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Analysis of an impact Duffing oscillator by means of a nonsmooth unfolding transformation

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Abstract

Forced vibrations of a one degree-of-freedom impact Duffing oscillator are considered in this paper. The nonsmooth unfolding transformation and the Van der Pol method are used together for vibration analysis and the forced vibrations in the region of the resonance family are treated in detail. In this resonance family, the system exhibits subharmonic vibrations. The stability and bifurcations of the periodic motions are also considered.

1. Introduction and problem formulation

Many efforts have been made to study the vibrations of impact systems due to the importance of this problem in engineering. Evidently, the book of Kobrinskii [1] was the first instance in which the theory of impact systems was treated. Holmes [2,3] showed analytically the importance of the Smale horseshoe in the formation of chaotic vibrations of impact systems. Thompson [4] analysed numerically the sequence of the period doubling bifurcations in impact oscillators. Experimental data of beam vibrations, impacting on a stop, are presented in the paper [5]. Moreover, a bilinear oscillator, which models the vibrations of this beam, is considered in this paper. Shaw and Holmes [6] studied the stability of the periodic motions of an impact system. Shaw [7] studied the dynamics of a one-degree-of-freedom oscillator with two-sided stops. The dynamics of the oscillator with zero stiffness are considered in the paper [8]. The three degree-of-freedom impacting system is investigated in Ref. [9]. The Hopf–Hopf bifurcation is analysed in detail. Xie [10] and Wen [11] investigated codimension two bifurcations corresponding to the double eigenvalue of -1. Nordmark [12] examined grazing bifurcations in vibro-impact systems. Nonideal dynamical systems with impact dampers are considered in Ref. [13]. Direct numerical simulations are used to study the system behaviour. Pilipchuk [14] suggested the use of saw-tooth time transformations to study impact systems. A detailed review of impact system dynamics is presented in Refs. [15,16], and the theory of impact systems is treated in the books [17,18].

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In this paper the Duffing oscillator, which impacts on stops during vibration, is considered. Nonsmooth unfolding transformations [19–23] are used to study the dynamics of this oscillator. These transformations are used jointly with the Van der Pol method and the resonance vibrations in this oscillator are studied in detail.

Basically the methods of lacing of linear solutions and the method of direct numerical integration are used to study impact systems. Both these methods need numerical computations, and therefore it is hard to detect qualitative behaviour of the system. Nonsmooth unfolding transformations in combination with the Van der Pol method, as suggested in this paper, allow one to obtain analytical solutions in the asymptotic limit, which match with the results from direct numerical integration. Moreover, it is possible to detect analytically the qualitative behaviour of the system.

2. Equations of motions and application of nonsmooth unfolding transformation

The impact Duffing oscillator is presented in the following form [15]:

$$m\ddot{s}_1 + \beta\dot{s}_1 + \gamma s_1 + \alpha s_1^3 = H_1 \cos(\Omega t); \tag{1a}$$

$$\dot{s}_1(t+0) = -\dot{s}_1(t-0); s_1 = \Delta, \tag{1b}$$

where $\gamma s_1 + \alpha s_1^3$ is a nonlinear restoring force; $H_1 \cos(\Omega\tau)$ is periodic force which acts on the mass and Δ is a value of the clearance. The dynamical system (1) contains both the ODE (1a) and the impact condition (1b).

System (1) is of relatively simple form and it is stressed, that this system has complex bifurcation behaviour, which is analysed only numerically [4,8]. The system of Eq. (1) describes vibrations hammers; machines, in which impacting motions are the basis of manufacturing [1]. The choice of system (1) for analysis is explained by the following.

1. The bifurcation behaviour can be analysed analytically by means of a nonsmooth unfolding transformation.
2. All the stages and the advantages of nonsmooth unfolding transformations are presented clearly for impacting system (1).
3. As this one degree-of-freedom dynamical system is successfully treated analytically in this paper, in future it is hoped that this method can be applied for analysis of dynamical systems with two and more degrees-of-freedom.

The nonsmooth unfolding transformation [19–22] has the following form:

$$s_1(t) = \Delta - z(t)\,\mathrm{sign}(z(t)). \tag{2}$$

Zhuravlev [19] suggested the form of transformations which can be used to analyse impact systems with arbitrary restitution coefficients of $0<r<1$. Unfortunately, such transformations are not used to analyse the dynamics of impact systems.

This is initial research devoted to the application of nonsmooth unfolding transformations to the analysis of impact dynamics and, therefore, the value of restitution coefficient $r = 1$ is considered. The arbitrary value of r will be considered in future work.

The method of linear solution piecing is used often to analyse impact oscillators [6]. It is impossible to obtain observable analytical solutions by this method. The method of nonsmooth unfolding transformations collectively with the asymptotic method, allows one to obtain analytical solutions. We stress that this nonlinear transformation satisfies the impact condition (1b). Therefore, after applying this transformation to the equation there is no further need to use the impact condition (1b). This is the essential advantage of this method.

