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ScienceDirect
Journal of Sound and Vibration 318 (2008) 1139–1153
JOURNAL OF SOUND AND VIBRATION
www.elsevier.com/locate/jsvi

Reduced-order models for nonlinear response prediction: Implicit condensation and expansion

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Received 14 July 2006; received in revised form 21 March 2008; accepted 21 April 2008
Handling Editor: S. Bolton
Available online 13 June 2008

Abstract

Accurate prediction of the response of aircraft skins to acoustic loading is important in the design of future air vehicles. Direct time integration of full-order, nonlinear, finite element models may be accurate, but is computationally expensive. Much work has been reported in recent years on prediction methods, which reduce a finite element model to a reduced-order system of nonlinear modal equations. The major difference among the methods is the means by which induced membrane displacements are modeled. One method is the implicit condensation method where the effects of membrane displacements are condensed into the nonlinear stiffness terms of the bending equations. Membrane displacements are not explicitly modeled, so membrane basis vectors are not needed. However, the lack of membrane displacements prohibits the recovery of membrane stresses and strains from the standard finite element strain–displacement relationships. Here, the implicit condensation method is improved by adding a step to recover membrane displacements using an estimated membrane basis. Examples are given that demonstrate the viability of the proposed method.
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1. Introduction

Future air vehicle designs will require structures that can withstand extreme acoustic and thermal environments. Examples include reusable space access vehicles exposed to launch, hypersonic flight, and re-entry or stealthy aircraft with buried engines and ducted exhaust. Predicting the structural response to acoustic and thermal loads will be critical to achieving reliable vehicles. Accurate prediction of the large amplitude nonlinear response of structures to random acoustic loading can be performed by direct time integration of finite element models. Unfortunately, the computational cost of this approach makes it impractical for use as a design tool. Recent research has focused on the construction of reduced-order models from finite element analysis. The nonlinear, finite element equations in physical degrees-of-freedom are transformed to a reduced-order model in generalized degrees-of-freedom. The transformation matrix or basis set is often constructed from a truncated set of the normal mode shapes.

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0022-460X/$ - see front matter Published by Elsevier Ltd.
doi:10.1016/j.jsv.2008.04.035

Modal models incorporating nonlinear terms were investigated by Nash [1] as a means of dramatically reducing the computational burden of numerical simulation for design and analysis. The approach uses a modal model including a few low frequency modes of the structure which are directly excited by the acoustic environment. Quadratic and cubic nonlinear terms in the modal coordinates are added. The coefficients of the nonlinear terms are only evaluated once rather than at every time step as required by direct integration of the full finite element model. A very large nonlinear finite element model can be reduced to a low-order model with constant coefficients, which can be integrated to produce response time histories in a matter of minutes on a personal computer.

The two primary challenges in developing accurate reduced-order models are the determination of the nonlinear stiffness coefficients and the selection of the basis vectors. The nonlinear stiffness coefficients can be computed using either direct or indirect approaches. In the direct approach, used by Nash [1] and Shi and Mei [2], the full-order nonlinear stiffness matrices are evaluated with combinations of the modal vectors and transformed to modal coordinates. This approach is effective but requires specialized finite element code. The indirect approaches use static nonlinear solutions of a full finite element model to determine stiffness coefficients. The static solutions can be performed in any commercial finite element code with nonlinear capability. The approach of Muravyov and Rizzi [3] computes the nonlinear stiffness coefficients from static solutions to a set of enforced displacements comprised of combinations of the basis vectors. McEwan et al. [4,5] developed a method which estimates the nonlinear coefficients from static solutions to a set of applied modal loads.

The selection of basis vectors is important in generating an accurate nonlinear modal model. The model must include all structural modes which are directly excited by the acoustic energy. These are typically low frequency normal modes with predominantly bending (transverse) deformation. However, for the nonlinear problem, this basis is inadequate. A fundamental characteristic of the large amplitude nonlinear response is the membrane displacement induced by finite bending displacement. Accurate determination of both the membrane and bending deformation is necessary for accurate finite element-based stress/strain recovery. Finite element-based stress/strain recovery refers to the direct substitution of nodal displacements as prescribed displacements into a finite element program to calculate stress and strain using existing post-processing capabilities.

One way to capture the membrane displacements is to explicitly include normal 'membrane' modes in the basis. This option has been termed the *bending and membrane modes method* [6,7]. Membrane modes are normal modes in which the deformation is primarily in the plane of the beam or plate. They tend to be much higher in frequency than bending modes and are not directly excited. Unfortunately, it can be difficult to identify membrane modes in a complex structure. Moreover, it is often not clear which or how many of them to include in the basis.

An alternative approach implicitly incorporates the membrane effects into the bending stiffness terms through an estimation routine. The method of McEwan et al. [4,5] referred to here as the *implicit condensation method*, takes advantage of the inherent benefits of static load solutions to compute nonlinear coefficients. Static load vectors are applied to the nonlinear finite element model. The static load vectors, which are linear combinations of the bending modes, induce both bending and membrane displacements in the solutions—the membrane displacements occur naturally as a result of the geometric nonlinearity. The reduced-order model, however, includes only bending modes. When the nonlinear stiffness terms are estimated from the static solutions, the membrane effects are implicitly condensed into the nonlinear coefficients in the model.

In contrast, the bending and membrane modes method uses enforced displacement solutions. The degrees-of-freedom of the nonlinear finite element model are displaced in linear combinations of the basis vectors and the applied forces are found. The nonlinear coefficients are determined from the applied force vectors. In this approach, for membrane displacements to be represented in the applied force vectors, membrane modes must be included explicitly, a priori, in the basis.

The major advantage of the implicit condensation method is that a membrane basis is not used in the construction of the reduced-order model. Modal bending amplitudes are computed from the time integration of the reduced-order model. Physical displacements spanned by the bending modes can be computed from the modal amplitudes. Physical displacements spanned by the membrane modes cannot be obtained. As a result, accurate strains cannot be recovered from the finite element strain–displacement equations. This shortcoming is overcome by estimating a nonlinear stress function from the nonlinear static solutions [8]. The nonlinear function computes stress directly from the modal bending amplitudes. A nonlinear strain function can also be estimated. The process of stress/strain recovery is a bit cumbersome and finite element-based stress/strain recovery is preferred.

