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Transverse vibration of viscoelastic rectangular plate with linearly varying thickness and multiple cracks

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Abstract

Based on the thin-plate theory and the two-dimensional viscoelastic differential constitutive relation, the differential equations of motion of the viscoelastic plate with linearly varying thickness and an arbitrary number of all-over part-through cracks are established, and the expressions of the additional rotation angle induced by the cracks are deduced. We assume that it is elastic in dilatation, but postulate the Kelvin–Voigt laws for distortion, the complex eigenvalue equations of the viscoelastic plate with linearly varying thickness and multiple cracks are derived by the differential quadrature method. The general eigenvalue equations of the viscoelastic plate with multiple cracks under different boundary conditions are calculated. The effects of various geometric parameters, dimensionless delay time and dimensionless crack parameters on the transverse vibration characteristics of a viscoelastic plate containing multiple all-over part-through cracks are analyzed.

1. Introduction

The analysis of the dynamic behavior of varying-thickness rectangular thin plates as the basic structures is of significant importance in practical engineering. Some of these plates with crack damage are not avoidable and are quite adverse for the normal work of the structures. Usually, the presence of cracks can result in changes of dynamic characteristics of the structures. In the existing literature, many methods have been applied to study the vibration problems of varying-thickness rectangular thin plates. Appl and Byers [1] studied the fundamental frequency of simply supported rectangular plates with linearly varying thickness. Soni and Rao [2] used the spline technique method to solve the vibration of rectangular plates with varying thickness. Tong [3] discussed the critical load and vibration characteristic of rectangular plates with varying thickness by the two-step series expansion method. Wang and Feng [4] adopted the Levy-type solution and a power series employing the method of Frobenius to obtain the exact solutions for the natural frequency of rectangular plates with linearly varying thickness in the x-direction, and the effects of aspect ratio and thickness ratio on the natural frequency were discussed.

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The analysis of the effects of vibration on a thin plate with crack has important theoretic significance. Much research work had been done on the vibration of plates with cracks [5,6]. Lee and Lim [7] determined the natural frequencies of a rectangular plate with a centrally located crack using the Rayleigh method. Solecki [8] studied the bending vibration of a rectangular plate with arbitrarily located rectilinear cracks. Liew et al. [9] presented a solution method for analysis of cracked plates under vibration. Khadem and Rezaee [10] took an analytical approach and investigated the vibration of the plate with an all-over part-through crack. Han and Ren [11] analyzed the effect of cracks on the dynamic characteristics of plates by means of a model of zero dimension element with crack. With the development of material science, viscoelastic plates and shells are widely applied. In the research on the dynamical problem of viscoelastic plate with crack Hu and Fu [12] introduced linear free vibration of viscoelastic plates with a crack and four edges simply supported using the Galerkin procedure. To the author's knowledge, few papers have been presented on varying-thickness viscoelastic plates containing an arbitrary number of all-over part-through cracks.

The aim of this paper is to construct the differential equations of motion of the viscoelastic plate with linearly varying thickness and an arbitrary number of all-over part-through cracks. The equations are suitable for various linear viscoelastic differential models. The complex eigenvalue equations of the cracked plate constituted elastic behavior in dilatation and the Kelvin–Voigt laws for distortion are presented by the differential quadrature method. The general eigenvalue equations of a viscoelastic plate with multiple cracks and different boundary conditions are calculated. The effects of various geometric parameters, dimensionless delay time and dimensionless crack parameters on the transverse vibration characteristics of viscoelastic plates containing multiple all-over part-through cracks are analyzed.

2. Differential equation of motion of linearly varying-thickness viscoelastic plates with multiple cracks

Consider a viscoelastic rectangular thin plate with linearly varying thickness and n all-over part-through cracks, and the crack is located at $x = x_c$ ($c = 1,2,\ldots,n$), crack depth h_c, as shown in Fig. 1. The thin plate is divided by cracks into I ($I = 1,2,\ldots,n+1$) domains. The plate has length a and width b in the x and y directions respectively, the thickness h_1 at $x = 0$ and h_2 at $x = a$ in the z direction. The density of the material is ρ. The varying relation of the thickness along x-direction is $h(x) = h_1[1 - (1 - h_2/h_1)x/a]$.

In three-dimension, the linear viscoelastic differential constitutive relation is

$$\begin{cases} P' s_{ij} = Q' e_{ij}, \\ P'' \sigma_{ii} = Q'' \varepsilon_{ii}, \end{cases} \tag{1}$$

where the differential operator $P' = \sum_{k=0}^{l} p'_k \mathrm{d}^k/\mathrm{d}t^k$, $Q' = \sum_{k=0}^{r} q'_k \mathrm{d}^k/\mathrm{d}t^k$, $P'' = \sum_{k=0}^{l_1} p''_k \mathrm{d}^k/\mathrm{d}t^k$, $Q'' = \sum_{k=0}^{r_1} q''_k \mathrm{d}^k/\mathrm{d}t^k$, p'_k, q'_k, p''_k, q''_k depend on the properties of the material; e_{ij} and s_{ij} are deviatoric tensor of stress and strain, σ_{ii} and ε_{ii} are spherical tensor of stress and strain, respectively.

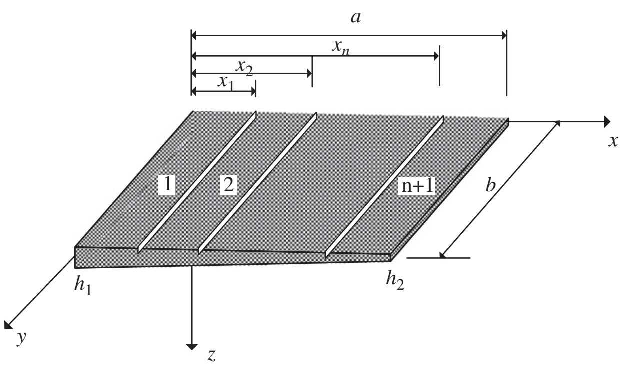

Fig. 1. A linearly varying thickness viscoelastic plate having n number of cracks.

For the plane stress problem, the constitutive equations of the linearly viscoelastic material in the Laplace domain [13] are

$$\begin{cases} \bar{P}'(\bar{P}'\bar{Q}'' + 2\bar{Q}'\bar{P}'')\bar{\sigma}_x = \bar{Q}'(2\bar{P}'\bar{Q}'' + \bar{Q}'\bar{P}'')\bar{\varepsilon}_x + \bar{Q}'(\bar{P}'\bar{Q}'' - \bar{Q}'\bar{P}'')\bar{\varepsilon}_y, \\ \bar{P}'(\bar{P}'\bar{Q}'' + 2\bar{Q}'\bar{P}'')\bar{\sigma}_y = \bar{Q}'(\bar{P}'\bar{Q}'' - \bar{Q}'\bar{P}'')\bar{\varepsilon}_x + \bar{Q}'(2\bar{P}'\bar{Q}'' + \bar{Q}'\bar{P}'')\bar{\varepsilon}_y, \\ \bar{Q}'\bar{\varepsilon}_{xy} = \bar{P}'\bar{\tau}_{xy}, \end{cases} \tag{2}$$

where $\bar{P}'$, $\bar{Q}'$, $\bar{P}''$, $\bar{Q}''$ is the Laplace transform of the differential operators P', Q', P'', Q''.

For convenience, let $\bar{P}_0 = \bar{P}'\bar{Q}'' + 2\bar{Q}'\bar{P}''$, $\bar{Q}_0 = \bar{Q}'(2\bar{P}'\bar{Q}'' + \bar{Q}'\bar{P}'')$, $\bar{Q}_1 = \bar{Q}'(\bar{P}'\bar{Q}'' - \bar{Q}'\bar{P}'')$, the polynomial $\bar{P}_0$, $\bar{Q}_0$ and $\bar{Q}_1$ about Laplace variable s_1 are independent of spatial coordinates. The relations between internal torque and the Laplace transformation $\bar{w}$ of deflection w^* are given by

$$\begin{cases} \bar{P}_0(\bar{M}_x) = -\int_{-h(x)/2}^{h(x)/2} z^2 \left[\bar{Q}_0 \dfrac{\partial^2 \bar{w}}{\partial x^2} + \bar{Q}_1 \dfrac{\partial^2 \bar{w}}{\partial y^2}\right] \mathrm{d}z, \\ \bar{P}_0(\bar{M}_y) = -\int_{-h(x)/2}^{h(x)/2} z^2 \left[\bar{Q}_1 \dfrac{\partial^2 \bar{w}}{\partial x^2} + \bar{Q}_0 \dfrac{\partial^2 \bar{w}}{\partial y^2}\right] \mathrm{d}z, \\ \bar{P}'(\bar{M}_{xy}) = \bar{P}'(\bar{M}_{yx}) = -\int_{-h(x)/2}^{h(x)/2} z^2 \bar{Q}' \left(\dfrac{\partial^2 \bar{w}}{\partial x\, \partial y}\right) \mathrm{d}z. \end{cases} \tag{3}$$

According to the D'Alembert principle, the equilibrium differential equation of the viscoelastic plate is deduced as

$$\frac{\partial^2 M_x}{\partial x^2} + 2\frac{\partial^2 M_{xy}}{\partial x\, \partial y} + \frac{\partial^2 M_y}{\partial y^2} - \rho h(x)\frac{\partial^2 w^*}{\partial t^2} = 0. \tag{4}$$

Multiplying the result of the Laplace transformation of Eq. (4) by $\bar{P}_0\bar{P}'$, if the partial derivative is continuous, Eq. (4) can be rewritten as

$$\bar{P}'\left(\frac{\partial^2[\bar{P}_0(\bar{M}_x)]}{\partial x^2}\right) + 2\bar{P}_0\left(\frac{\partial^2[\bar{P}'(\bar{M}_{xy})]}{\partial x\, \partial y}\right) + \bar{P}'\left(\frac{\partial^2[\bar{P}(\bar{M}_y)]}{\partial y^2}\right) - \rho h(x)\bar{P}_0\bar{P}'\bar{s}^2\bar{w} = 0. \tag{5}$$

Substituting Eq. (3) into Eq. (5), the differential equation of the viscoelastic rectangular plate with linearly varying thickness in the Laplace domain is obtained as

$$\begin{aligned} &\frac{h^3(x)}{12}\bar{P}'\bar{Q}_0\nabla^4\bar{w} + \frac{6h^2(x)}{12}\frac{\mathrm{d}h(x)}{\mathrm{d}x}\bar{P}'\bar{Q}_0\frac{\partial}{\partial x}\nabla^2\bar{w} + \frac{6h(x)}{12}\left(\frac{\mathrm{d}h(x)}{\mathrm{d}x}\right)^2\left[\bar{P}'\bar{Q}_0\frac{\partial^2\bar{w}}{\partial x^2} + \bar{P}'\bar{Q}_1\frac{\partial^2\bar{w}}{\partial y^2}\right] \\ &\quad + \bar{P}_0\bar{P}'\rho h(x)\bar{s}^2\bar{w} = 0. \end{aligned} \tag{6}$$

Eq. (6) is suitable for various linear viscoelastic differential constitutive relations, and the corresponding differential equations are obtained by introducing the Laplace transform $\bar{P}'$, $\bar{Q}'$, $\bar{P}''$, $\bar{Q}''$ of the differential operator.