Due to transformation (2), the variable $z(t)$ replaces the impact condition (1b). Therefore, the dynamical system (1) with respect to the variable $z(t)$ has the following form:

$$m\ddot{z} + \beta\dot{z} + (\gamma + 3\alpha\Delta^2)z - (\gamma\Delta + \Delta^3\alpha)\mathrm{sign}(z) + \alpha z^3 - 3\Delta\alpha z^2\,\mathrm{sign}(z) = -\mathrm{sign}(z)H\cos(\Omega t). \tag{3}$$

Thus, the impact condition (1b) can be rejected.

The dimensionless variables and parameters are introduced as

$$s=\frac{z}{z_*};\quad \tau=pt;\quad p^2=\frac{\gamma}{m};\quad \frac{\Delta}{z_*}=\varepsilon\delta;\quad \varepsilon\chi=\frac{\beta}{\sqrt{\gamma m}};\quad \varepsilon\eta=z_*^2\frac{\alpha}{\gamma};\quad \varepsilon H=\frac{H_1}{\gamma z_*},\tag{4}$$

where $\varepsilon\ll 1$. Then the dynamical system (3) has the following form:

$$\ddot{s}+\varepsilon\chi\dot{s}+s-\varepsilon\delta\,\mathrm{sign}(s)+\varepsilon\eta s^3+O(\varepsilon^2)=-\varepsilon\,\mathrm{sign}(s)H\cos(\Omega\tau).\tag{5}$$

Instead of solving Eq. (1a) with the restriction (1b), the weakly nonlinear oscillator (5) is obtained. This is the major advantage of a nonsmooth unfolding transformation. This system can be analysed by asymptotic methods.

3. Periodic motions analysis by Van der Pol method

The dynamics of system (5) are considered in the region of the following resonances:

$$\Omega=2m+\varepsilon\sigma,\tag{6}$$

where $m=1,2,\dots$ and σ is a detuning parameter. We stress that Eq. (6) describes the family of resonances. The following Van der Pol transformation is applied to system (5):

$$(s,\dot{s})=a\left[\cos\left(\frac{\Omega t-\theta}{2m}\right);-\frac{\Omega}{2m}\sin\left(\frac{\Omega t-\theta}{2m}\right)\right].\tag{7}$$

As a result, nonautonomous dynamical system of the first order are derived. The average procedure is applied to this system. Then a new autonomous dynamical system is obtained:

$$\begin{aligned}a'&=F_a(a,\theta)=-\frac{\chi a}{2}+\frac{4mH(-1)^m}{\pi(4m^2-1)}\sin\theta;\\ \theta'&=F_\theta(a,\theta)=\frac{4m\delta}{\pi a}-\frac{3\eta m}{4}a^2+\sigma+\frac{4mH(-1)^m}{\pi a(4m^2-1)}\cos\theta;\end{aligned}\tag{8}$$

where $a'=\mathrm{d}a/\mathrm{d}T_1;\ T_1=\varepsilon t$.

The dynamical system of Eq. (8) has fixed points which are denoted by $a_*\theta_*$. Small motions $(\Delta a,\Delta\theta)$ close to these fixed points are analysed in order to study the stability of the overall system. These motions are described by the following:

$$\begin{aligned}\Delta a'&=-\frac{\chi}{2}\Delta a+\frac{4Hm(-1)^m}{\pi(4m^2-1)}\cos\theta_*\Delta\theta;\\ \Delta\theta'&=-\left(\frac{4m\delta}{\pi a_*^2}+\frac{3\eta m}{2}a_*+\frac{4mH(-1)^m}{\pi a_*^2(4m^2-1)}\cos\theta_*\right)\Delta a-\frac{4mH(-1)^m}{\pi a(4m^2-1)}\sin\theta_*\Delta\theta.\end{aligned}\tag{9}$$

Characteristic exponents λ_i of Eq. (9) are determined as the roots of the following quadratic equation:

$$\lambda^2-\lambda\,\mathrm{tr}+\det=0,\tag{10}$$

where $\mathrm{tr}=-(\chi/2)-((4mH(-1)^m)/(\pi a(4m^2-1)))\sin\theta$;

$$\det=\frac{\chi^2 mH(-1)^m}{\pi a(4m^2-1)}\sin\theta+\frac{4mH(-1)^m}{\pi(4m^2-1)}\left[\frac{4m\delta}{\pi a^2}+\frac{3\eta m}{2}a+\frac{4mH(-1)^m}{\pi a^2(4m^2-1)}\cos\theta\right]\cos\theta.$$

3.1. Analysis of the system vibrations without damping

The case without damping $\chi=0$ is considered. The fixed points of Eq. (8) are analysed. Then the variable θ of the fixed points satisfies the equation: $\sin\theta=0$. Two types of solutions exist in Eq. (8), which have the following form:

$$\sigma_a = \frac{3\eta m}{4}a^2 - \frac{4m}{\pi a}\left(\delta + \frac{H(-1)^m}{4m^2-1}\right);$$
$$\sigma_b = \frac{3\eta m}{4}a^2 - \frac{4m}{\pi a}\left(\delta - \frac{H(-1)^m}{4m^2-1}\right). \quad (11)$$

The case $m = 2l$; $l = 1,2,\ldots$ is considered for which the solutions (11) have the following form:

$$\sigma_a = \frac{3\eta l}{2}a^2 - \frac{8l}{\pi a}\left(\delta + \frac{H}{16l^2-1}\right); \quad (12a)$$

$$\sigma_b = \frac{3\eta l}{2}a^2 - \frac{8l}{\pi a}\left(\delta - \frac{H}{16l^2-1}\right). \quad (12b)$$

The frequency response of system (5) is described by Eq. (12). Let us consider two cases:

$$\text{I.} \quad \delta - \frac{H}{16l^2-1} > 0;$$

$$\text{II.} \quad \delta - \frac{H}{16l^2-1} < 0. \quad (13)$$

In case I, the frequency response according to resonance (6), with $m = 2l$, is shown qualitatively in Fig. 1, and in case II, the frequency response is presented in Fig. 2. The branches of the frequency responses, which are characterized by Eqs. (12a) and (12b), are denoted by a and b in Figs. 1 and 2.

