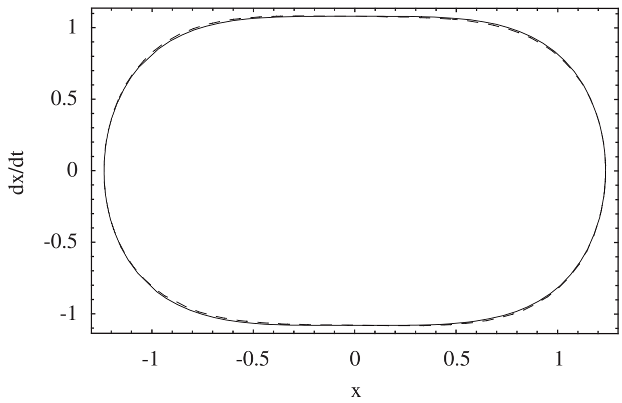

Fig. 1. Limit cycle solutions of Eq. (64) obtained analytically (____) and numerically (- - -).

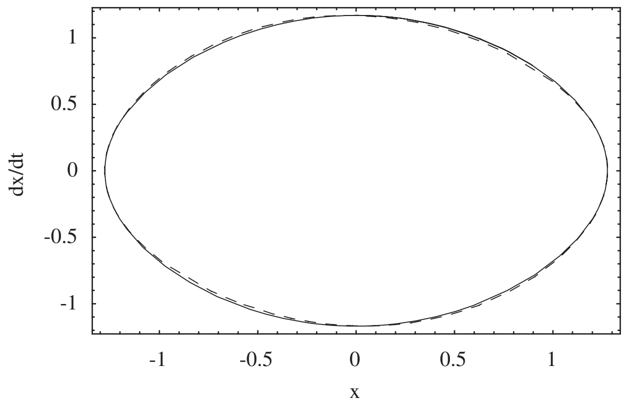

Fig. 2. Limit cycle solutions of Eq. (68) obtained analytically (____) and numerically (- - -).

Applying the Runge–Kutta method the numerical solution of Eq. (64) is calculated and also plotted in Fig. 1. The difference between the analytical and numerical results is negligible.

Example 2. To illustrate case II, we consider the equation

$$\ddot{x} + x - 0.2x^3 - \varepsilon(1 - \dot{x}^2)\dot{x} = 0. \tag{68}$$

The analytical solution in the first approximation is

$$x = 1.3288sn(0.8905t, 0.189). \tag{69}$$

Using the first time derivative

$$\dot{x} = 1.1833cn(0.8905t, 0.189)dn(0.8905t, 0.189) \tag{70}$$

and (69) the limit cycle in $x - \dot{x}$ (Fig. 2) is

$$\dot{x}^2 = 1.4002(1 - 0.56634x^2)(1 - 0.10704x^2). \tag{71}$$

Comparing result (71) with that obtained by numerical integration of Eq. (68) it is shown that the solutions are in good agreement.

Example 3. Case III is illustrated by the following equation:

$$\ddot{x} - 2.4\,x + x^3 - \varepsilon(1 - \dot{x}^2)\dot{x} = 0. \tag{72}$$

The approximate analytical solution according to Eqs. (45) and (46) is

$$x = 2.080\,1\ dn(1.470\,8t, 0.890\,63) \tag{73}$$

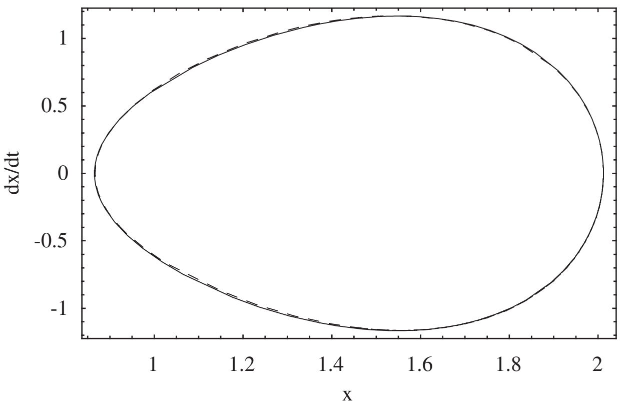

Fig. 3. Limit cycle solutions of Eq. (72) obtained analytically (____) and numerically (- - -).

and the limit cycle is

$$\dot{x}^2 = 9.36(1 - 0.23112x^2)(0.23112x^2 - 0.10937). \tag{74}$$

Eq. (72) is numerically solved and the result is compared with the analytic one (74). In Fig. 3 both the solutions are plotted. Comparing the solutions it is evident that the difference is negligible.