Assuming elastic behavior in dilatation and the Kelvin–Voigt law for distortion, the constitutive equations are as follows [14]:

$$\begin{cases} s_{ij} = 2Ge_{ij} + 2\eta\dot{e}_{ij}, \\ \sigma_{ii} = 3K\varepsilon_{ii}, \end{cases} \tag{7}$$

where G, η, and K are the shear elastic modulus, viscoelastic coefficient, and bulk elastic modulus, respectively. From the Laplace transformation of Eq. (7), one can obtain the differential operator $\bar{P}' = 1$, $\bar{Q}' = 2G + 2\eta\bar{s}$, $\bar{P}'' = 1$, $\bar{Q}'' = 3K$. Substituting the above polynomial to Eq. (6), and carrying out the Laplace inverse transformation, the differential equation of motion of linearly varying thickness viscoelastic plates constituted by the Kelvin–Voigt model is

$$\frac{h_1^2}{12}\left[1-\left(1-\frac{h_2}{h_1}\right)\frac{x}{a}\right]^2\left(A_3+A_4\frac{\partial}{\partial t}+A_5\frac{\partial^2}{\partial t^2}\right)\nabla^4 w_I^*+\frac{6h_1^2}{12a}\left[1-\left(1-\frac{h_2}{h_1}\right)\frac{x}{a}\right]\left(\frac{h_2}{h_1}-1\right)$$
$$\times\left(A_3+A_4\frac{\partial}{\partial t}+A_5\frac{\partial^2}{\partial t^2}\right)\frac{\partial}{\partial x}\nabla^2 w_I^*+\frac{6h_1^2}{12a^2}\left(\frac{h_2}{h_1}-1\right)^2\left[\left(A_3+A_4\frac{\partial}{\partial t}+A_5\frac{\partial^2}{\partial t^2}\right)\frac{\partial^2 w_I^*}{\partial x^2}\right.$$
$$\left.+\left(A_6+A_7\frac{\partial}{\partial t}-A_5\frac{\partial^2}{\partial t^2}\right)\frac{\partial^2 w_I^*}{\partial y^2}\right]+\rho\left(A_1+A_2\frac{\partial}{\partial t}\right)\frac{\partial^2 w_I^*}{\partial t^2}=0 \quad (I=1,2,\ldots,n+1), \tag{8}$$

where

$$A_1=3K+4G,\quad A_2=4\eta,\quad A_3=2G(6K+2G),\quad A_4=8G\eta+12K\eta,\quad A_5=4\eta^2,\quad A_6=2G(3K-2G),$$
$$A_7=6K\eta-8G\eta,\quad G=\frac{E}{2(1+\mu)},\quad K=\frac{E}{3(1-2\mu)},$$

μ is Poisson's ratio,

$$\nabla^4 w_I^*=\frac{\partial^4 w_I^*}{\partial x^4}+2\frac{\partial^4 w_I^*}{\partial x^2\,\partial y^2}+\frac{\partial^4 w_I^*}{\partial y^4},\quad \nabla^2 w_I^*=\frac{\partial^2 w_I^*}{\partial x^2}+\frac{\partial^2 w_I^*}{\partial y^2}.$$

3. Vibration model of a cracked plate

At the crack location $x=x_c$, one can write continuity conditions [5,10,12] as follows:

$$w_I^*(x_I^-,y,t)=w_{I+1}^*(x_I^+,y,t),\quad M_{Ix}(x_I^-,y,t)=M_{(I+1)x}(x_I^+,y,t),$$
$$V_{Ix}(x_I^-,y,t)=V_{(I+1)x}(x_I^+,y,t),\quad w_{I,x}^*(x_I^-,y,t)+\Theta-w_{I+1,x}^*(x_I^+,y,t)=0. \tag{9}$$

The above equations express the equality of deflections, bending moment and equivalent shear forces at the two sides of the crack location, respectively, and the slope compatibility condition of the cracks, the Θ is the additional rotation angle induced by the crack, $V_{Ix}=Q_{Ix}+M_{Ixy,y}$.

Performing Laplace transform of Eq. (9), and multiplying by the polynomial $\bar{P}_0$ and $\bar{P}_0\bar{P}'$, one yields

$$\bar{w}_I(x_I^-,y)=\bar{w}_{I+1}(x_I^+,y),\quad \bar{P}_0\left(\bar{M}_{Ix}(x_I^-,y)\right)=\bar{P}_0\left(\bar{M}_{(I+1)x}(x_I^+,y)\right),$$
$$\bar{P}_0\bar{P}'\left(\bar{Q}_{Ix}(x_I^-,y)\right)+\bar{P}_0\bar{P}'\left(\bar{M}_{Ixy,y}(x_I^-,y)\right)=\bar{P}_0\bar{P}'\left(\bar{Q}_{(I+1)x}(x_I^+,y)\right)+\bar{P}_0\bar{P}'\left(\bar{M}_{(I+1)xy,y}(x_I^+,y)\right),$$
$$\bar{w}_{I,x}(x_I^-,y)+\bar{\Theta}-\bar{w}_{I+1,x}(x_I^+,y)=0. \tag{10}$$

In Eq. (10), the $\bar{\Theta}$ is the additional rotation angle in the Laplace domain

$$3\bar{Q}'\bar{Q}''(2\bar{P}'\bar{Q}''+\bar{P}''\bar{Q}')\bar{\Theta}=12\left[(2\bar{P}'\bar{Q}''+\bar{P}''\bar{Q}')^2-(\bar{P}'\bar{Q}''-\bar{P}''\bar{Q}')^2\right]\alpha_{bb}\bar{\sigma}_b, \tag{11}$$

where $\bar{\sigma}_b$ is the Laplace transform of σ_b. One obtains the relations between $\bar{\sigma}_b$ and the Laplace transformation $\bar{w}_I$ of the deflection w_I^*

$$2\bar{P}'(\bar{P}'\bar{Q}''+2\bar{Q}'\bar{P}'')\bar{\sigma}_b=-h(x)\bar{Q}'\left[(2\bar{P}'\bar{Q}''+\bar{P}''\bar{Q}')\frac{\partial^2\bar{w}_I}{\partial x^2}+(\bar{P}'\bar{Q}''-\bar{P}''\bar{Q}')\frac{\partial^2\bar{w}_I}{\partial y^2}\right], \tag{12}$$

α_{bb} expresses the non-dimensional compliance coefficient, characterizing the crack is given by the following equation:

$$\alpha_{bb}=\int_0^{s_c} g_b^2\,\mathrm{d}\varsigma, \tag{13}$$

where g_b is a dimensionless function of the relative crack depth $s_c=h_c(x)/h(x)$ defined as

$$g_b=\varsigma^{1/2}(1.99-2.47\varsigma+12.97\varsigma^2-23.17\varsigma^3+24.80\varsigma^4). \tag{14}$$

Eq. (11) is suitable for various linear viscoelastic differential constitutive relations. Substituting the polynomial $\bar{P}'$, $\bar{Q}'$, $\bar{P}''$, $\bar{Q}''$ to Eq. (11), the corresponding values $\bar{\Theta}$ are obtained. Substituting Eq. (14) into Eq. (11), by carrying out the inverse transformation, Θ can obtained. Assuming elastic behavior in dilatation and the Kelvin–Voigt law for distortion, one may obtain

$$\left(A_8 + A_9\frac{\partial}{\partial t}\right)\Theta = -6h_1\left[1-\left(1-\frac{h_2}{h_1}\right)\frac{x}{a}\right]\alpha_{bb}\left[\left(A_8 + A_9\frac{\partial}{\partial t}\right)\frac{\partial^2 w_I^*}{\partial x^2} + \left(A_{10} - A_9\frac{\partial}{\partial t}\right)\frac{\partial^2 w_I^*}{\partial y^2}\right], \tag{15}$$

where $A_8 = 6K+2G$, $A_9 = 2\eta$, $A_{10} = 3K-2G$.

Introducing dimensionless parameters and variables

$$\xi = \frac{x}{a},\quad \zeta = \frac{y}{b},\quad w_I = \frac{w_I^*}{a},\quad c = \frac{a}{b},\quad r = \frac{h_1}{a},\quad \lambda = \frac{h_2}{h_1},$$
$$\tau = \frac{th_1}{a^2}\sqrt{\frac{E}{12\rho(1-\mu^2)}},\quad H = \frac{h_1}{a^2}\sqrt{\frac{E}{12\rho(1-\mu^2)}}\frac{\eta}{E}. \tag{16}$$

Substituting Eq. (16) into Eq. (8), one obtains

$$\begin{aligned}
&\left[1-\left(1-\frac{h_2}{h_1}\right)\xi\right]^2\left[1+\frac{4}{3}(2-\mu)(1+\mu)H\frac{\partial}{\partial\tau}+\frac{4}{3}(1-2\mu)(1+\mu)^2H^2\frac{\partial^2}{\partial\tau^2}\right]\nabla^4 w_I\\
&+6\left[1-\left(1-\frac{h_2}{h_1}\right)\xi\right]\left(\frac{h_2}{h_1}-1\right)\left[1+\frac{4}{3}(2-\mu)(1+\mu)H\frac{\partial}{\partial\tau}+\frac{4}{3}(1-2\mu)(1+\mu)^2H^2\frac{\partial^2}{\partial\tau^2}\right]\frac{\partial}{\partial\xi}\nabla^2 w_I\\
&+6\left(\frac{h_2}{h_1}-1\right)^2\left[\left(1+\frac{4}{3}(2-\mu)(1+\mu)H\frac{\partial}{\partial\tau}+\frac{4}{3}(1-2\mu)(1+\mu)^2H^2\frac{\partial^2}{\partial\tau^2}\right)\frac{\partial^2 w_I}{\partial\xi^2}\right.\\
&\left.+\left(\mu+\frac{2}{3}(5\mu-1)(1+\mu)H\frac{\partial}{\partial\tau}-\frac{4}{3}(1-2\mu)(1+\mu)^2H^2\frac{\partial^2}{\partial\tau^2}\right)c^2\frac{\partial^2 w_I}{\partial\zeta^2}\right]\\
&+\left[1+\frac{4(1+\mu)(1-2\mu)}{3(1-\mu)}H\frac{\partial}{\partial\tau}\right]\frac{\partial^2 w_I}{\partial\tau^2} = 0\quad (I = 1,2,\ldots,n+1),
\end{aligned} \tag{17}$$

where τ is dimensionless time, H is dimensionless delay time,

$$\nabla^2 w_I = \frac{\partial^2 w_I}{\partial\xi^2} + c^2\frac{\partial^2 w_I}{\partial\zeta^2},\quad \nabla^4 w_I = \frac{\partial^4 w_I}{\partial\xi^4} + 2c^2\frac{\partial^4 w_I}{\partial\xi^2\,\partial\zeta^2} + c^4\frac{\partial^4 w_I}{\partial\zeta^4}.$$