Note that the backlash is denoted by δ in system (5). Therefore, the difference in the system behaviour, as presented in Figs. 1 and 2, shows the crucial influence of backlash on the steady vibrations.

The stability of the fixed points (Figs. 1 and 2) is studied and stability of the equilibria of branches a is determined by the characteristic exponents λ, which satisfy the following equation:

$$\lambda^2 = -\frac{8lH}{\pi(16\,l^2-1)}\frac{d\sigma_a}{da}, \quad (14)$$

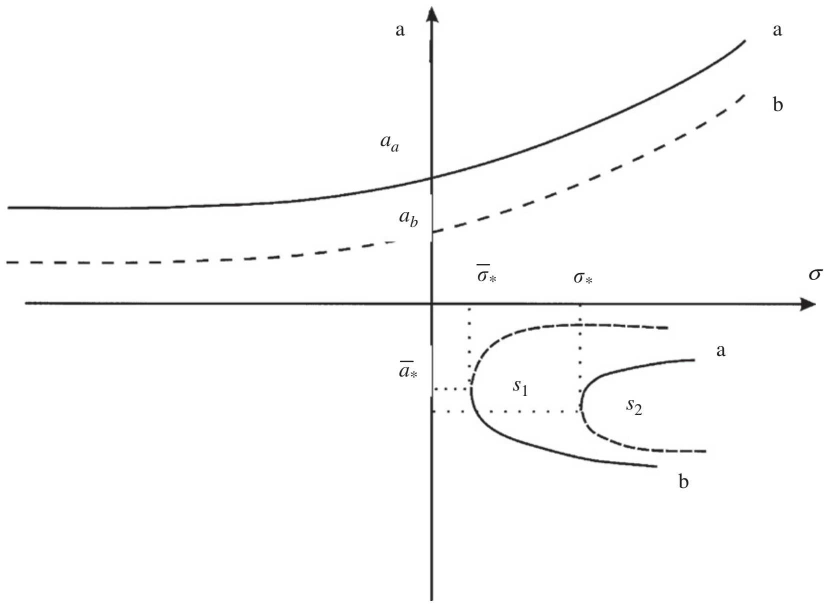

Fig. 1. Frequency response for case I of the inequalities (13).

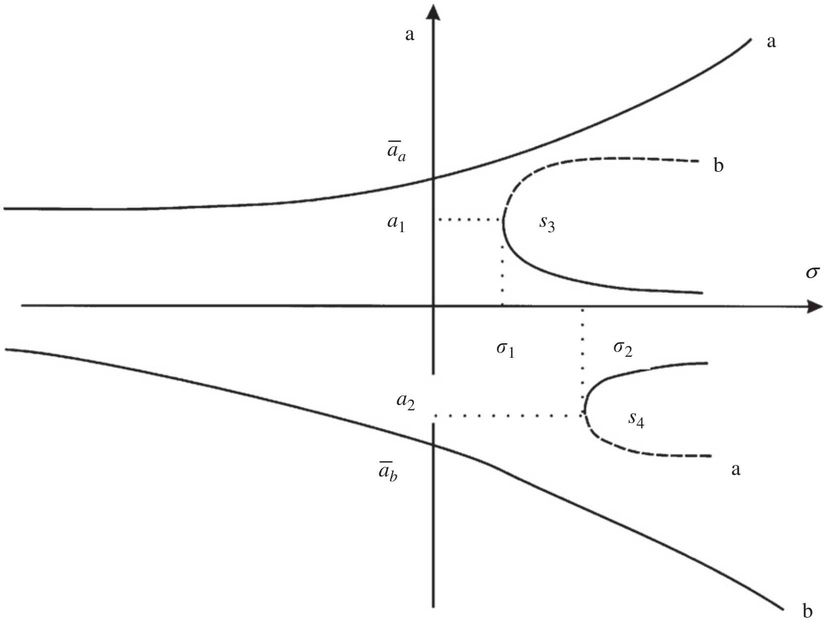

Fig. 2. Frequency response for case II of the inequalities (13).

where $d\sigma_a/da$ is determined for the branches of the frequency response. Then the stability condition of the equilibria of the branches a has the following form:

$$\frac{d\sigma_a}{da} > 0.$$

The characteristic exponents for the stability determination of the equilibria, which belong to the branches b, are obtained from the equation:

$$\lambda^2 = \frac{8lH}{\pi(16l^2-1)}\frac{d\sigma_b}{da}. \tag{15}$$

Then the stability condition of the fixed points, which belongs to branches b, have the following form:

$$\frac{d\sigma_b}{da} < 0.$$

The stable fixed points and unstable fixed points are shown in Figs. 1 and 2 by solid and dotted lines, respectively.