The effects of curvature also need to be considered. Straight beams and flat plates have bending and membrane displacements that are coupled only through nonlinear stiffness terms. Mode shapes involve either pure bending or membrane displacements. The situation is different for curved beams or shells. Curvature causes linear coupling of bending and membrane displacements in addition to the nonlinear coupling. Normal modes are not strictly bending or membrane, but a combination of the two. The lower frequency modes which respond directly to external loading tend to be bending dominated. To avoid confusion, the term *bending modes* will be used to describe these modes. The higher frequency modes that tend to be membrane dominated will be referred to generically as *membrane modes*.

A second effect of curvature is the emergence of nonlinear softening behavior. For straight beams, the predominant effect of large displacements is increasing stiffness (hardening). For curved beams, however, stiffness can either increase or decrease (soften) depending on the specific geometry and loading. The hardening behavior is manifested through cubic nonlinear terms in the modal equations whereas softening occurs through quadratic nonlinear terms.

Elevated temperatures often accompany acoustic loading in the sonic fatigue problem. Elevated temperatures can change material properties, change the stress state, and buckle the structure. It will be assumed that the temperature environment is constant with time (but not necessarily spatially) and the structure is not buckled.

The objective of this paper is to present an extension of the implicit condensation method. The method is improved so that membrane displacements can be obtained, making traditional finite element-based stress/strain recovery possible. A procedure is developed to estimate a membrane basis set. Physical membrane displacements are expanded directly from the modal bending amplitudes. The effects of curvature are handled through the addition of quadratic terms to the model.

There are many approaches to verify the proposed method. The most obvious approach would be to compare predictions from the method to experimental data. Difficulties arise with modeling uncertainties associated with boundary conditions and other physical phenomena. Comparisons to numerical simulations are more tractable. However, numerical simulations require the integration of the full finite element model. If the integration is a simulation of the random, forced response problem, long time records are needed to obtain averaged statistics such as averaged spectra. The computational expense can be reduced by shortening the simulation, for instance, simulating the nonlinear, large amplitude, free vibration response rather than forced response. Frequencies and deflections of the free response could be then compared to those from the reduced-order model to verify the proposed method. Alternatively, the computational expense of a forced vibration simulation can be controlled by selecting an example problem with a small number of degrees-of-freedom. That is the approach used in this paper.

2. Reduced-order models

Finite element discretization of the acoustic response problem results in the nonlinear equations of motion,

$$\mathbf{M}\ddot{\mathbf{w}} + [\mathbf{K} + \mathbf{K1}(\mathbf{w}) + \mathbf{K2}(\mathbf{w},\mathbf{w})]\mathbf{w} = \mathbf{f}(t), \tag{1}$$

where $\mathbf{M}$ is the mass matrix ($m \times m$, where m is the number of degrees-of-freedom), $\mathbf{K}$ is the linear stiffness matrix ($m \times m$), $\mathbf{w}$ is the displacement vector ($m \times 1$), and $\mathbf{f}(t)$ is a vector ($m \times 1$) of external time varying forces. Quadratic and cubic nonlinear stiffness matrices, $\mathbf{K1}$ and $\mathbf{K2}$ (both, $m \times m$), are linear and quadratic functions of the nodal displacements, respectively. All of the quantities are in physical coordinates.

The equations of motion can be transformed into generalized coordinates by selection of any suitable transformation matrix, $\mathbf{T}$,

$$\mathbf{w} = \mathbf{T}\mathbf{p}, \tag{2}$$

where $\mathbf{p}$ is a vector ($n \times 1$) of generalized displacements or amplitudes and the dimensions of $\mathbf{T}$ are $m \times n$. The columns of $\mathbf{T}$ are basis vectors. The number of basis vectors is much less than the number of physical degrees-of-freedom, resulting in a much smaller number of generalized displacements than physical displacements. The reduced equations of motion become

$$\bar{\mathbf{M}}\,\ddot{\mathbf{p}} + \bar{\mathbf{C}}\,\dot{\mathbf{p}} + \bar{\mathbf{K}}\,\mathbf{p} + \boldsymbol{\theta}(p_1, p_2, \ldots, p_n) = \bar{\mathbf{f}}(t), \tag{3}$$

where

$$\bar{\mathbf{M}} = \mathbf{T}^{\mathrm{T}}\mathbf{M}\,\mathbf{T}, \quad \bar{\mathbf{K}} = \mathbf{T}^{\mathrm{T}}\mathbf{K}\,\mathbf{T} \quad \text{and} \quad \bar{\mathbf{f}}(t) = \mathbf{T}^{\mathrm{T}}\mathbf{f}(t). \tag{4}$$

A linear damping matrix, $\bar{\mathbf{C}}$, has been introduced into the equations at this point. The dimensions of the reduced matrices, $\bar{\mathbf{M}}, \bar{\mathbf{C}}$, and $\bar{\mathbf{K}}$ are $n \times n$. The reduced force vector, $\bar{\mathbf{f}}(t)$, is time varying with dimensions of $n \times 1$.

The nonlinearity appears as an internal force vector function, $\boldsymbol{\theta}$, with dimensions of $n \times 1$. The nonlinearity is expressed as a function of the n generalized displacements (scalar functions of time). The function for "rth" equation can be written as

$$\theta_r = \sum_{i=1}^{n}\sum_{j=i}^{n} B_r(i,j)p_i p_j + \sum_{i=1}^{n}\sum_{j=i}^{n}\sum_{k=j}^{n} A_r(i,j,k)p_i p_j p_k, \tag{5}$$

where $B_r(i,j)$ and $A_r(i,j,k)$ are scalars that express quadratic and cubic modal stiffness, respectively. The quadratic nonlinear terms are important only if the structure is curved or if membrane modes are included explicitly in the model.

The reduced equations are in generalized coordinates. If normal mode shapes are used as the basis set, the reduced equations become modal equations, and the reduced mass and stiffness matrices become diagonal. The reduced equations will be coupled through the nonlinear stiffness function. If the basis vectors are not normal mode shapes, the reduced equations are also coupled through off-diagonal terms in the reduced mass and stiffness matrices.

2.1. *The implicit condensation method*

One of the methods for determining the nonlinear coefficients is the implicit condensation method. This method is intended for general structures where partitioning of the mode shapes into bending and membrane degrees-of-freedom may not be feasible. The reduced equations are obtained in this method by using a basis set that includes only a truncated set of the bending mode shapes,

$$\mathbf{T} = \boldsymbol{\Phi}_b = [\phi_1 \phi_2 \ldots \phi_n], \tag{6}$$

where the individual bending mode shapes, $\boldsymbol{\phi}_i$, are vectors ($m \times 1$). The dimensions of $\boldsymbol{\Phi}_b$ are $m \times n$. The mode shapes can be obtained from any commercial finite element package. Note that the mode shapes will contain an entry for every degree-of-freedom. For flat structures, the degree-of-freedom entries that correspond to membrane displacements will be zero. For curved structures modeled in the Cartesian coordinate system, the bending modes shapes will be fully populated. The implicit condensation method uses the complete mode shapes, because for general structures, it is impossible to partition the mode shape into bending and membrane degrees-of-freedom.