The dimensionless continuity conditions at the location ($\xi = \xi_c$) is

$$w_I(\xi_I^-,\zeta,\tau) = w_{I+1}(\xi_I^+,\zeta,\tau),$$

$$\left(B_1\frac{\partial^2 w_I(\xi_I^-,\zeta,\tau)}{\partial\xi^2} + B_2c^2\frac{\partial^2 w_I(\xi_I^-,\zeta,\tau)}{\partial\zeta^2}\right) = \left(B_1\frac{\partial^2 w_{I+1}(\xi_I^+,\zeta,\tau)}{\partial\xi^2} + B_2c^2\frac{\partial^2 w_{I+1}(\xi_I^+,\zeta,\tau)}{\partial\zeta^2}\right),$$

$$\left[1-\left(1-\frac{h_2}{h_1}\right)\xi_c^-\right]\left(B_1\frac{\partial^3 w_I(\xi_I^-,\zeta,\tau)}{\partial\xi^3} + B_3c^2\frac{\partial^3 w_I(\xi_I^-,\zeta,\tau)}{\partial\xi\,\partial\zeta^2}\right) + 3\left(\frac{h_2}{h_1}-1\right)\left(B_1\frac{\partial^2 w_I(\xi_i^-,\zeta,\tau)}{\partial\xi^2} + B_2c^2\frac{\partial^2 w_I(\xi_I^-,\zeta,\tau)}{\partial\zeta^2}\right)$$

$$\left[1-\left(1-\frac{h_2}{h_1}\right)\xi_I^+\right]\left(B_1\frac{\partial^3 w_{I+1}(\xi_I^+,\zeta,\tau)}{\partial\xi^3} + B_3c^2\frac{\partial^3 w_{I+1}(\xi_I^+,\zeta,\tau)}{\partial\xi\,\partial\zeta^2}\right) + 3\left(\frac{h_2}{h_1}-1\right)\left(B_1\frac{\partial^2 w_{I+1}(\xi_I^+,\zeta,\tau)}{\partial\xi^2} + B_2c^2\frac{\partial^2 w_{I+1}(\xi_I^+,\zeta,\tau)}{\partial\zeta^2}\right),$$

$$B_1\left(\frac{\partial w_I(\xi_I^-,\zeta,\tau)}{\partial\xi} - \frac{\partial w_{I+1}(\xi_I^+,\zeta,\tau)}{\partial\xi}\right) = -6r\left[1-\left(1-\frac{h_2}{h_1}\right)\xi_I^-\right]\alpha_{bb}\left(B_1\frac{\partial^2 w_I(\xi_I^-,\zeta,\tau)}{\partial\xi^2} + B_2c^2\frac{\partial^2 w_I(\xi_I^-,\zeta,\tau)}{\partial\zeta^2}\right), \tag{18}$$

where

$$B_1 = 1+\frac{2}{3}(1-2\mu)(1+\mu)H\frac{\partial}{\partial\tau},\quad B_2 = \mu-\frac{2}{3}(1-2\mu)(1+\mu)H\frac{\partial}{\partial\tau},\quad B_3 = (2-\mu)+2(1-2\mu)(1+\mu)H\frac{\partial}{\partial\tau}.$$

Suppose that the solution to Eq. (14) takes the form $w_I(\xi,\zeta,\tau)=W_I(\xi,\zeta)\mathrm{e}^{\mathrm{j}\omega\tau}$, a dimensionless differential equation of the linearly varying thickness viscoelastic plate with the Kelvin–Voigt model is expressed as

$$\left[1-\left(1-\frac{h_2}{h_1}\right)\xi\right]^2 D_1\nabla^4 W_I+6\left[1-\left(1-\frac{h_2}{h_1}\right)\xi\right]\left(\frac{h_2}{h_1}-1\right)c^2 D_1\frac{\partial}{\partial\xi}\nabla^2 W_I$$
$$+6\left(\frac{h_2}{h_1}-1\right)^2\left(D_1\frac{\partial^2 W_I}{\partial\xi^2}+D_2c^2\frac{\partial^2 W_I}{\partial\zeta^2}\right)+D_3j^2\omega^2 W_I=0, \tag{19}$$

where

$$D_1=1-\frac{4(1-2\mu)(1+\mu)^2}{3}H^2\omega^2+\frac{4(2-\mu)(1+\mu)}{3}H\omega\mathrm{j},\quad D_2=\mu+\frac{4(1-2\mu)(1+\mu)^2}{3}H^2\omega^2$$
$$+\frac{2(5\mu-1)(1+\mu)}{3}H\omega\mathrm{j},\quad D_3=1+\frac{4(1-2\mu)(1+\mu)}{3(1-\mu)}H\omega\mathrm{j},\quad \mathrm{j}=\sqrt{-1}$$

and ω is the dimensionless complex frequency.

Substituting $w_I(\xi,\zeta,\tau)=W_I(\xi,\zeta)\mathrm{e}^{\mathrm{j}\omega\tau}$ into Eq. (18) we obtain the following:

$$W_I(\xi_I^-,\zeta)=W_{I+1}(\xi_I^+,\zeta),$$
$$\left(D_4\frac{\partial^2 W_I(\xi_I^-,\zeta)}{\partial\xi^2}+D_5c^2\frac{\partial^2 W_{I+1}(\xi_I^-,\zeta)}{\partial\zeta^2}\right)=\left(D_4\frac{\partial^2 W_{I+1}(\xi_I^+,\zeta)}{\partial\xi^2}+D_5c^2\frac{\partial^2 W_{I+1}(\xi_I^+,\zeta)}{\partial\zeta^2}\right),$$
$$\left[1-\left(1-\frac{h_2}{h_1}\right)\xi_c^-\right]\left(D_4\frac{\partial^3 W_I(\xi_I^-,\zeta)}{\partial\xi^3}+D_6c^2\frac{\partial^3 W_I(\xi_I^-,\zeta)}{\partial\xi\,\partial\zeta^2}\right)+3\left(\frac{h_2}{h_1}-1\right)\left(D_4\frac{\partial^2 W_I(\xi_I^-,\zeta)}{\partial\xi^2}+D_5c^2\frac{\partial^2 W_I(\xi_I^-,\zeta)}{\partial\zeta^2}\right)$$
$$=\left[1-\left(1-\frac{h_2}{h_1}\right)\xi_c^+\right]\left(D_4\frac{\partial^3 W_{I+1}(\xi_I^+,\zeta)}{\partial\xi^3}+D_6c^2\frac{\partial^3 W_{I+1}(\xi_I^+,\zeta)}{\partial\xi\,\partial\zeta^2}\right)$$
$$+3\left(\frac{h_2}{h_1}-1\right)\left(D_4\frac{\partial^2 W_{I+1}(\xi_I^+,\zeta)}{\partial\xi^2}+D_5c^2\frac{\partial^2 W_{I+1}(\xi_I^+,\zeta)}{\partial\zeta^2}\right),$$
$$D_4\left(\frac{\partial W_I(\xi_I^-,\zeta)}{\partial\xi}-\frac{\partial W_{I+1}(\xi_I^+,\zeta)}{\partial\xi}\right)=-6r\left[1-\left(1-\frac{h_2}{h_1}\right)\xi_I^-\right]\alpha_{bb}\left(D_4\frac{\partial^2 W_I(\xi_I^-,\zeta)}{\partial\xi^2}+D_5c^2\frac{\partial^2 W_I(\xi_I^-,\zeta)}{\partial\zeta^2}\right). \tag{20}$$

In the above relation,

$$D_4=1+\frac{2}{3}(1-2\mu)(1+\mu)H\mathrm{j}\omega,\quad D_6=(2-\mu)+2(1-2\mu)(1+\mu)H\mathrm{j}\omega,$$
$$D_5=\mu-\frac{2}{3}(1-2\mu)(1+\mu)H\mathrm{j}\omega.$$

4. Complex eigenvalue equation

The complex eigenvalue equations of the viscoelastic plate constituted by the Kelvin–Voigt model with cracks are established by the differential quadrature method. The differential quadrature method [16] is to approximate the partial derivatives of a function with respect to a spatial variable at any discrete point as the weighted linear sum of the function values at all the discrete points chosen in the solution domain of the spatial variable. Postulating smooth function $f(x, y)$ in region $0\leqslant x\leqslant a$, $0\leqslant y\leqslant b$, the partial derivative of the rth order with respect to x of function $f(x, y)$ at the point (x_i, y_i), the partial derivative of the sth order with respect to y, the mixed partial derivative of the sth order with respect to y and the rth order with respect to x are defined as follows, respectively [15]:

$$\frac{\partial^r f(x_i,y_j)}{\partial x^r}=\sum_{k=1}^{N}A_{ik}^{(r)}f(x_k,y_j)\quad (i=1,2,\ldots,N;\quad r=1,2,\ldots,N-1),$$
$$\frac{\partial^s f(x_i,y_j)}{\partial y^s}=\sum_{m=1}^{M}A_{jm}^{(s)}f(x_i,y_m)\quad (j=1,2,\ldots,M;\quad s=1,2,\ldots,M-1),\tag{21}$$

$$\frac{\partial^{r+s} f(x_i,y_j)}{\partial x^r\,\partial y^s}=\sum_{k=1}^{N}A_{ik}^{(r)}\sum_{m=1}^{M}A_{jm}^{(s)}f(x_i,y_m),$$

where N and M are the number of nodes in the x and y directions, respectively, $A_{ik}^{(r)}$ and $A_{jm}^{(s)}$ are weight coefficients, and they are determined by [16]

$$A_{ik}^{(1)}=\begin{cases}\prod\limits_{\substack{\mu=1\\ \mu\neq i,k}}^{N}(x_i-x_\mu)\Big/\prod\limits_{\substack{\mu=1\\ \mu\neq k}}^{N}(x_k-x_\mu) & (i,k=1,2,\ldots,N;\ k\neq i),\\ \sum\limits_{\substack{\mu=1\\ \mu\neq i}}^{N}\frac{1}{x_i-x_\mu} & (i,k=1,2,\ldots,N;\ k=i),\end{cases}\tag{22}$$

$$A_{jm}^{(1)}=\begin{cases}\prod\limits_{\substack{\mu=1\\ \mu\neq j,m}}^{M}(y_j-y_\mu)\Big/\prod\limits_{\substack{\mu=1\\ \mu\neq m}}^{M}(y_m-y_\mu) & (j,m=1,2,\ldots,M;\quad m\neq j),\\ \sum\limits_{\substack{\mu=1\\ \mu\neq j}}^{M}\dfrac{1}{y_j-y_\mu} & (j,m=1,2,\ldots,M;\quad m=j).\end{cases}\tag{23}$$