Four saddle-node bifurcation points S_1, S_2, S_3, S_4 are shown in Figs. 1 and 2. The coordinates of these points on the bifurcation diagrams are as follows:

$$S_1(\bar{\sigma}_*, \bar{a}_*);\ S_2(\sigma_*, a_*);\ S_3(\sigma_1, a_1);\ S_4(\sigma_2, a_2); \tag{16}$$

$$\bar{\sigma}_* = 6l\sqrt[3]{\frac{3\eta}{\pi^2}}\left(\delta - \frac{H}{16l^2-1}\right)^{2/3}; \quad \bar{a}_* = -2\sqrt[3]{\frac{1}{3\eta\pi}\left(\delta - \frac{H}{16l^2-1}\right)};$$

$$\sigma_* = 6l\sqrt[3]{\frac{3\eta}{\pi^2}}\left(\delta + \frac{H}{16l^2-1}\right)^{2/3}; \quad a_* = -2\sqrt[3]{\frac{1}{3\eta\pi}\left(\delta + \frac{H}{16l^2-1}\right)}; \quad \sigma_2 = \sigma_*; \quad a_2 = a_*;$$

$$\sigma_1 = 6l\sqrt[3]{\frac{3\eta}{\pi^2}}\left(\frac{H}{16l^2-1} - \delta\right)^{2/3}; \quad a_1 = \sqrt[3]{\frac{8}{3\eta\pi}\left(\frac{H}{16l^2-1} - \delta\right)}.$$

Note, that the coordinates of the bifurcation points satisfy the following inequalities:

$$\bar{\sigma}_* < \sigma_*; \quad a_* < \bar{a}_*. \tag{17}$$

The coordinates of the intersections of the frequency responses with y-axis (Figs. 1 and 2) are determined as

$$\begin{aligned} a_a &= \sqrt[3]{\frac{16}{3\pi\eta}\left(\delta + \frac{H}{16l^2-1}\right)}; \\ a_b &= \sqrt[3]{\frac{16}{3\pi\eta}\left(\delta - \frac{H}{16l^2-1}\right)}; \\ \bar{a}_a &= a_a; \\ \bar{a}_b &= -\sqrt[3]{\frac{16}{3\pi\eta}\left(\frac{H}{16l^2-1} - \delta\right)}. \end{aligned} \tag{18}$$

Note that these coordinates satisfy the following inequality:

$$a_a > a_b. \tag{19}$$

Now the case $m = 2l-1$ is treated. This case will be considered in less detail than the previous one. Two groups of solutions, which are denoted by c and d, are observed. These fixed points are determined from the equations:

$$\sigma_c = \frac{3\eta(2l-1)}{4}a^2 - \frac{4(2l-1)}{\pi a}\left(\delta - \frac{H}{4(2l-1)^2-1}\right); \tag{20a}$$

$$\sigma_d = \frac{3\eta(2l-1)}{4}a^2 - \frac{4(2l-1)}{\pi a}\left(\delta + \frac{H}{4(2l-1)^2-1}\right). \tag{20b}$$

Two cases are considered:

$$\text{III.} \quad \delta - \frac{H}{4(2l-1)^2-1} > 0;$$

$$\text{IV.} \quad \delta - \frac{H}{4(2l-1)^2-1} < 0; \tag{21}$$

If Eqs. (20) and (11) are compared then the following conclusion can be made. The behaviour of branches c (20a) and branches b (Figs. 1 and 2) are qualitatively the same. Moreover, the behaviour of branches d and the curves a are qualitatively the same too.

Now the stability of the fixed points is considered. The stability of the fixed points which belong to branch c is described by the characteristic exponents $\lambda_{1,2}$:

$$\lambda^2 = \frac{4(2l-1)H}{\pi[4(2l-1)^2-1]}\frac{d\sigma_c}{da} \tag{22}$$

and the stability of the fixed points, which belong to branch d, is described by the characteristic exponents:

$$\lambda^2 = -\frac{4(2l-1)H}{\pi[4(2l-1)^2-1]}\frac{d\sigma_d}{da}. \tag{23}$$

The frequency response of the vibrations (Figs. 1 and 2) for $m = 2l$ is qualitatively the same as the frequency response for $m = 2l-1$.

Using the approach suggested in this paper, it is impossible to obtain asymptotic behaviour of the backbone curves. The analysis of Eq. (5) by the Van der Pol method is true for $s = O(1)$, but the asymptotic behaviour of the backbone curves is fulfilled for greater values of s.