5. Conclusion

The following conclusions' are made:

1. Both the approximate analytical methods, the homotopy averaged method introduced in the paper and the generalized harmonic balance method, are suitable for solving the generalized Rayleigh differential equation.

2. The results obtained using the both analytical procedures are equal, i.e., give the same values for the amplitude and frequency of orbital motion and also the modulus of the elliptic function which describes the motion.

3. In the paper it is shown that the generalized harmonic balance method is also applicable for the systems with soft cubic nonlinearity and the system where the linear term has a negative coefficient and not only for the oscillator with hard cubic nonlinearity as it was previously concluded.

4. Comparing the analytically obtained results for orbital motion with numerical ones it is concluded that they are in good agreement.

5. The steady-state orbital motion is independent of the initial conditions.

6. Analyzing the relations for the frequency and modulus of the elliptic function it is seen that their form is the same as that for the cubic equation where the dissipation is neglected. The amplitude a of the limit cycle strongly depends on modulus k, i.e., on the parameter properties of the system: c_0, c_1, c_2 and c_3.

7. The properties of the limit cycle do not depend on the value of ε, but strongly depend on the parameter ratios c_0/c_2 and c_1^2/c_3.

8. For the case of a pure cubic oscillator ($c_1 = 0$ and $c_3 > 0$) the amplitude of the limit cycle depends on c_3: for higher values of c_3 the amplitude is smaller and vice versa. The frequency of motion ω also depends on the coefficient c_3

$$\omega = 1.2359\sqrt[4]{c_3 \frac{c_0}{c_2}}. \tag{75}$$

For a higher value of c_3 the frequency is higher and the period of vibration is smaller (35).

9. Comparing the linear and the pure cubic case it is evident that the frequency of orbital motion and also the period of motion of the nonlinear system depend on the c_0/c_2 ratio while for the linear system this is not the case.

10. Comparing the linear (31) and the general case (29) it can be concluded that the value of the amplitude of limit cycle is a product of $\sqrt{(1/c_1)(c_0/c_2)}$ and an invariant constant value for the linear oscillator and of the value $\sqrt{(1/c_1)(c_0/c_2)}$ and a constant which depends on the properties of the system for the nonlinear oscillator. The motion frequency of the linear oscillator is $\sqrt{c_1}$ and for the nonlinear oscillator it is corrected with a multiplier $1/\sqrt{1-2k^2}$.

Appendix A. Elliptic functions

For the convenience of our readers, we gather some facts on Jacobian elliptic functions (see Ref. [18]) for details. Jacobian elliptic functions satisfy algebraic relations which are analogous to those for trigonometric functions. The fundamental three elliptic functions are $cn\ (\tau,k)$, $sn(\tau,k)$ and $dn(\tau,k)$. Each of the elliptic functions depends on the modulus k as well as the argument τ. Note that the elliptic functions sn and cn may be thought of as generalizations of sin and cos where their period depends on the modulus k.

The elliptic functions satisfy the following identities, which are analogous to $\sin^2+\cos^2=1$:

$$sn^2+cn^2=1,\quad k^2\,sn^2+dn^2=1,\quad k^2\,cn^2+1-k^2=dn^2.$$

Before averaging, it is very convenient to transform all the elliptic functions into a sinus elliptic function

$$sn^2\,dn^2=sn^2-k^2\,sn^4,$$

$$sn^4\,dn^4=sn^4-2k^2\,sn^6+k^4\,sn^8,$$

$$sn^2\,cn^2\,dn^2=sn^2-(k^2+1)\,sn^4+k^2\,sn^6,$$

$$sn^2\,cn^4\,dn^2=sn^2-(k^2+2)\,sn^4+(2k^2+1)\,sn^6-k^2\,sn^8,$$

$$sn^2\,cn^6\,dn^2=sn^2-(k^2+3)\,sn^4+(3k^2+3)\,sn^6-(3k^2+1)\,sn^8+k^2\,sn^{10}.$$