The original implementation of the implicit condensation method [4] neglected the quadratic terms and any cubic term involving three modes. The nonlinear function with these restrictions can be written as

$$\theta_r = \sum_{i=1}^{n} A_r(i,i,i)p_i^3 + \sum_{i=1}^{n-1}\sum_{j=i+1}^{n} \{A_r(i,i,j)p_i^2 p_j + A_r(i,j,j)p_i p_j^2\}. \tag{7}$$

These restrictions are suitable only for a flat structure. Quadratic coupling among modal displacements occurs in a curved structure, so no restrictions on Eq. (5) will be made for these cases. Given the form of the nonlinear function, the nonlinear coefficients can be estimated from nonlinear static load solutions using a least squares procedure [4,8].

The original procedure suggested by McEwan [8] sought to achieve the smallest possible nonlinear function by eliminating insignificant terms in Eq. (7) and re-estimating the nonlinear function. Various techniques and criteria were used to retain or eliminate terms. Depending on the criteria, important terms could be erroneously eliminated [9]. Here, all the terms either in Eq. (5) or (7) will be retained. The computational penalty for the inclusion of these terms occurs in the numerical integration of the reduced-order model. The penalty is insignificant in the context of the entire problem.

The model can only predict those displacements spanned the basis set. The predicted displacement is an approximation

$$\mathbf{w} \approx \mathbf{w}_b = \mathbf{\Phi}_b \mathbf{p}, \tag{8}$$

where $\mathbf{p}$ is the vector $(n \times 1)$ of modal bending amplitudes and $\mathbf{w}_b$ is the vector $(m \times 1)$ of the physical displacements spanned by the bending mode shapes.

2.2. The estimated membrane basis

The implicit condensation method does not have a means of predicting membrane stretching. If the membrane displacements could be predicted, finite element-based stress/strain recovery would be possible. An expansion process is sought to obtain the membrane displacements. The process requires a membrane basis. In this section, a procedure is developed to estimate a membrane basis from the static nonlinear solutions.

The approximation in Eq. (8) can be improved by adding a vector to represent membrane stretching not spanned by the bending modes

$$\mathbf{w} \approx \mathbf{w}_b + \mathbf{w}_m, \tag{9}$$

where $\mathbf{w}_m$ is a full length vector $(m \times 1)$ in physical coordinates. Notice that the displacement vectors in Eq. (9) are not partitioned. Rather, the displacement vector is separated into two additive vectors. Each physical degree-of-freedom has an entry in both vectors. Any structure can be modeled. In the case of a planar structure, many of the entries will be zero in either $\mathbf{w}_b$or $\mathbf{w}_m$. In the case of a curved structure modeled in a Cartesian coordinate system, the vectors will be fully populated.

The added displacement vector can be decomposed into a basis set and generalized amplitudes

$$\mathbf{w}_m = \mathbf{T}_m \mathbf{q}, \tag{10}$$

where $\mathbf{q}$ is the vector of generalized membrane amplitudes, and $\mathbf{T}_m$ is a matrix whose columns are the membrane basis vectors. The membrane basis vectors are orthogonal to the columns of $\mathbf{\Phi}_b$. The basis vectors are not necessarily normal membrane mode shapes. The number of generalized membrane amplitudes has not been determined yet. For convenience, let that number be k. Therefore, the vector dimension of $\mathbf{q}$ is $k \times 1$ and the dimensions of $\mathbf{T}_m$ are $m \times k$.

The expansion process by which $\mathbf{T}_m$ and $\mathbf{q}$ are derived is summarized here. Recall that a set of static solutions are used in the implicit condensation method to estimate the nonlinear parameters. Those static solutions are used here to estimate the membrane basis set. A single static solution is represented by

$$\mathbf{w} \approx \mathbf{w}_b + \mathbf{w}_m = \mathbf{\Phi}_b \mathbf{p} + \mathbf{T}_m \mathbf{q}. \tag{11}$$

Considering many static solutions, the equation becomes

$$\mathbf{W} \approx \mathbf{\Phi}_b \mathbf{P} + \mathbf{T}_m \mathbf{Q}, \tag{12}$$

where the columns of the matrices, $\mathbf{W}$, $\mathbf{P}$, and $\mathbf{Q}$, correspond to the individual static solution cases. If the number of static solutions is represented by l, the dimensions of $\mathbf{W}$, $\mathbf{P}$, and $\mathbf{Q}$, are $m \times l$, $n \times l$, and $k \times l$, respectively.

Noting that columns of $\mathbf{T}_m$ are orthogonal to the columns of $\mathbf{\Phi}_b$, the modal bending amplitudes are found by

$$\mathbf{P} = (\mathbf{\Phi}_b^{+})\mathbf{W}, \tag{13}$$

where $\mathbf{\Phi}_b{}^+$ is the pseudo-inverse of $\mathbf{\Phi}_b$. Eq. (12) can then be rearranged as

$$\mathbf{T}_m\mathbf{Q} \approx \mathbf{W} - \mathbf{\Phi}_b\mathbf{P}. \tag{14}$$

If the matrix of generalized membrane amplitudes, $\mathbf{Q}$, were known, a membrane basis set could be estimated from

$$\mathbf{T}_m \approx [\mathbf{W} - \mathbf{\Phi}_b\mathbf{P}]\mathbf{Q}^+. \tag{15}$$

However, the generalized membrane amplitudes are not defined at this point.

Since membrane effects are condensed into the bending equations in the implicit condensation method, it is assumed that the generalized membrane amplitudes are directly related to the modal bending amplitudes. The form of the relationship must be defined to compute the estimated membrane basis. In the case of the bending and membrane modes method, an explicit process is available to condense membrane equations into bending equations [6]. In the condensation process, the modal membrane amplitudes are approximated by linear combinations of quadratic terms involving the modal bending amplitudes. The combinations are defined by the nonlinear quadratic coefficients and the linear stiffness coefficients of the membrane modal equations.