3.2. Vibrations of the system with dissipation

The stationary vibrations of the system with dissipation are determined by the fixed points of system (8). These fixed points satisfy the system of two nonlinear algebraic equations:

$$\frac{4Hm(-1)^m}{\pi(4m^2-1)}\sin\theta = \frac{\chi a}{2};$$
$$\frac{4Hm(-1)^m}{\pi(4m^2-1)}\cos\theta = \frac{3\eta m}{4}a^3 - \sigma a - \frac{4m\delta}{\pi}. \tag{24}$$

Two groups of the fixed points are obtained from Eq. (24):

$$\sigma = \frac{3\eta m}{4}a^2 - \frac{4m\delta}{\pi a} \pm \sqrt{\frac{16m^2H^2}{\pi^2(4m^2-1)^2a^2} - \frac{\chi^2}{4}}. \tag{25}$$

Note that the general coordinate of system (1a) and the variables of system (8) are connected such that:

$$s_1 = \Delta - \left| z_* a\cos\left(\frac{\Omega}{2m}t - \frac{\theta}{2m}\right)\right|. \tag{26}$$

If a change of variable of the form $a \to -a$ is applied, then the solutions of system (1a) are unchanged.

Eq. (26) shows that system (1) behaviour in the resonance family (6) is subharmonic.

Now, the backbone curve for free vibrations, which is described by Eq. (25), with $\chi = H = 0$, is considered. The equation of the backbone curve is the following:

$$\sigma = \frac{3\eta m}{4}a^2 - \frac{4m\delta}{\pi a}. \tag{27}$$

The saddle-node bifurcation of free vibrations is observed at

$$\tilde{a}_b = -2\sqrt[3]{\frac{\delta}{3\pi\eta}} \tag{28}$$

The backbone curves are shown in Figs. 3 and 4 by the chain lines.

Now the forced vibrations at $H \neq 0$; $\chi \neq 0$ are analysed. The results of the analysis are shown on the frequency response. As follows from Eq. (25), forced vibrations take place if the inequality satisfies the following:

$$|a| < a_0; \quad a_0 = \frac{8mH}{\pi(4m^2-1)\chi}. \tag{29}$$

Now, two types of frequency response, which differ by qualitative branch arrangement, are considered. Fig. 3 shows the first type of the frequency response, which satisfies the following inequality:

$$-a_0 > \tilde{a}_b. \tag{30}$$

Fig. 4 shows the second type of the frequency response. In this case, the following inequality is true:

$$\tilde{a}_b > -a_0. \tag{31}$$

From this it is seen that the stability of the fixed points of system (8) can be analysed. The evolution of small perturbations close to the fixed points in time is described by system (9). The characteristic exponents of this system are determined by Eq. (10). Differentiating the system of two nonlinear algebraic equations for the fixed points of Eq. (8) with respect to σ, the following equations are derived:

$$\frac{\partial F_a}{\partial a}\frac{da}{d\sigma} + \frac{\partial F_a}{\partial\theta}\frac{d\theta}{d\sigma} = 0;$$
$$\frac{\partial F_\theta}{\partial a}\frac{da}{d\sigma} + \frac{\partial F_\theta}{\partial\theta}\frac{d\theta}{d\sigma} + \frac{\partial F_\theta}{\partial\sigma} = 0, \tag{32}$$

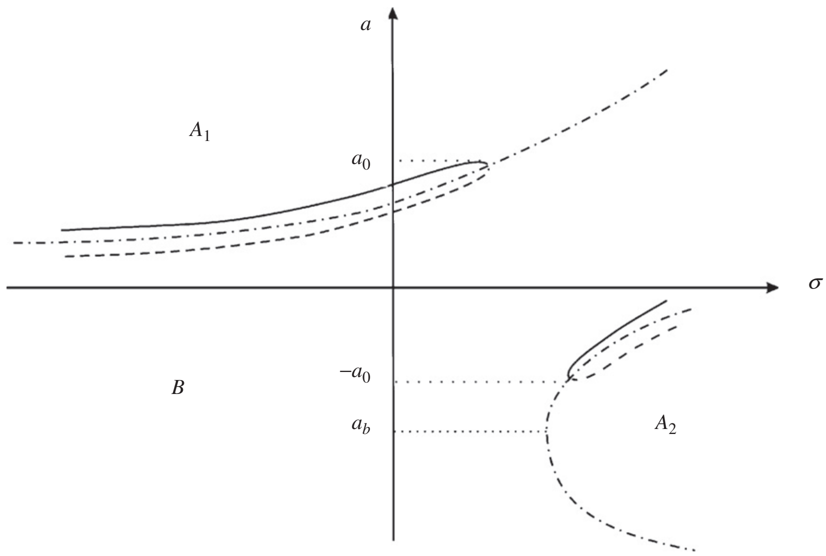

Fig. 3. Frequency response of the forced vibrations for case (30).