Averaging the sinus elliptic functions according to Bryd and Friedman [18] one obtains

$$\begin{aligned}
M_2 &= \int_0^{4K} sn^2\,\mathrm{d}\tau=\frac{4}{k^2}[K-E],\\
M_4 &= \int_0^{4K} sn^4\,\mathrm{d}\tau=\frac{4}{3k^4}[(2+k^2)K-2(1+k^2)E],\\
M_6 &= \int_0^{4K} sn^6\,\mathrm{d}\tau=\frac{4}{15k^6}[(8+3k^2+4k^4)K-(8+7k^2+8k^4)E],\\
M_8 &= \int_0^{4K} sn^8\mathrm{d}\tau=\frac{4}{105k^8}[(48+16k^2+17k^4+24k^6)K-(48+40k^2+40k^4+48k^6)E],\\
&\ldots\\
M_{2m+2} &= \int_0^{4K} sn^{2m+2}\,\mathrm{d}\tau=\frac{2m(1+k^2)M_{2m}+(1-2m)M_{2m-2}}{(2m+1)k^2}.
\end{aligned}$$

References

[1] A.H. Nayfeh, D.T. Mook, *Nonlinear Oscillators*, Wiley, New York, 1979.
[2] A.H. Nayfeh, *Introduction to Perturbation Techniques*, Wiley, New York, 1981.
[3] N.N. Bogoliubov, Y.S. Mitropolski, *Asymptotic Methods in the Theory of Non-linear Oscillations*, Gordon & Breach, New York, 1961.
[4] A.H. Nayfeh, *Perturbation Method*, Wiley, New York, 1973.

[5] J.G. Margallo, J.D. Bejarano, The limit cycles of generalized Rayleigh–Liénard oscillator, *Journal of Sound and Vibration* 156 (1992) 283–301.
[6] J.D. Bejarano, A.M. Sanchez, Generalized Fourier transforms for nonlinear systems, *Journal of Mathematical Physics* 30 (1989) 1871–1876.
[7] J.-H. He, Homotopy perturbation technique, *Computer Methods in Applied Mechanics and Engineering* 178 (1999) 257–262.
[8] J.-H. He, Coupling method of a homotopy technique and a perturbation technique for non-linear problems, *International Journal of Non-Linear Mechanics* 35 (2000) 37–43.
[9] L. Cveticanin, The homotopy–perturbation method applied for solving complex-valued differential equations with strong cubic nonlinearity, *Journal of Sound and Vibration* 285 (2005) 1171–1179.
[10] L. Cveticanin, Homotopy–perturbation method for pure nonlinear differential equation, *Chaos, Solitons and Fractals* 30 (2006) 1221–1230.
[11] J.G. Margallo, J.D. Bejarano, S.B. Yuste, Generalized Fourier series for the study of limit cycles, *Journal of Sound and Vibration* 125 (1988) 13–21.
[12] J.G. Margallo, J.D. Bejarano, Generalized Fourier series and limit cycles of generalized Van der Pol oscillators, *Journal of Sound and Vibration* 136 (1990) 453–466.
[13] P.A.T. Christopher, An approximate solution to a strongly nonlinear, second order differential equation, *International Journal of Control* 17 (1973) 597–608.
[14] S.B. Yuste, J.D. Bejarano, Extension and improvement to the Krylov–Bogoliubov method using elliptic functions, *International Journal of Control* 49 (1989) 1127–1141.
[15] V.T. Coppola, R.H. Rand, Averaging using elliptic functions, Approximations of limit cycle, *Acta Mechanica* 81 (1990) 125–142.
[16] S.H. Chen, Y.K. Cheung, An elliptic perturbation method for certain strongly non-linear oscillators, *Journal of Sound and Vibration* 192 (1996) 453–464.
[17] M. Belhaq, F. Lakrad, On the elliptic harmonic balance method for mixed parity non-linear oscillators, *Journal of Sound and Vibration* 233 (2000) 935–937.
[18] P. Byrd, M. Friedman, *Handbook of Elliptic Integrals for Engineers and Physicists*, Springer, Berlin, 1954.
[19] R.E. Mickens, A generalized of the method of harmonic balance, *Journal of Sound and Vibration* 111 (1987) 515–518.
[20] P.J. Holmes, J. Marsden, A partial differential equation with infinitely many periodic orbits: Chaotic oscillations of a forced beam, *Archive for Rational Mechanics and Analysis* 76 (1981) 135–166.
[21] F.C. Moon, P.J. Holmes, Strange attractors and chaos in nonlinear systems, *ASME, Journal of Applied Mechanics* 50 (1983) 1021–1032.
[22] F.C. Moon, S.W. Shaw, Chaotic vibrations of a beam with nonlinear boundary conditions, *International Journal of Nonlinear Mechanics* 18 (1983) 465–477.