Here, a quadratic relationship between modal bending amplitudes and the generalized membrane amplitudes is assumed in order to estimate a membrane basis. The generalized membrane amplitudes are thereby defined as

$$\mathbf{q} = [p_1^2 \quad p_1p_2 \quad p_1p_3 \quad \cdots \quad p_1p_n \quad p_2^2 \quad p_2p_3 \quad \cdots \quad p_2p_n \quad \cdots \quad p_{n-1}^2 \quad p_{n-1}p_n \quad p_n^2]^{\mathrm{T}}, \tag{16}$$

where the terms p_1 through p_n refer to the individual modal bending amplitudes. This set of generalized membrane amplitudes spans all the possible quadratic combinations of the modal bending amplitudes. The number of quadratic combinations is equal to $(n/2)(n+1)$, which defines the number of generalized membrane amplitudes, k. Given the modal bending amplitudes from the static load cases, a set of generalized membrane amplitudes can be generated from Eq. (16) and the membrane basis set can then be estimated from Eq. (15).

2.3. *Description of the proposed method*

The availability of an estimated membrane basis allows for an extension of the implicit condensation method. The implicit condensation method produces a condensed model without a membrane basis set. The condensed model is attractive because it is the lowest order model possible. Integration of this model produces time histories of modal bending amplitudes. The proposed extension of the implicit condensation method uses the estimated membrane basis in the post-processing of the integration results to synthesize physical membrane displacements. The steps in the extended method are listed in Table 1. The extended method will be called the *implicit condensation and expansion method*. Steps 1–5 are directly from the implicit condensation method. Steps 6–8 are the expansion procedure. The product of the expansion process is a set of physical

Table 1
Steps in the implicit condensation and expansion method

1.	Obtain the bending modes from a normal modes analysis. Select the appropriate bending modes for the basis.
2.	Determine the reduced stiffness and mass matrices, Eq. (4), for the selected modes.
3.	Scale linear combinations of the selected modes to construct a set of applied forces. Obtain a set of nonlinear static solutions to the applied loads.
4.	Estimate the nonlinear coefficients for the reduced-order model using the static solutions.
5.	Apply the external, time-varying, force to the model and integrate the model to determine the structural response. The response will be in terms of modal bending amplitudes as a function of time.
6.	Use the nonlinear static cases in Step 3 to estimate a membrane basis set. This is done by first obtaining the modal membrane amplitudes for each static solution from the static modal bending amplitudes by using Eq. (16). The modal membrane amplitudes for each individual static solution define a column in **Q**. Eq. (15) is then used to estimate the basis.
7.	Determine the generalized membrane amplitudes as a function of time using Eq. (16) from the model response obtained in Step 5.
8.	Determine the physical displacements from Eq. (11).

displacements as a function of time that can be used with a commercial FEA package to recover stress and strains.

3. Example problems

The sonic fatigue prediction problem requires methods that will work for complex aircraft structures. The implicit condensation and expansion method is intended for general structures and not necessarily restricted to flat plates and panels. Simple structures are used in the two examples so that response data can be obtained directly from a full-order finite element model in a reasonable time. The results from the finite element model will be compared to results from the reduced-order model. The first example problem is a flat beam. The second example is a beam with a shallow curvature.

3.1. The flat beam example

The flat beam chosen for the example is from Rizzi and Przekop [7]. The beam's dimensions are 18 in $\times$ 1 in $\times$ 0.09 in. The beam is clamped at both ends and is made from an aluminum alloy. The material properties for the beam are: $E = 10.6 \times 10^6$ psi, $G = 4.0 \times 10^6$ psi, and $\rho = 2.588 \times 10^{-4}$ lb s^2 in^{-4}, where E is the modulus of elasticity, G is the shear modulus, and ρ is the material density. The forcing function is a spatially uniform load whose magnitude is random. The force magnitude is a band-limited (0–1500 Hz), Gaussian random variable. Two loading cases are considered. The first is a low level, nominally linear case (0.0072 lb in^{-1} rms), and the second is a high level case (0.9216 lb in^{-1} rms) which results in considerable nonlinear behavior.

A symmetric half-model of the beam was constructed using 36 beam elements of equal length. The implicit condensation and expansion method was used to obtain a reduced-order model. Predictions from the integrated model are compared to direct integration results from a full finite element model. The various static nonlinear solutions and normal modal analyses required to construct the reduced-order model were performed in NASTRAN. The full-order model was run in an internal code written in Matlab. Mass proportional damping was used in all the simulations. The damping level was chosen so that damping for the first bending mode corresponded to 2.0% of critical.

The basis set for the implicit condensation and expansion method included the mode shapes for the first four symmetric bending modes. The linear natural frequencies for these modes are listed in Table 2. The standard implicit condensation method was used to determine a condensed bending modes only model. The nonlinear stiffness coefficients were estimated for this model from a set of 32 nonlinear, static, applied force solutions. Since the beam is flat, the reduced model was restricted to the form given by Eq. (7). The reduced-order model had four equations with 16 cubic coefficients in each equation. The external input force was applied to the reduced-order model and the model was integrated to obtain the time history of modal bending amplitudes. The reduced-order predictions were obtained using a Newmark-Beta integration scheme. The time step for the integration was 1.25×10^{-5} s. Time records of 100 s were obtained.

There are ten quadratic combinations defined by Eq. (16) given four bending modes. Therefore, there were ten membrane basis vectors estimated for the implicit condensation and expansion method from the static

Table 2
Frequencies of the modes retained in the reduced-order models

Mode no.	Flat beam (Hz)	Curved beam (Hz)
1	57.8	158.3
2	312.1	258.0
3	770.3	399.6
4	1431.2	513.5
5		774.3
6		1071.4
7		1428.1
8		1831.5

applied force solutions. The results of the time integration were expanded to obtain the time history of the generalized membrane amplitudes and the full physical displacement time record.

A Newmark-Beta approach was also used to integrate the full-order model. The time step for the integration was 1×10^{-5} s. Time records of 60 s were obtained. The time records are processed to obtain the power spectral density of the displacement or stress at the physical locations of interest.

Predictions from the implicit condensation and expansion method are shown in comparison to the full model predictions in Figs. 1–4. Predictions for both excitations levels are shown in each figure. Membrane (in-plane) displacement at a point 4 in from the clamped end is shown in Fig. 1. Transverse displacement at the same location is shown in Fig. 2. Membrane stress at the center of an element 3.625 in from the clamped end is shown in Fig. 3. Total stress at the same location is shown in Fig. 4. These locations were chosen since they are close to those reported by Rizzi and Przekop [7].