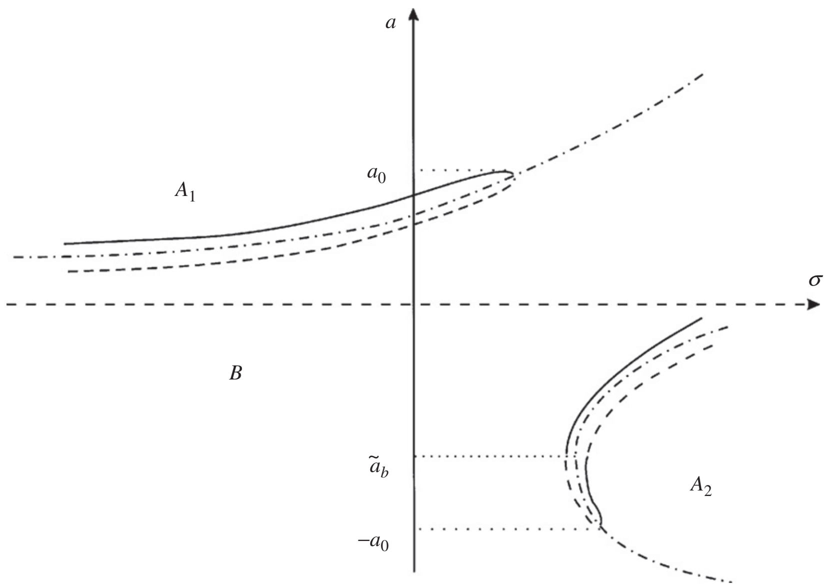

Fig. 4. Frequency response of the forced vibrations for case (31).

where $(\partial F_\theta/\partial\sigma) = 1$. The parameter det, which is determined by the formulas (10), satisfies the following equation:

$$\det = \frac{\partial F_\theta}{\partial\theta}\frac{\partial F_a}{\partial a} - \frac{\partial F_a}{\partial\theta}\frac{\partial F_\theta}{\partial a} = \frac{\mathrm{d}\sigma}{\mathrm{d}a}\frac{\partial F_a}{\partial\theta}, \tag{33}$$

where $(\partial F_a/\partial\theta) = a\tilde{f}$; $\tilde{f} = (3\eta m/4)a^2 - (4m\delta/\pi a) - \sigma$. Note that the derivative $\mathrm{d}\sigma/\mathrm{d}a$ is determined along the branches of the frequency response (Figs. 3 and 4). Thus the characteristic exponents of the variational equation (10) can be presented in the following form:

$$2\lambda_{1,2} = -\chi \pm \sqrt{\chi^2 - 4a\tilde{f}\,\frac{\mathrm{d}\sigma}{\mathrm{d}a}}. \tag{34}$$

Finally the analysis of the characteristic exponents (34) of equations can be carried out. Note that the backbone curves (Figs. 3 and 4) separate the plane (σ,a) on the three regions A_1, A_2, B. In the regions A_1, A_2, the following inequality is true:

$$\tilde{f} > 0. \tag{35}$$

Moreover, in region B the following inequality is satisfied:

$$\tilde{f} < 0. \tag{36}$$

Note that the sign of $d\sigma/da$ can be determined by the qualitative arrangement of the frequency response branches. The stability of the fixed points can be determined using the analysis presented above and using Eq. (34). The branches of the stable and unstable fixed points are shown in Figs. 3 and 4 by solid and dotted lines, respectively.

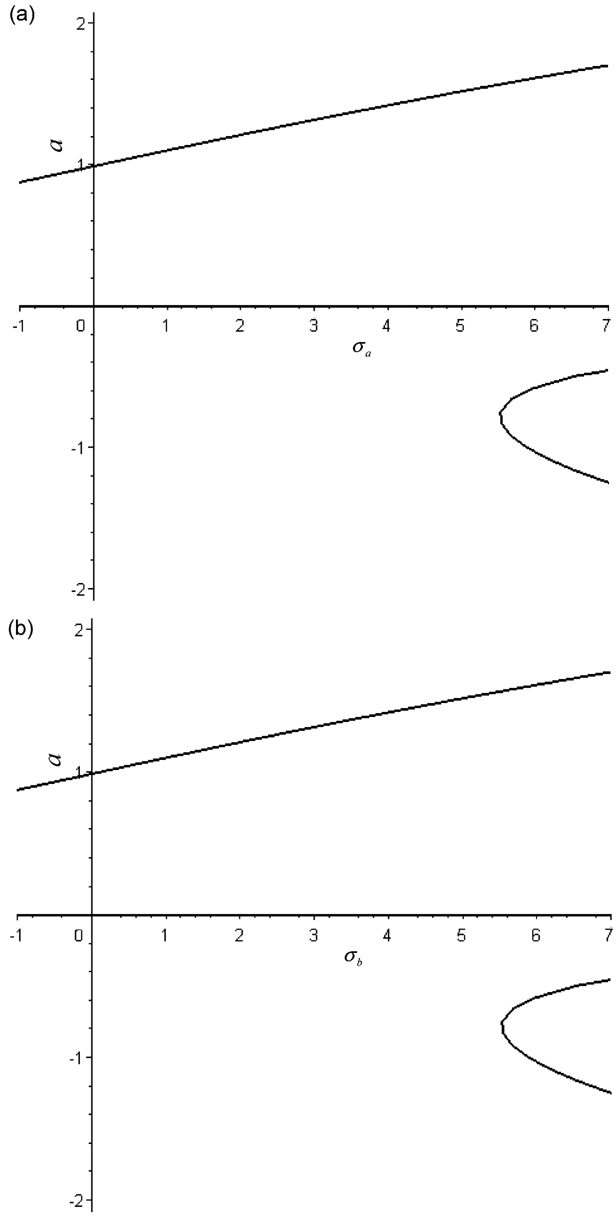

Fig. 5. Backbone curves of free vibrations. (a) and (b) are calculated from Eqs. (12a) and (12b), respectively.