In the case of $r=2,3,\ldots,N-1$; $s=2,3,\ldots,M-1$, they are as follows:

$$A_{ik}^{(r)}=\begin{cases}r\left(A_{ii}^{(r-1)}A_{ik}^{(1)}-\dfrac{A_{ik}^{(r-1)}}{x_i-x_k}\right) & (i,k=1,2,\ldots,N;\quad k\neq i),\\ -\sum\limits_{\substack{\mu=1\\ \mu\neq i}}^{N}A_{i\mu}^{(r)} & (i=1,2,\ldots,N;\quad 1\leqslant r\leqslant (N-1)),\end{cases}\tag{24}$$

$$A_{jm}^{(s)}=\begin{cases}s\left(A_{jj}^{(s-1)}A_{jm}^{(1)}-\dfrac{A_{jm}^{(s-1)}}{y_j-y_m}\right) & (j,m=1,2,\ldots,M;\quad m\neq j),\\ -\sum\limits_{\substack{\mu=1\\ \mu\neq j}}^{M}A_{j\mu}^{(s)} & (j=1,2,\ldots,M;\quad 1\leqslant s\leqslant (M-1)).\end{cases}\tag{25}$$

According to the procedures of the differential quadrature method, Eq. (19) can be given in the form:

$$\left[1-\left(1-\frac{h_2}{h_1}\right)\xi\right]^2\left(\sum_{k=1}^{N}A_{ik}^{(4)}W_{kj}+2c^2\sum_{m=1}^{M}A_{jm}^{(2)}\sum_{k=1}^{N}A_{ik}^{(2)}W_{km}+c^4\sum_{m=1}^{M}A_{jm}^{(4)}W_{im}\right)+6\left[1-\left(1-\frac{h_2}{h_1}\right)\xi\right]$$
$$\times\left(\frac{h_2}{h_1}-1\right)\left(\sum_{k=1}^{N}A_{ik}^{(3)}W_{kj}+c^2\sum_{m=1}^{M}A_{jm}^{(2)}\sum_{k=1}^{N}A_{ik}^{(1)}W_{km}\right)+6\left(\frac{h_2}{h_1}-1\right)^2\left(\mu c^2\sum_{m=1}^{M}A_{jm}^{(2)}W_{im}+\sum_{k=1}^{N}A_{ik}^{(2)}W_{kj}\right)$$
$$+\left\{\left[1-\left(1-\frac{h_2}{h_1}\right)\xi\right]^2a_1\left(\sum_{k=1}^{N}A_{ik}^{(4)}W_{kj}+2c^2\sum_{m=1}^{M}A_{jm}^{(2)}\sum_{k=1}^{N}A_{ik}^{(2)}W_{km}+c^4\sum_{m=1}^{M}A_{jm}^{(4)}W_{im}\right)Hj+6\left[1-\left(1-\frac{h_2}{h_1}\right)\xi\right]\right.$$

$$\times\left(\frac{h_2}{h_1}-1\right)a_1\left(c^2\sum_{m=1}^{M}A_{jm}^{(2)}\sum_{k=1}^{N}A_{ik}^{(1)}W_{km}+\sum_{k=1}^{N}A_{ik}^{(3)}W_{kj}\right)Hj+6\left(\frac{h_2}{h_1}-1\right)^2\left(a_1\sum_{k=1}^{N}A_{ik}^{(2)}W_{kj}+a_3c^2\sum_{m=1}^{M}A_{jm}^{(2)}W_{im}\right)Hj\Bigg\}\omega$$
$$+\Bigg\{\left[1-\left(1-\frac{h_2}{h_1}\right)\xi\right]^2a_2\left(\sum_{k=1}^{N}A_{ik}^{(4)}W_{kj}+2c^2\sum_{m=1}^{M}A_{jm}^{(2)}\sum_{k=1}^{N}A_{ik}^{(2)}W_{km}+c^4\sum_{m=1}^{M}A_{jm}^{(4)}W_{im}\right)H^2j^2$$
$$+6\left[1-\left(1-\frac{h_2}{h_1}\right)\xi\right]\left(\frac{h_2}{h_1}-1\right)a_2\left(c^2\sum_{m=1}^{M}A_{jm}^{(2)}\sum_{k=1}^{N}A_{ik}^{(1)}W_{km}+\sum_{k=1}^{N}A_{ik}^{(3)}W_{kj}\right)H^2j^2+j^2W$$
$$+6\left(\frac{h_2}{h_1}-1\right)^2\left(a_2\sum_{k=1}^{N}A_{ik}^{(2)}W_{kj}-a_2c^2\sum_{m=1}^{M}A_{jm}^{(2)}W_{im}\right)H^2j^2\Bigg\}\omega^2+a_4Hj^3W\omega^3=0, \tag{26}$$

where

$$a_1=\frac{4}{3}(2-\mu)(1+\mu),\quad a_2=\frac{4}{3}(1-2\mu)(1+\mu)^2,\quad a_3=\frac{2}{3}(1+\mu)(5\mu-1),\quad a_4=\frac{4(1-2\mu)(1+\mu)}{3(1-\mu)}.$$

The differential quadrature forms of crack continuity conditions (20) are

$$W_{cj}-W_{cj}=0,$$
$$D_4\sum_{k=1}^{N}A_{c+1,k}^{(2)}W_{kj}+D_5c^2\sum_{m=1}^{M}A_{jm}^{(2)}W_{km}=D_4\sum_{k=1}^{N}A_{c+1,k}^{(2)}W_{kj}+D_5c^2\sum_{m=1}^{M}A_{jm}^{(2)}W_{km},$$
$$\left[1-\left(1-\frac{h_2}{h_1}\right)\xi\right]\left(D_4\sum_{k=1}^{N}A_{c+2,k}^{(3)}W_{kj}+D_6c^2\sum_{m=1}^{M}A_{jm}^{(2)}\sum_{k=1}^{N}A_{c+2,k}^{(1)}W_{km}\right)$$
$$+3\left(\frac{h_2}{h_1}-1\right)\left(D_4\sum_{k=1}^{N}A_{c+2,k}^{(2)}W_{kj}+D_5c^2\sum_{m=1}^{M}A_{jm}^{(2)}W_{km}\right)$$
$$=\left[1-\left(1-\frac{h_2}{h_1}\right)\xi\right]\left(D_4\sum_{k=1}^{N}A_{c+2,k}^{(3)}W_{kj}+D_6c^2\sum_{m=1}^{M}A_{jm}^{(2)}\sum_{k=1}^{N}A_{c+2,k}^{(1)}W_{km}\right)$$
$$+3\left(\frac{h_2}{h_1}-1\right)\left(D_4\sum_{k=1}^{N}A_{c+2,k}^{(2)}W_{kj}+D_5c^2\sum_{m=1}^{M}A_{jm}^{(2)}W_{km}\right),$$
$$D_4\sum_{k=1}^{N}A_{c+3,k}^{(1)}W_{kj}+D_4\sum_{k=1}^{N}A_{c+3,k}^{(1)}W_{kj}+6r\left[1-\left(1-\frac{h_2}{h_1}\right)\xi\right]\alpha_{bb}\left(D_4\sum_{k=1}^{N}A_{c+3,k}^{(2)}W_{kj}+D_5c^2\sum_{k=1}^{N}A_{c+3,k}^{(2)}W_{kj}\right)=0. \tag{27}$$

In this paper, $N=M$, it is noted that the boundary conditions are applied and continuity conditions at the crack line are needed. The simply supported plate adopts the weight coefficient method to treat the boundary, the clamped-supported plate adopts the δ method and the crack location conditions adopt the δ method to treat. The differential quadrature forms of boundary conditions of the clamped–simply–clamped–simply edge support and clamped–simply–simply–simply edge support are as follows:

$$\begin{cases} W_{1j}=W_{Nj}=W_{i1}=W_{iN}=0 & (i,j=1,2,\ldots,N), \\ \sum_{k=1}^{N}A_{ik}^{(1)}W_{kj}=0 & (i=2,N-1;\quad j=2,3,\ldots,N-2), \\ \sum_{k=1}^{N}A_{jk}^{(2)}W_{ik}=0 & (j=1,N;\quad i=1,2,\ldots,N), \end{cases} \tag{28}$$

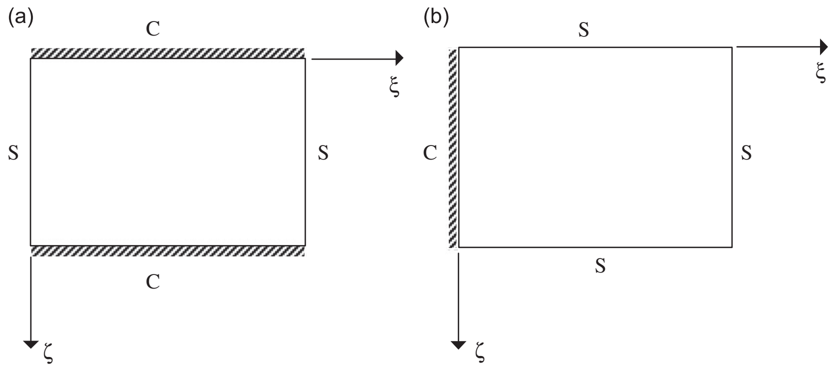

Fig. 2. A plate with simply–clamped–simply–clamped edge support (S–C–S–C) and clamped–simply–simply–simply edge support (C–S–S–S).