The membrane and transverse displacement predictions for the low level excitation shown in Figs. 1 and 2 agree very well, virtually lying on top of one another. The stress and displacement predictions for the high level excitation shown in Figs. 1–4 also agree very well with only slight variations due to the random nature of the response. A small discrepancy can be found in the membrane stress predictions at the low excitation level. These predictions are plotted in Fig. 3. The implicit condensation and expansion method membrane stress prediction is somewhat lower in amplitude than those from the full finite element model at the high frequency end of the spectrum. The reason for this minor difference is presumed to be due to numerical round-off errors. The discrepancy is not seen in the total stress prediction because the bending stress dominates the total stress prediction at this excitation level. At the high excitation levels, the membrane stress prediction from the reduced-order model agrees with the full-order prediction.

It is beneficial to use the example problem to illustrate the improvement of the proposed method over the implicit condensation method. Recall that there are no membrane modes in the basis set of the implicit condensation method and therefore, membrane displacements of a flat beam or panel cannot be predicted. The improvement, termed the implicit condensation and expansion method, provides the ability to predict the *membrane* displacement as shown in Fig. 1. The *transverse* displacements of the flat beam, shown in Fig. 2, are identical whether the implicit condensation or the improved method is used.

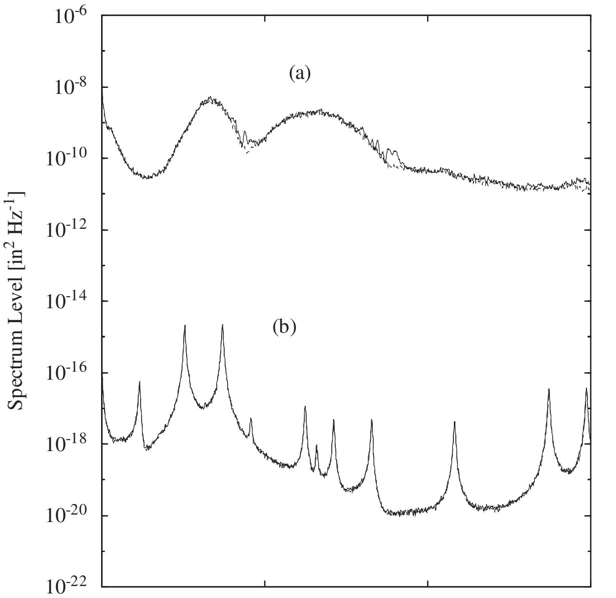

Fig. 1. The in-plane displacement for the flat beam at a location 4 in from the clamped end from a full finite element model; —, and the implicit condensation and expansion method; - - -, for two random loading conditions: (a) 0.9216 lb in^{-1} rms, and (b) 0.0072 lb in^{-1} rms.

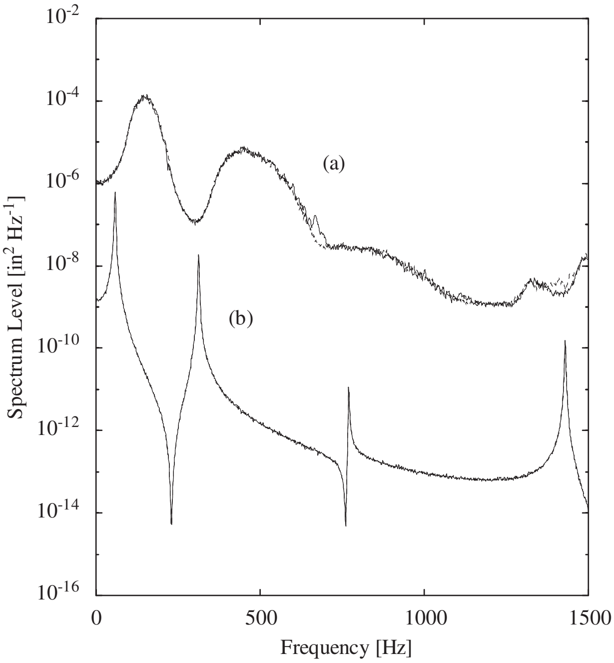

Fig. 2. The transverse displacement for the flat beam at a location 4 in from the clamped end from a full finite element model; —, and the implicit condensation and expansion method; - - -, for two random loading conditions: (a) 0.9216 lb in^{-1} rms, and (b) 0.0072 lb in^{-1} rms.

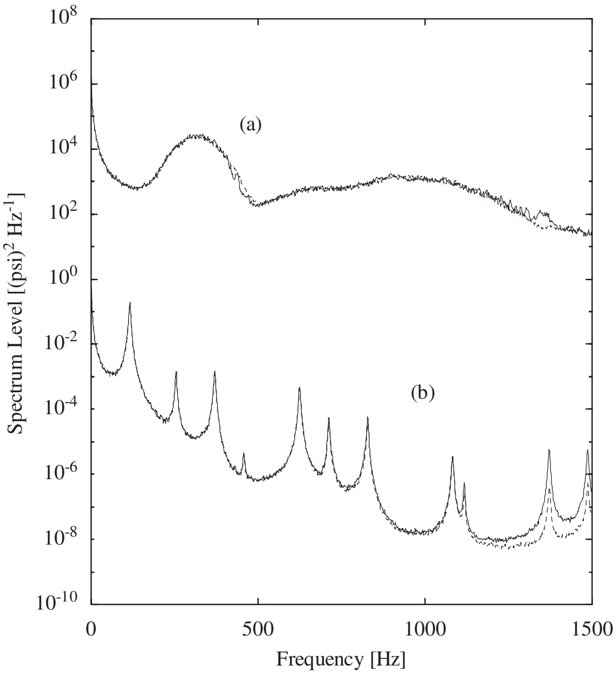

Fig. 3. The membrane stress component for the flat beam at a location 3.625 in. from the clamped end from a full finite element model; —, and the implicit condensation and expansion method; - - -, for two random loading conditions: (a) 0.9216 lb in^{-1} rms, and (b) 0.0072 lb in^{-1} rms.