The fixed points of the modulation equation (8) considered above correspond to the periodic vibrations of Eq. (26). Moreover, the stable and unstable fixed points correspond to the stable and unstable periodic vibrations, respectively.

Thus, due to cooperative use of the nonsmooth unfolding transformations and the Van der Pol method, the motions of impact system (1) have been studied analytically.

4. Numerical analysis of vibrations

The frequency responses which have been considered qualitatively in the previous section are now analysed numerically for particular values of system parameters. The following values of system (5) parameters are

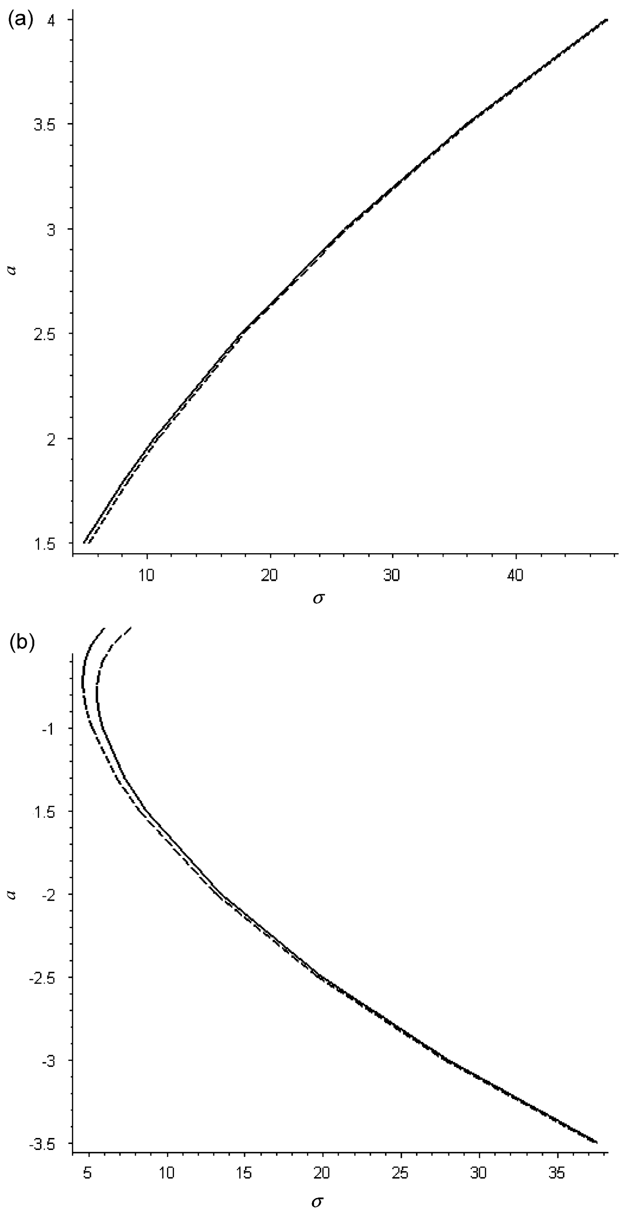

Fig. 6. Frequency response of the forced vibrations. (a) $-a \in [1.5; 4]$ and (b) $-a \in [-3.5; -1]$.

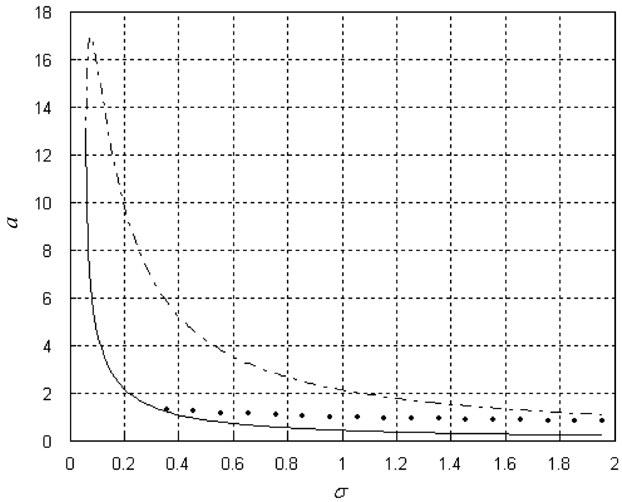

Fig. 7. Frequency response of forced vibrations for parameters $\eta = 0$; $H = 2$; $\delta = 1$; $\varepsilon = 0.1$; $\chi = 0.1$; $m = 1$.

considered:

$$\varepsilon = 0.1; \quad \delta = 1; \quad \eta = 2; \quad H = 2; \quad \chi = 0.1; \quad l = 1; \quad m = 2 \tag{37}$$

At first the frequency response of the free vibrations is considered. Parameters (37) correspond to case I of Eq. (13). Therefore, the coordinates of the characteristic points of the frequency response (Fig. 1) are the following:

$$a_a = 0.98717; \quad a_b = 0.90273; \quad \overline{\sigma}_* = 4.62029; \quad \sigma_* = 5.551; \quad \overline{a}_* = -0.7164; \quad a_* = -0.7835.$$

Fig. 5 shows the backbone curves of free vibrations. Figs. 5a and b are calculated by Eqs. (12a) and (12b), respectively.