$$\begin{cases} W_{1j} = W_{Nj} = W_{i1} = W_{iN} = 0 & (i,j = 1,2,\ldots,N), \\ \sum_{k=1}^{N} A_{ik}^{(1)} W_{kj} = 0 & (i = 2; \quad j = 2,3,\ldots,N-2), \\ \sum_{k=1}^{N} A_{ik}^{(2)} W_{kj} = 0 & (i = N-1; \quad j = 2,3,\ldots,N-2), \\ \sum_{m=1}^{M} B_{jm}^{(2)} W_{im} = 0 & (j = 2, N-1; \quad i = 3,4,\ldots,N-2). \end{cases} \tag{29}$$

Eqs. (26) and (27) and Eq. (28) or (29) can be written in the matrix form as

$$(\omega^3[\boldsymbol{Q}] + \omega^2[\boldsymbol{R}] + \omega[\boldsymbol{G}] + [\boldsymbol{K}]) \quad \{W_{kj}\} = \{0\}, \tag{30}$$

where the matrix $[\boldsymbol{Q}]$, $[\boldsymbol{R}]$, $[\boldsymbol{G}]$, $[\boldsymbol{K}]$ involve such parameters as dimensionless delay time H, aspect ratio of the plate, thickness parameter of the plate and crack parameter. Eq. (30) is a generalized eigenvalue problem. Then, the complex eigenvalue equations of the linearly varying thickness viscoelastic plate with linearly varying thickness and an arbitrary number of all-over part-through cracks is

$$\left|\omega^3[\boldsymbol{Q}] + \omega^2[\boldsymbol{R}] + \omega[\boldsymbol{G}] + [\boldsymbol{K}]\right| = 0. \tag{31}$$

By solving the eigenvalue equations, one can obtain the complex frequency and eigenvalue curves of the viscoelastic plate with cracks (see Fig. 2).

5. Numerical results and analysis

If $H \to 0$, $s = 0$, one may obtain the equation of transverse free vibration of the linearly varying thickness elastic plate. To validate the present theory and to check the correctness of the program, the first three natural frequencies of the transverse free vibration of the intact linearly varying thickness elastic plate with different boundaries are calculated first. Tables 1 and 2 show the results for the simply–clamped–simply–clamped edge support and clamped–simply–simply–simply edge support, respectively. It can be seen that the present results agree very well with those from Ref. [17], and the natural frequencies of the elastic plate increase with the increase of the aspect ratio and thickness ratio.

In the present work, in order to perform the vibration analysis of the linearly varying-thickness viscoelastic rectangular plate containing an arbitrary number of all-over part-through cracks, one may consider the viscoelastic rectangular plate containing one, two, three or four cracks constituted by the Kelvin–Voigt model under two different types of boundary conditions.

Table 1
The first three natural frequencies of the transverse free vibration with S–C–S–C supported plate

Aspect ratio c	Thickness ratio λ	Present solution			Aspect ratio c	Thickness ratio λ	Present solution		
		ω_1	ω_2	ω_3			ω_1	ω_2	ω_3
0.5	1.0	23.8201	28.9551	39.0932	1.5	1.0	39.0931	79.5373	102.2192
	0.8	21.3774	25.9875	35.0835		0.8	35.0834	71.3839	91.5520
	0.6	18.7798	22.8355	30.8175		0.6	30.8166	62.7202	79.7650
	0.4	15.9391	19.3944	26.1487		0.4	26.1487	53.2584	66.3297
	0.2	12.6303	15.3976	20.7100		0.2	20.7100	42.2586	50.0892
Results in [17]	1.0	23.82	28.95	39.09					
1.0	1.0	28.9551	54.7466	69.3393	2.0	1.0	54.7466	94.5971	154.7996
	0.8	25.9874	49.1132	62.2240		0.8	49.1132	84.9155	138.9283
	0.6	22.8351	43.0743	54.6451		0.6	43.0743	74.6655	122.0559
	0.4	19.3944	36.4057	46.3416		0.4	36.4057	63.5226	103.6185
	0.2	15.3977	28.5431	36.6445		0.2	28.5430	50.6400	77.3924
Results in [17]	1.0	28.95	54.74	69.33					

Table 2
The first three natural frequencies of the transverse free vibration with C–S–S–S supported plate

Aspect ratio c	Thickness ratio λ	Present solution			Aspect ratio c	Thickness ratio λ	Present solution		
		ω_1	ω_2	ω_3			ω_1	ω_2	ω_3
0.5	1.0	17.3339	23.6482	35.0527	1.5	1.0	35.0527	69.9194	100.2711
	0.8	15.8843	21.3411	31.3544		0.8	31.3544	62.8436	89.1599
	0.6	14.3342	18.9213	27.4679		0.6	27.4679	55.3808	76.8536
	0.4	12.6240	16.3240	23.2986		0.4	23.2986	47.3164	62.9263
	0.2	10.5955	13.3760	18.6072		0.2	18.6072	38.1023	46.5785
1.0	1.0	23.6482	51.6758	58.6534	2.0	1.0	51.6757	86.1406	140.8584
	0.8	21.3411	46.0620	52.8844		0.8	46.0620	77.2470	126.4652
	0.6	18.9213	40.0864	46.7691		0.6	40.0864	67.9159	111.2325
	0.4	16.3240	33.5819	40.1162		0.4	33.5819	57.8957	94.6979
	0.2	13.3760	26.1604	32.4401		0.2	26.1604	46.5279	72.4041

5.1. A S– C– S– C supported plate with a crack

Figs. 3 and 4 show the variation of the first two-order dimensionless complex frequencies ω of the plate with crack depth for $H = 10^{-3}$, $r = 0.15$, $c = 1.0$ and location $\xi_1 = 0.1, 0.28, 0.3$. This means that a comparison is made with the results known in the elastic plate, it should be noted that the frequencies of the viscoelastic plate decrease with the increase of dimensionless delay time, and the change in frequencies decreases as a crack increases. For a certain value of the depth, as the crack gets closer to the center of the plate, the real part of the dimensionless complex frequency ω decreases, while its imaginary part increases. For a certain value of the crack location, by increasing the depth of the crack, the real part of the dimensionless complex frequency ω decreases, while its imaginary part increases, and the corresponding value of the variation ratio increases by increasing the crack, the same situation for the second vibration mode. As a result, the introduction of an all-over part-through crack in a viscoelastic plate decreases the stiffness of the plate; by increasing the crack, the stiffness of the structure decreasing.

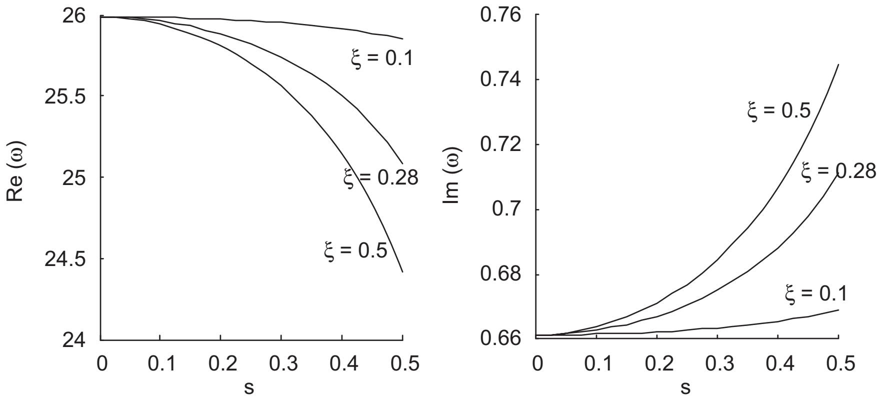

Fig. 3. The first dimensionless complex frequency varied with the crack depth ($H = 0.001$, $\lambda = 0.8$, $c = 1$).

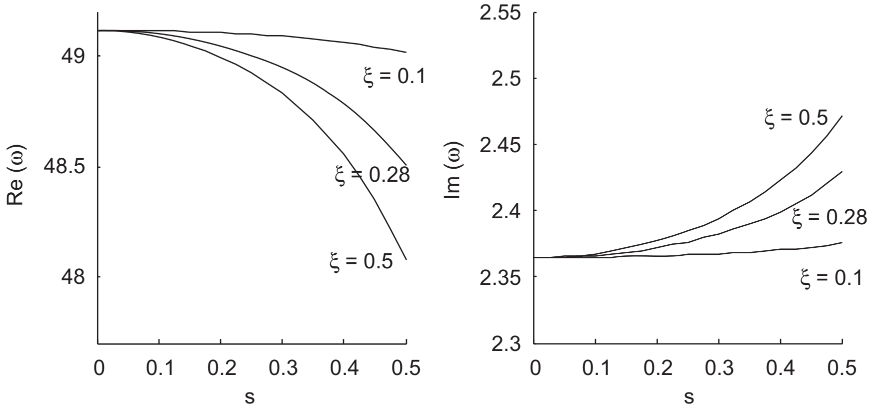

Fig. 4. The second dimensionless complex frequency varied with the crack depth ($H = 0.001$, $\lambda = 0.8$, $c = 1$).

Figs. 5–7 show the variation of dimensionless complex frequencies ω of the plate with crack depth for $H = 10^{-3}$, $r = 0.15$, $c = 1.0$, $\xi = 0.5$ and location $\lambda = 0.2, 0.4, 0.6, 0.8, 1.0$. For the case of $s = 0$ and by referring to Table 1, with the increase of dimensionless delay time, the real part of the dimensionless complex frequency ω decreases, its imaginary part changes positive values from zero, and the imaginary values increase with the increase of H and mode order. By increasing the depth of the crack, the real part of the dimensionless complex frequency ω decreases, while its imaginary parts increase. By increasing the thickness ratio λ, the real part of the dimensionless complex frequency ω increases, and its imaginary parts increase.

Figs. 8 and 9 show the variation of the first two-order dimensionless complex frequencies ω of the plate with crack location for $H = 10^{-3}$, $r = 0.15$, $c = 1.0$, $\lambda = 1$, $s = 0.1$ and the location $\xi = 0.1, 0.28, 0.3$, respectively. As shown in the figure, as the crack gets to the right end from the left end, the real part of the dimensionless complex frequency ω decreases, while its imaginary part increases. When the crack location is $\xi = 0.5$, the real part of ω is least, while its imaginary part remains max. When the location further gets closer to the right end, the real part of ω begins to increase, while its imaginary part remains decreases. For the case of $\xi = 1$, the dimensionless complex frequency is the equivalence relation with the case of $\xi = 0$, and the result is the same as the value of the case of $s = 0$.