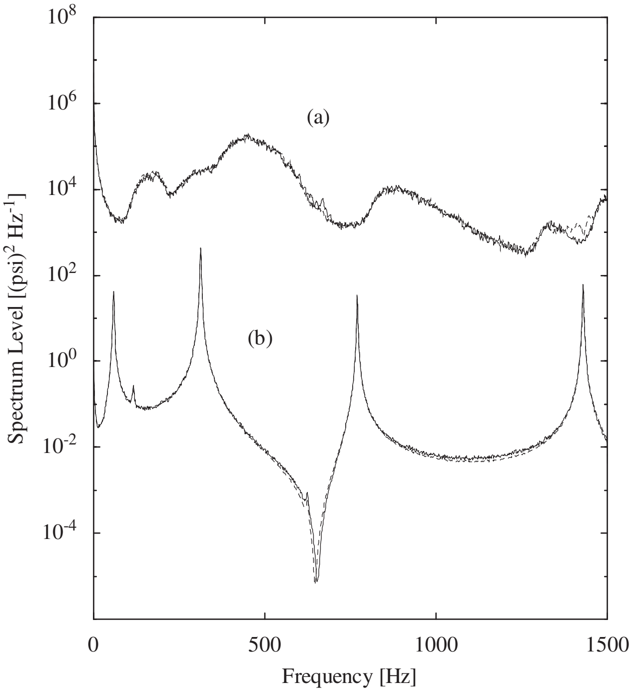

Fig. 4. The total stress at the top of the flat beam at a location 3.625 in from the clamped end from a full finite element model; —, and the implicit condensation and expansion method; - - -, for two random loading conditions: (a) 0.9216 lb in^{-1} rms, and (b) 0.0072 lb in^{-1} rms.

The major benefit of the improved method is that the full displacement field is predicted, therefore, the stress and strain fields can be recovered using standard finite element strain–displacement relationships. In contrast, the implicit condensation method cannot use the standard recovery procedure because there are no membrane displacements. This method must use an estimated mapping function [4,5,8]. Stress and strain predictions using this mapping function can be accurate but they are not standard finite element procedures.

The major benefit of reduced-order methods is computational savings. The savings depend on the number of degrees-of-freedom in the full-order model. This particular example is a small problem. The custom coded full-order integration required approximately 45 h on a desktop personal computer (2.8 GHz processor). Integration of the same model using ABAQUS with the same time step and record length requires an estimated 186 h on the same computer. Integration of the reduced-order model required only 15 min. The computational savings become more dramatic as the size of the model increases. The computation expense of full-order integration increases geometrically with the number of degrees-of-freedom. The expense of the reduced-order model integration does not change (assuming that the number of modes in the model does not change). The implicit condensation and expansion method uses the time integration of the model provided by the implicit condensation method. The improved method requires an expansion of the integration results to determine the membrane displacements. The computation of this expansion, in this example, required only 3 s.

3.2. *The curved beam example*

The curved beam configuration investigated in this study is taken from Prezekop and Rizzi [10] and is shown in Fig. 5. The beam is similar to the one used in the first example with curvature added. The aluminum alloy beam had dimensions of 18.0 in long (projected length) $\times$ 1.0 in $\times$ 0.09 in thick with an 81.25-in radius of curvature. Both ends of the beam were clamped. The beam material properties were the same as those used for the flat beam.

A finite element model of the beam was constructed in ABAQUS using 80 B31 beam elements. This model was used to compute mode shapes and frequencies and perform the nonlinear applied loads solutions.

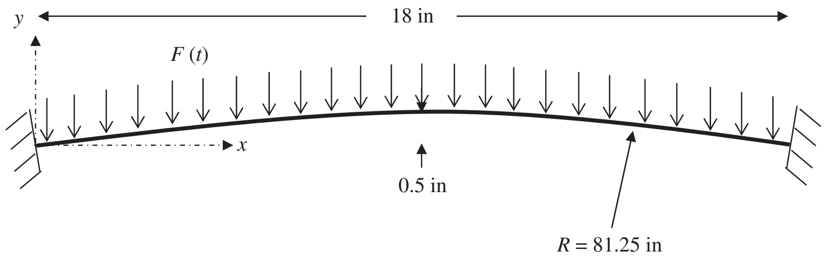

Fig. 5. The geometry of the curved beam.

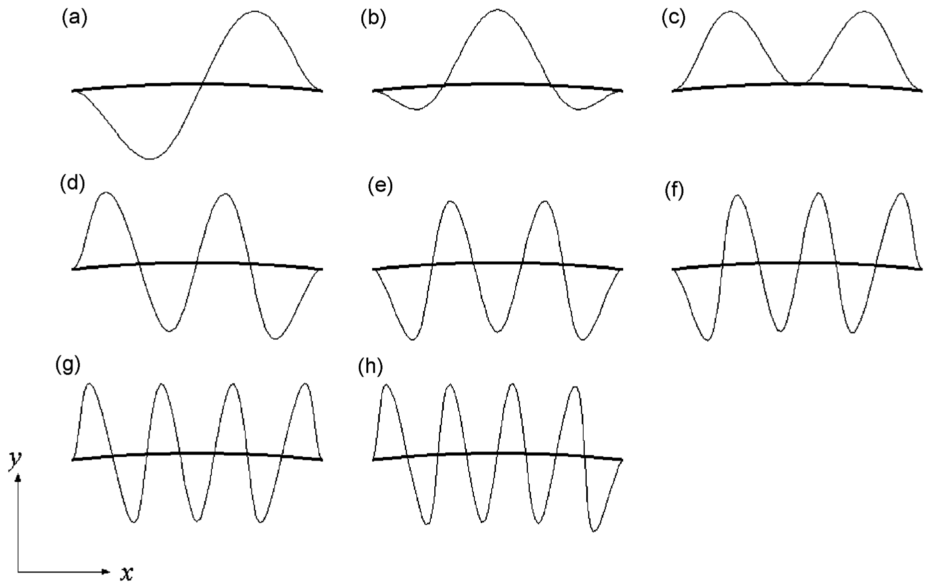

Fig. 6. The first eight mode shapes of the curved beam from the finite element model. Mode shapes 1–8 are shown in (a)–(h) respectively.

The mode shapes for the first eight modes of the beam are shown in Fig. 6. The linear natural frequencies for these modes are listed in Table 2.

Direct time integration of a full-order finite element model was performed using an in-house code written in Matlab. This model also used 80 two-noded beam elements. A mass proportional damping matrix was used to provide 2.0% critical damping at 258 Hz—the frequency of the first symmetric mode of the beam. A band-limited random excitation was applied as a uniform load on the model with a 0–1500 Hz frequency range. Two load levels were applied—0.2304 and 0.6517 lb in^{-1}—giving minimal and significant nonlinear response, respectively. Simulations were performed for 50 s at each load level with a time step of 1×10^{-5} s.

A reduced-order nonlinear model of the beam was computed using the implicit condensation and expansion method from a set of nonlinear static solutions. The form of the nonlinear function is defined by Eq. (5), which includes quadratic terms necessary to model curved structures. The lowest 8 normal modes of the beam were retained in the model. All of these modes had predominantly transverse displacements. Notice that the reduced-order model includes anti-symmetric modes (modes 1, 4, 6, and 8). These modes cannot be excited directly by a symmetric load, but can be autoparametrically excited by the symmetric modes [10]. These types of modes were unnecessary for the flat beam example. The curved beam however has coupling between the symmetric and anti-symmetric modes that is manifested in the quadratic terms in the model. The first seven modes were within the excitation bandwidth. Although the eighth mode was above the excitation bandwidth, it was included to be consistent with Ref. [10]. No membrane dominated modes were retained in the model.

The reduced-order model was subjected to the distributed random loading used with the full model simulation. Fifty-second time records were computed using Newmark time integration with a time step of 2×10^{-5} s.

A set of 36 estimated membrane basis vectors was computed from the static nonlinear solutions using the implicit condensation and expansion method. The modal bending amplitudes were then expanded to produce the generalized membrane amplitudes. The modal bending amplitudes and the generalized membrane amplitudes were transformed to physical coordinates.

Power spectral densities of displacement from the implicit condensation and expansion method are compared to full-order, finite element model simulation results in this section. The comparisons are shown in Figs. 7–10. Results are presented for four degrees-of-freedom: x- and y-direction displacements at the center of the beam and x- and y-direction displacements at the quarter point. Note that at the quarter point these displacements are not identical to the in-plane and transverse components, respectively.

No significant nonlinear effects are shown in the y-direction displacement at the center point (Fig. 7). The implicit condensation and expansion method results are virtually identical to the full-order results. Nonlinear effects are clearly shown in the x-direction displacements at the center point (Fig. 8). For a flat beam, there would be no x-direction displacements at the center point. However, the curved beam does have displacements and a large increase in the displacement occurs for a modest increase in the excitation. The implicit condensation and expansion method results capture these nonlinear effects with nearly identical agreement with the full-order model. The autoparametric excitation of the first mode is demonstrated in the displacements at the quarter point (Figs. 9 and 10). At the low excitation level, the first anti-symmetric mode is not excited, but as the excitation is increased, the mode is quadratically excited by the higher frequency symmetric modes. Again, the implicit condensation and expansion method results capture the autoparametric response and provide nearly identical agreement with the full-order model.

In the flat beam example, the benefits of the implicit condensation and expansion method over the implicit condensation method are obvious, but in the curved beam example, those benefits are more subtle. The curved beam is modeled in Cartesian coordinates. The bending mode shapes have both x and y-direction components.

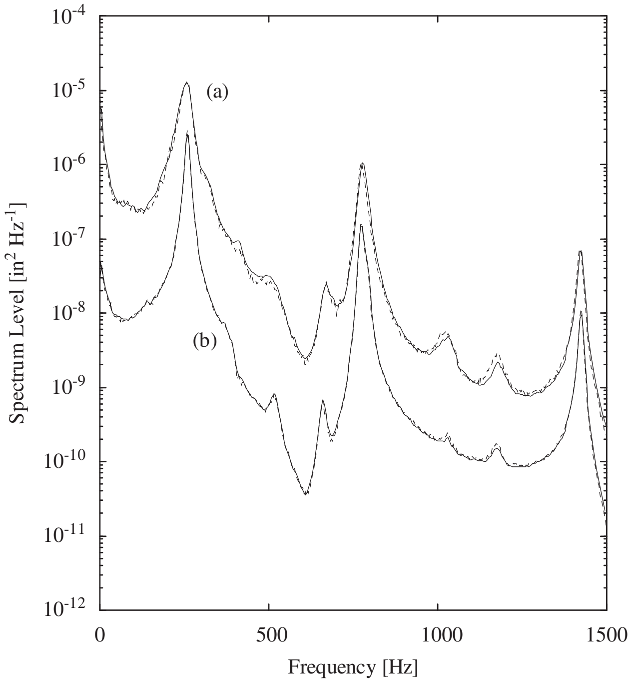

Fig. 7. The y-direction displacement for the curved beam at the beam center from a full finite element model; —, and the implicit condensation and expansion method; - - -, for two random loading conditions: (a) 0.6517 lb in^{-1} rms, and (b) 0.2304 lb in^{-1} rms.

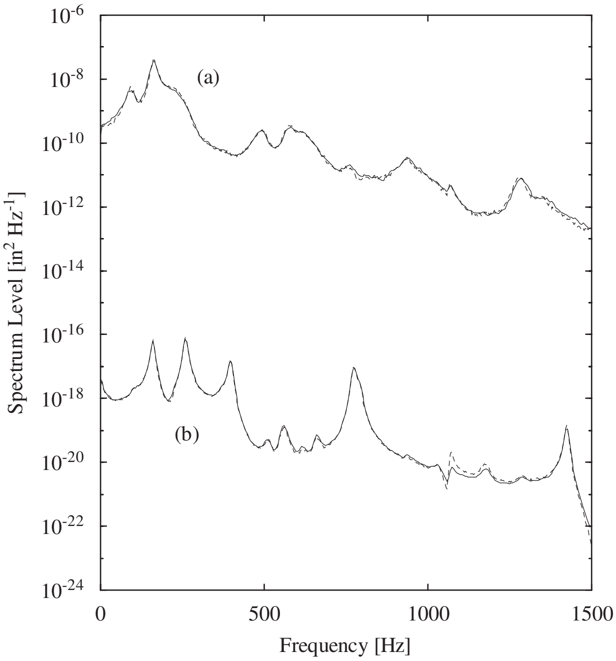

Fig. 8. The x-direction displacement for the curved beam at the beam center from a full finite element model; —, and the implicit condensation and expansion method; - - -, for two random loading conditions: (a) 0.6517 lb in^{-1} rms, and (b) 0.2304 lb in^{-1} rms.