Figs. 10 and 11 show the variation of the first two-order dimensionless complex frequencies ω of the plate with crack location for $H = 10^{-3}$, $r = 0.15$, $c = 1.0$, $\lambda = 0.8$, $s = 0.1, 0.2, 0.3$, respectively. As shown in the figure, as the crack gets closer to the center of the plate and the depth of the crack increases, the real part of the

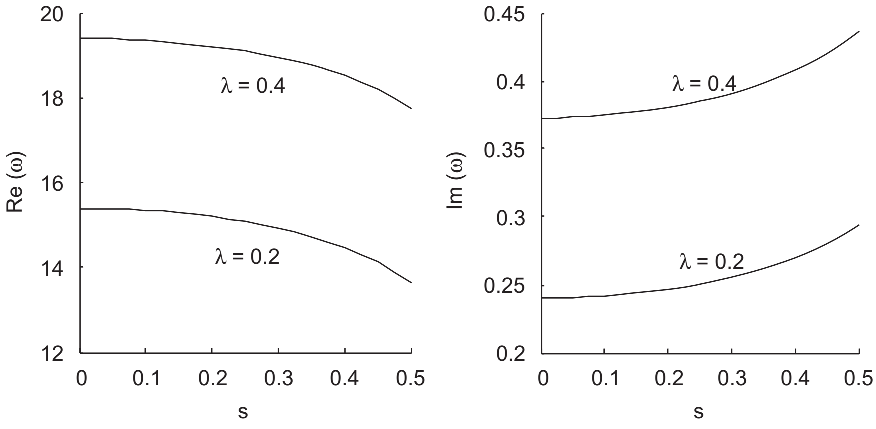

Fig. 5. The first dimensionless complex frequency varied with crack depth ($H = 0.001$, $\lambda = 0.2$, 0.4 $c = 1$).

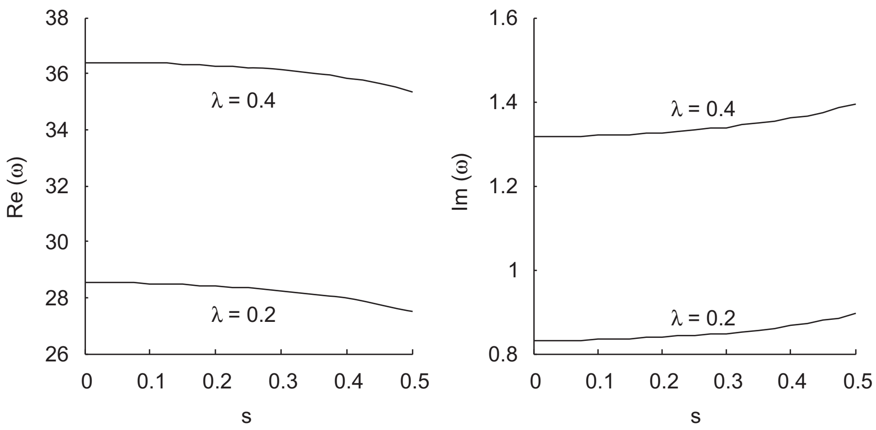

Fig. 6. The second dimensionless complex frequency varied with crack depth ($H = 0.001$, $\lambda = 0.2$, 0.4 $c = 1$).

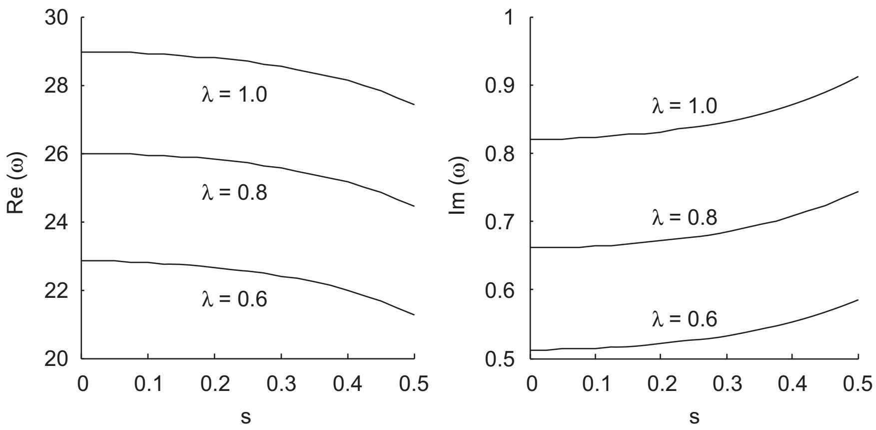

Fig. 7. The first dimensionless complex frequency varied with crack depth ($H = 0.001$, $\lambda = 0.6$, 0.8, 1.0, $c = 1$).

dimensionless complex frequency ω of the linearly varying-thickness plate decreases, while its imaginary part increases. For the case of $\xi = 0.5$, the real part of the dimensionless complex frequency ω remains least, while its imaginary part is max. The same situation is applicable to the second vibration mode.

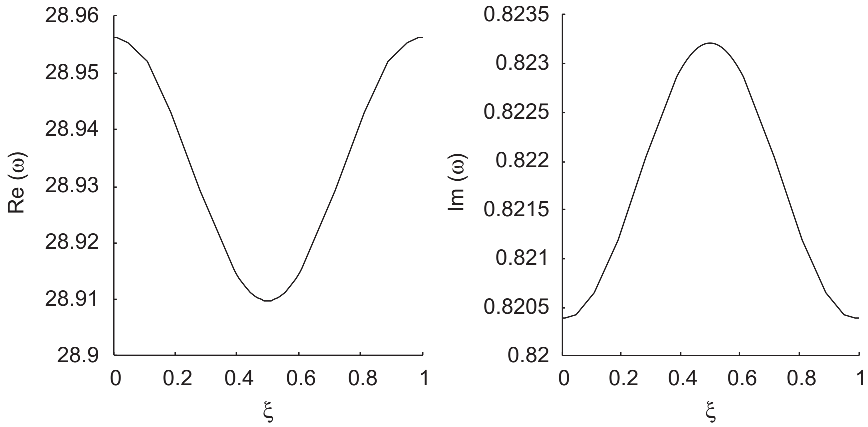

Fig. 8. The first dimensionless complex frequency varied with crack location ($H = 0.001$, $\lambda = 1.0$, $s = 0.1$, $c = 1$).

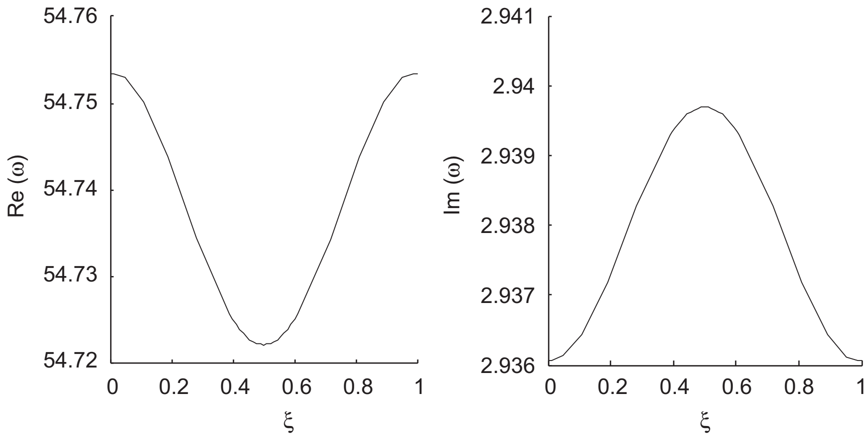

Fig. 9. The second dimensionless complex frequency varied with crack location ($H = 0.001$, $\lambda = 1.0$, $s = 0.1$, $c = 1$).

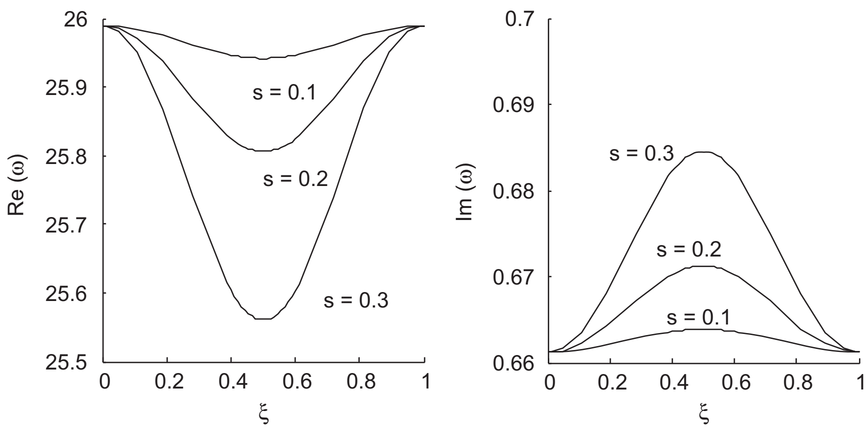

Fig. 10. The first dimensionless complex frequency varied with crack location ($H = 0.001$, $\lambda = 0.8$, $c = 1$).

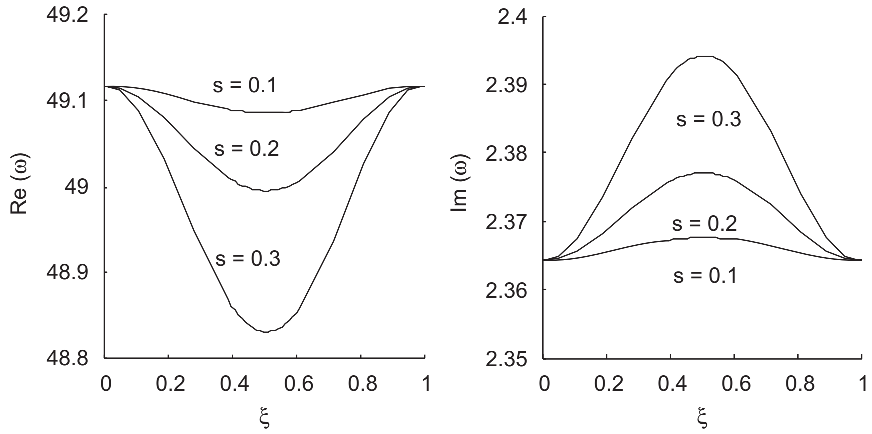

Fig. 11. The second dimensionless complex frequency varied with crack location ($H = 0.001$, $\lambda = 0.8$, $c = 1$).

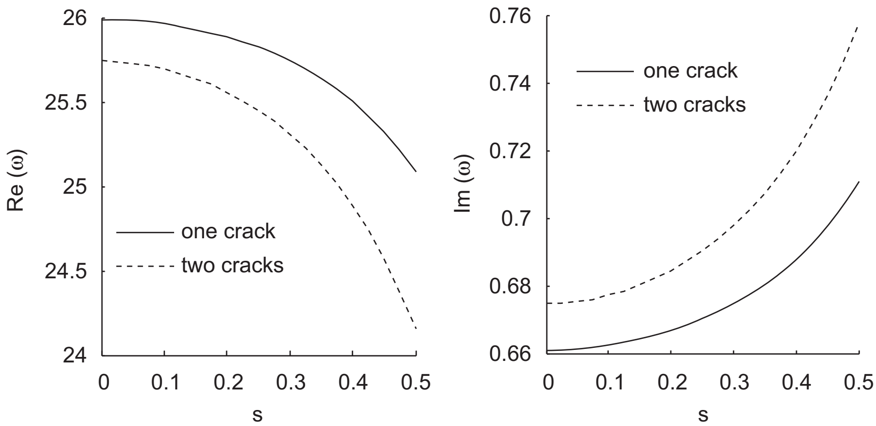

Fig. 12. The first dimensionless complex frequency varied with crack location ($H = 0.001$, $\lambda = 0.8$, $c = 1$).

5.2. A S–C–S–C supported plate with two cracks

Figs. 12 and 13 show the variation of the first two-order dimensionless complex frequencies ω of the plate with crack depth, the geometrical properties of the plate are used, that is $H = 10^{-3}$, $r = 0.15$, $c = 1.0$, $\lambda = 0.8$. The real line expresses the curve of ω of the plate with a crack that is at location $\xi_1 = 0.28$. The broken line is the curve of ω of the plate with two cracks, the first crack is at location $\xi_1 = 0.28$ and has a depth $s_1 = 0.3$, the second crack's location is at $\xi_2 = 0.5$ and its depth varies from $s_2 = 0$ to 0.5. As shown in Figure, the real part of the dimensionless complex frequency ω of the plate with two cracks is lower than one crack, while its imaginary part is larger than one crack. As a result, the introduction of two cracks in a viscoelastic plate decreases the stiffness of the plate; with increase in the crack, the stiffness of the structure diminishes more.

5.3. A S–C–S–C supported plate with three cracks

Figs. 14 and 15 show the variation of the first two-order dimensionless complex frequencies ω of the plate containing three cracks with crack location, for $H = 10^{-3}$, $r = 0.15$, $c = 1.0$, $\lambda = 0.8$ and $s = 0.1$, 0.2, 0.3, respectively. Both the first and the second cracks are at left locations ($\xi_1 = 0.01$, $\xi_2 = 0.049$) and have a depth ($s_1 = 0.1$, $s_2 = 0.3$). The third crack's location varies from $\xi_3 = 0.1$ to 0.99and its depth also varies ($s = 0.1$, 0.2 or 0.3). It can be seen that the effects of the third crack on the dimensionless complex frequencies are quite

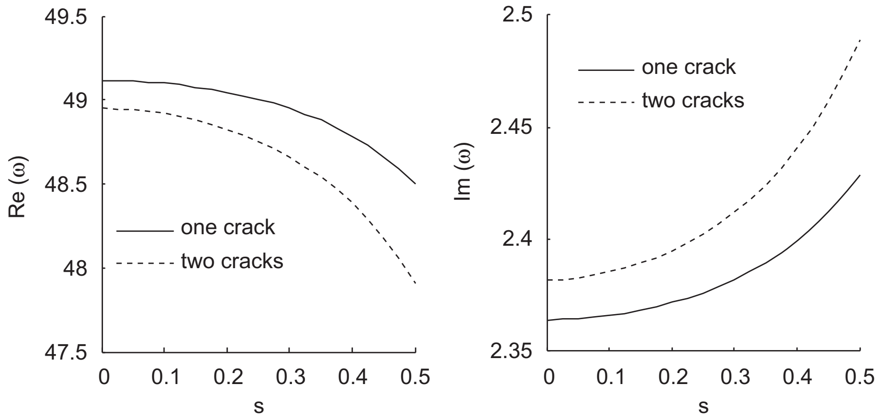

Fig. 13. The second dimensionless complex frequency varied with crack location ($H = 0.001$, $\lambda = 0.8$, $c = 1$).

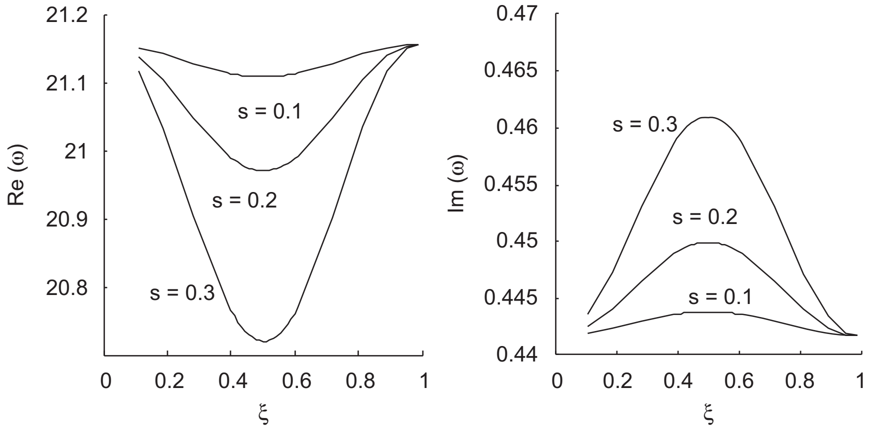

Fig. 14. The first dimensionless complex frequency varied with crack location ($H = 0.001$, $\lambda = 0.5$, $c = 1$).

obvious, it is clear that on $\xi = 0.5$, the dimensionless complex frequency ω has the least value, and its imaginary part has the peak value.

Fig. 16 shows the effect of the third crack on the first-order dimensionless complex frequencies ω, the geometrical properties of the plate are used, that is $H = 10^{-3}$, $r = 0.15$, $s_3 = 0.3$, $c = 1.0$ and $\lambda = 0.5$, 0.8, respectively. As shown in the figure, by increasing the thickness ratio λ, the real part of the dimensionless complex frequencyω increases, and its imaginary parts increase.

5.4. A C–S–S–S supported plate with four cracks

Figs. 17 and 18 show the variation of the first two-order dimensionless complex frequencies ω of the linearly varying thickness viscoelastic plate containing four cracks with crack location, for $H = 10^{-3}$, $r = 0.15$, $s_3 = 0.3$, $c = 1.0$ and $\lambda = 0.5$. The locations and depths of the first, second and third cracks are all predefined ($\xi_1 = 0.01$, $s_1 = 0.3$, $\xi_2 = 0.1$, $s_2 = 0.2$, $\xi_3 = 0.188$, $s_3 = 0.1$). The fourth crack's location varies from $\xi_4 = 0.28$ to 1 and its depth varies, $s_4 = 0.15$ or 0.3. Figs. 19 and 20 show the variation of the first two-order dimensionless complex frequencies ω of the linearly varying-thickness viscoelastic plate containing four cracks, for $H = 10^{-3}$, $r = 0.15$, $\xi_4 = 0.5$, aspect ratio $c = 0.5$ and $\lambda = 0.8$ or 1. Figs. 21 and 22 show the variation of the first-order dimensionless complex frequencies ω of the linearly varying thickness viscoelastic plate containing four cracks, for $H = 10^{-3}$, $r = 0.15$, $\xi_4 = 0.5$, $\lambda = 0.5$ and an aspect ratio $c = 0.5$ or 1.0 or 1.5. As shown in the above figures, by considering the boundary condition for the case of the clamped–simply–simply–simply supported plate, the value of variation ratio of the real part and imaginary

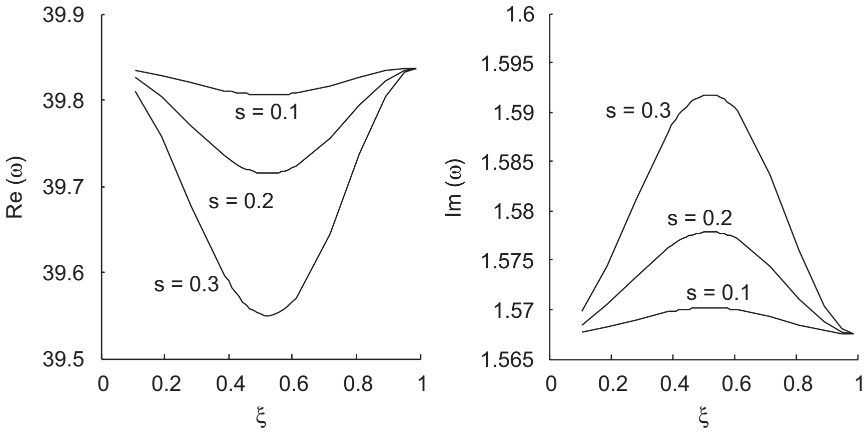

Fig. 15. The second dimensionless complex frequency varied with crack location ($H = 0.001$, $\lambda = 0.5$, $c = 1$).

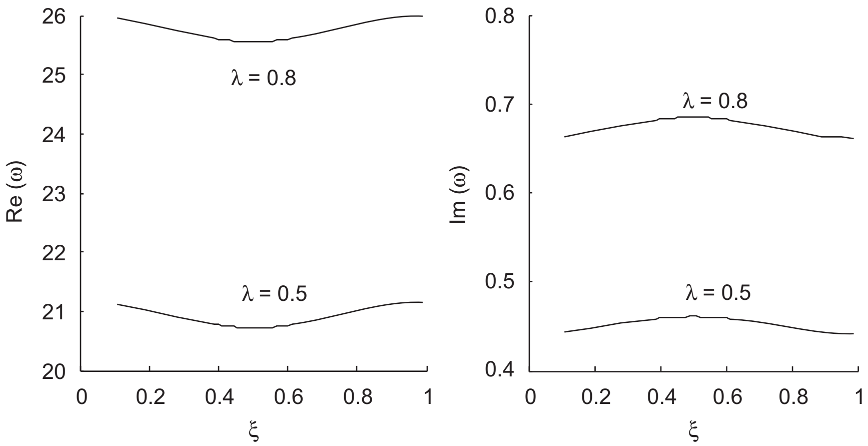

Fig. 16. The first dimensionless complex frequency varied with crack location ($H = 0.001$, $s_3 = 0.3$, $c = 1$).