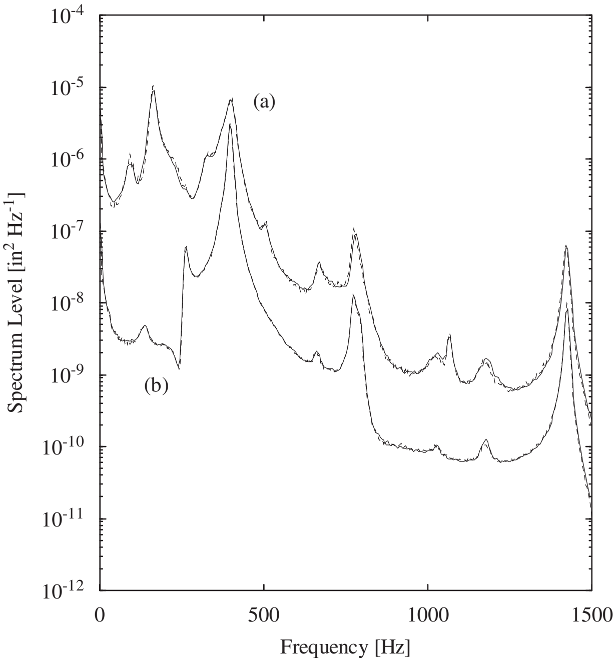

Fig. 9. The y-direction displacement for the curved beam at the quarter point from a full finite element model; —, and the implicit condensation and expansion method; - - -, for two random loading conditions: (a) 0.6517 lb in^{-1} rms, and (b) 0.2304 lb in^{-1} rms.

The implicit condensation method, which relies on those mode shapes as its basis, predicts some displacements in the x-direction. The so-called membrane basis provided by the implicit condensation and expansion method improves those predictions. The degree of improvement depends on location and the degree of nonlinearity in

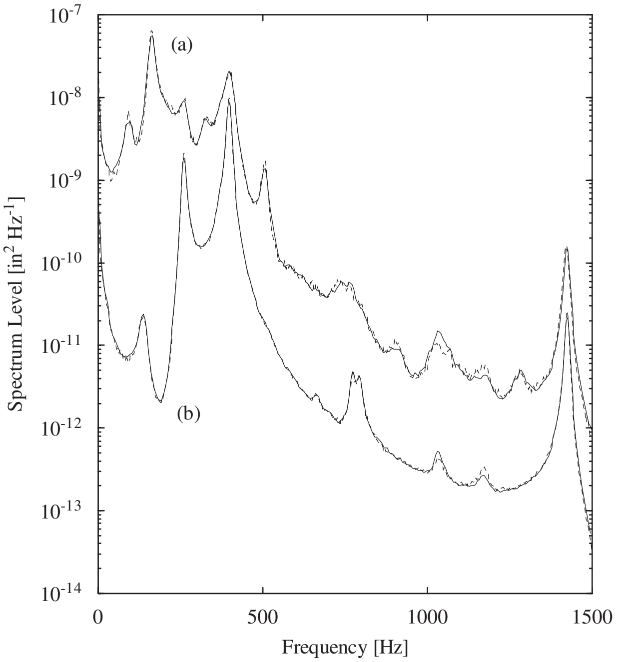

Fig. 10. The *x*-direction displacement for the curved beam at the quarter point from a full finite element model; —, and the implicit condensation and expansion method; - - -, for two random loading conditions: (a) $0.6517\,\text{lb}\,\text{in}^{-1}$ rms, and (b) $0.2304\,\text{lb}\,\text{in}^{-1}$ rms.

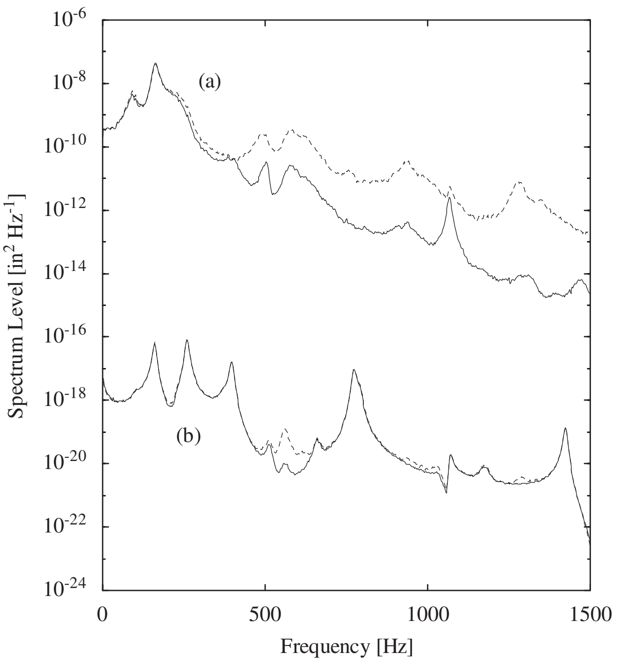

Fig. 11. The *x*-direction displacement for the curved beam at the beam center from the implicit condensation method; —, and the implicit condensation and expansion method; - - -, for two random loading conditions: (a) $0.6517\,\text{lb}\,\text{in}^{-1}$ rms, and (b) $0.2304\,\text{lb}\,\text{in}^{-1}$ rms.

the response. In our example, the improvement provided at the quarter point is very small. The improvement at the beam center is more pronounced and illustrated in Fig. 11.

4. Conclusion

A significant improvement to a method for predicting the nonlinear, random response of shell-like aerospace structures was developed and verified. The original implicit condensation method reduces a nonlinear finite element model to a low-order system of nonlinear equations. Time responses can then be efficiently computed by direct time integration of the low-order set of equations. The reduction is accomplished through a transformation to modal coordinates using a set of bending modes as basis vectors. Nonlinear terms are then added to the modal equations through an estimation process using static nonlinear solution results. The approach has a drawback—physical in-plane (membrane) displacements cannot be directly computed. The implicit condensation method is improved here by developing a procedure to estimate a membrane basis set. Membrane displacements then can be expanded from the bending mode results using the estimated membrane basis. This extension of the implicit condensation method has been termed the implicit condensation and expansion method. Traditional finite element-based stress recovery can be used with the implicit condensation and expansion method since the full displacement field can be recovered.

The accuracy of the implicit condensation and expansion method was verified using two example problems—straight and curved beams with clamped ends. Numerical simulation results were computed with low-order models using the implicit condensation and expansion method and compared to results from time simulations using full finite element models of the beams. Displacement and stress power spectral densities of the flat beam computed with the implicit condensation and expansion method model agreed very closely with full model results. Displacement power spectral densities from the curved beam implicit condensation and expansion method model also agreed very closely with the full model. In both example problems, membrane displacements were accurately recovered from a low-order model containing only bending modes.

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