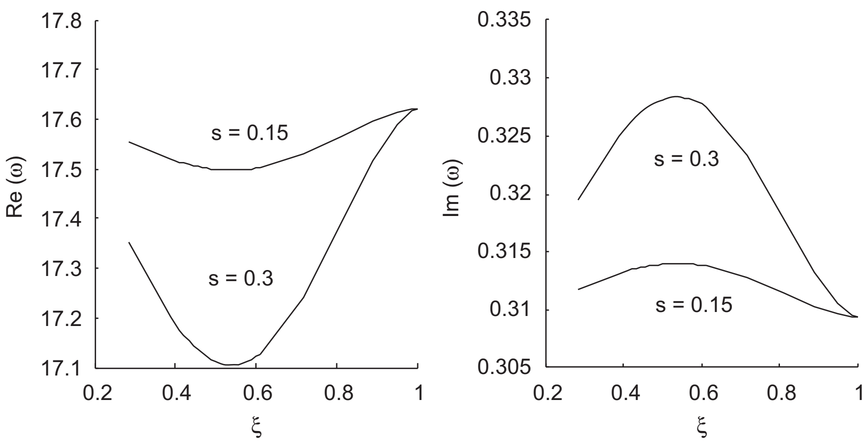

Fig. 17. The first dimensionless complex frequency varied with crack location ($H = 0.001$, $\lambda = 0.5$, $\xi_4 = 0.28$–1, $c = 1$).

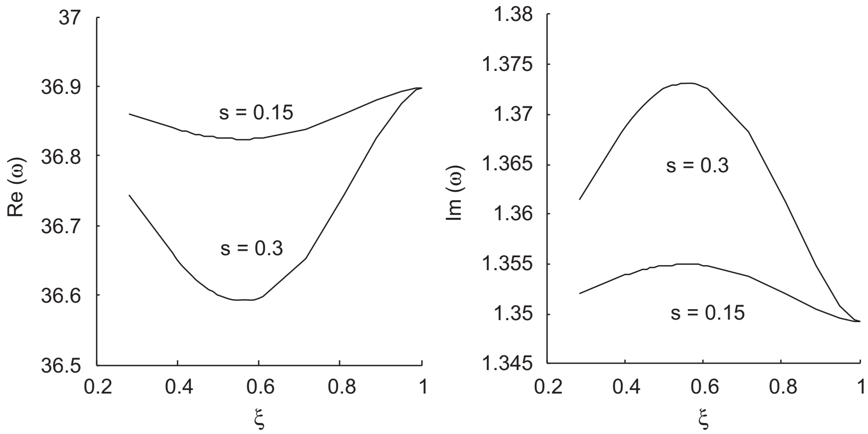

Fig. 18. The second dimensionless complex frequency varied with crack location ($H = 0.001$, $\lambda = 0.5$, $\xi_4 = 0.28$–1, $c = 1$).

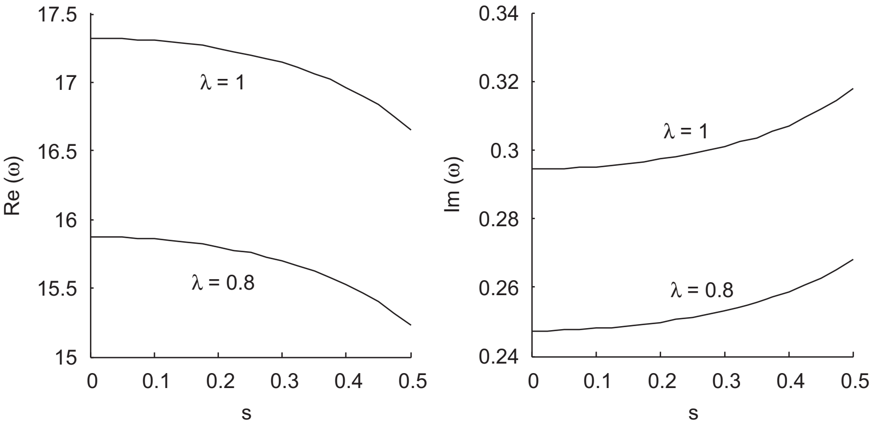

Fig. 19. The first dimensionless complex frequency varied with crack depth ($H = 0.001$, $\xi_4 = 0.5$, $c = 0.5$).

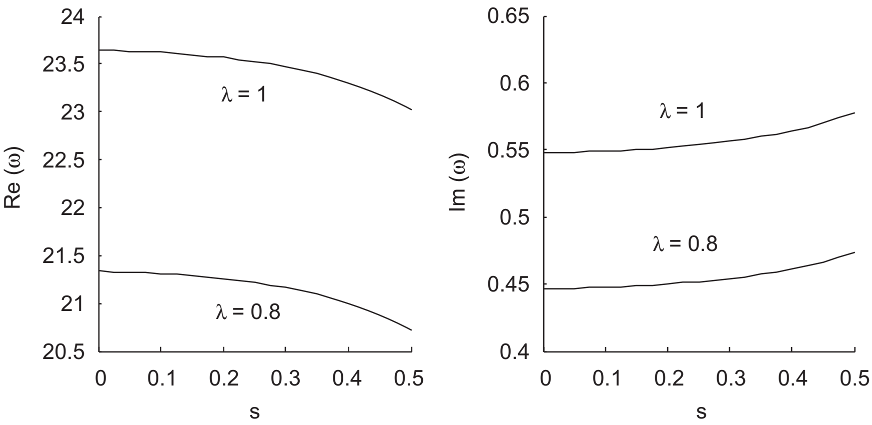

Fig. 20. The second dimensionless complex frequency varied with crack depth ($H = 0.001$, $\xi_4 = 0.5$, $c = 0.5$).

part of the dimensionless complex frequency ω will increase. With the increase in crack number, the corresponding variation ratios of the real parts and imaginary part of the dimensionless complex frequencies increase. However, on increasing the aspect ratio and thickness ratio, the real part and imaginary part of

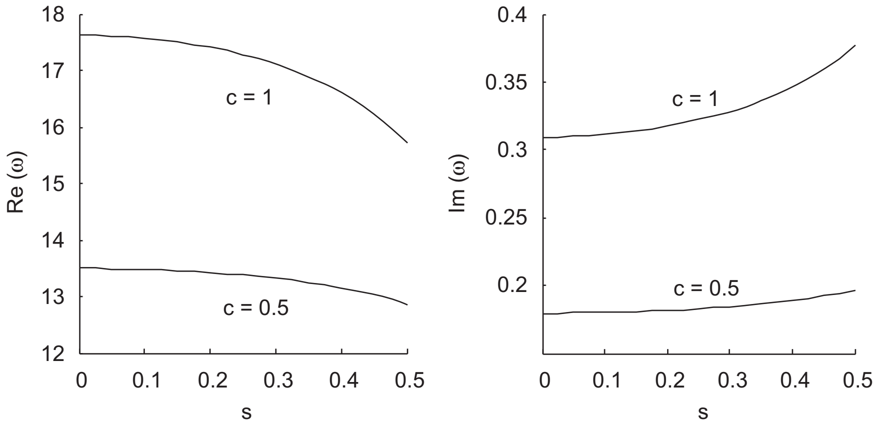

Fig. 21. The first dimensionless complex frequency varied with crack depth ($H = 0.001$, $\xi_4 = 0.5$, $\lambda = 0.5$).

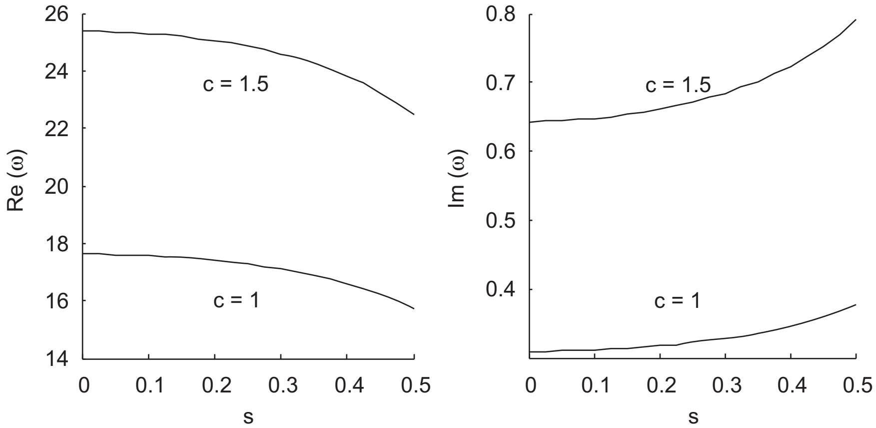

Fig. 22. The first dimensionless complex frequency varied with crack depth ($H = 0.001$, $\xi_4 = 0.5$, $\lambda = 0.5$).

dimensionless complex frequency ω will increase. In addition, by increasing the crack depth, the less the aspect ratio of the plate is, the less the variation ratio of the real part and imaginary part of the dimensionless complex frequencies is.

6. Conclusions

The differential equations of motion of the linearly varying thickness viscoelastic plate containing an arbitrary number of all-over part-through cracks and the expressions of the additional rotation induced by the cracks are established. The complex eigenvalue equations of the linearly varying thickness viscoelastic plate with an arbitrary number of cracks are obtained by the differential quadrature method. The numerical method of calculation and investigating the changes of the dimensionless complex frequencies of a viscoelastic rectangular plate due to the presence of cracks are obtained. By referring to the complex eigenvalue equations, Dimensionless complex frequencies of the system have direct relations with the dimensionless delay time of the material, the aspect ratio, the thickness ratio and the crack parameter. This means that one can obtain a quantitative analysis evidence of the influence of the crack damage on mode parameter.

It can be seen from the results of the numerical calculation, that in the case of the presence of a crack, when $H \to 0$ the real part of the dimensionless complex frequency ω has positive values, its imaginary part is zero. With the increase of dimensionless delay time, its imaginary parts change positive values from zero and increase with mode order. In the case of the presence of a crack, by increasing the depth of the crack and getting closer to the center of the plate from the edge of the plate, the real part of the dimensionless complex frequency ω decreases, and its imaginary parts increase. This means that the presence of the all-over part-through crack would affect the frequencies of natural vibration of the viscoelastic rectangular plate of the plate. Thus, the local flexibility increases. As a result, the corresponding stiffness of the plate decreases and the stiffness of the structure diminishes with increase in the crack. With the increase in crack number, the corresponding variation ratios of the real and imaginary parts of dimensionless complex frequencies increase. By increasing the aspect ratio and thickness ratio, the real and imaginary parts of the dimensionless complex frequency ω will increase. In addition, the less the aspect ratio of the plate, the less the variation ratio of the real part and imaginary part of dimensionless complex frequencies with increase in crack depth.

The present approach can be extended to study the vibration characteristics and dynamic stability of viscoelastic cracked plate with other types of varying-thickness and boundary conditions.

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