The following parameters are obtained to calculate the forced vibrations:

$$\tilde{a}_b = -0.7515; \quad a_0 = 6.7906.$$

These values of parameters correspond to the inequality of Eq. (31). Fig. 6 shows the frequency response of the forced vibrations.

Direct numerical integrations of oscillator (5) with the parameters:

$$\eta = 0; \quad H = 2; \quad \delta = 1; \quad \varepsilon = 0.1; \quad \chi = 0.1; \quad m = 1;$$

were carried out in order to confirm the method. Fig. 7 shows the frequency response and the results of the direct numerical simulations, which are presented by dots in the figure. As examples, the waveforms of the vibrations are shown in Fig. 8.

5. Concluding remarks

The method suggested in this paper can be divided into the following phases.

(1) applying nonsmooth unfolding transformations, which allows to exclude the impact condition;
(2) applying an asymptotic, or another approximate method, to study the new dynamical system;
(3) analysis of the obtained results.

Note that different methods of nonlinear dynamics can be used to study Eq. (3). For example, the harmonic balance method or the Melnikov technique can also be applied.

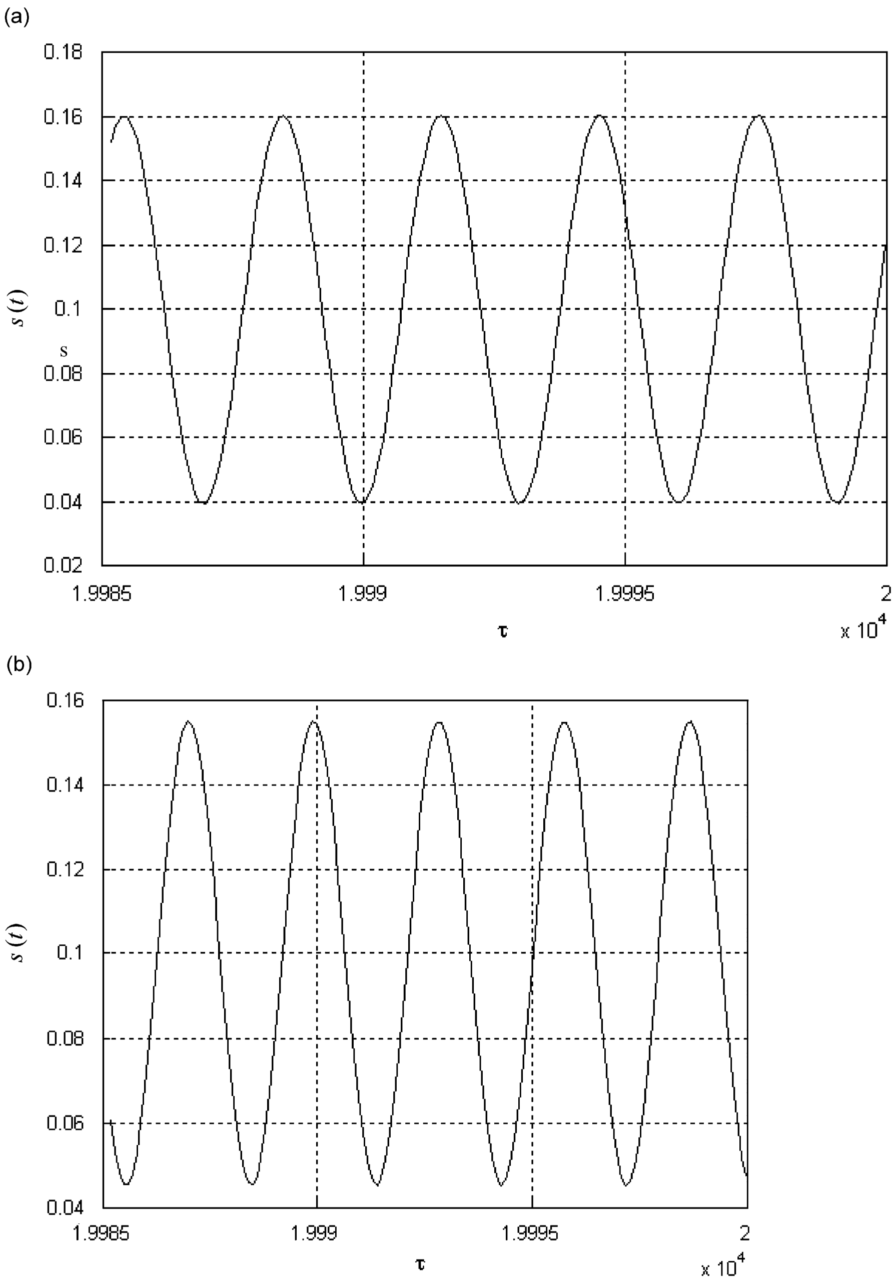

Fig. 8. Waveform of vibrations at (a) $-\sigma = 0.7559$ and (b) $-\sigma = 1.5559$.

The novelty of the results obtained in this paper is the following:

- the nonsmooth unfolding transformations and the Van der Pol method are used together to analyse the impact system;
- the results of the dynamical behaviour analysis are also proposed as new information.

Application of nonsmooth unfolding transformations jointly with the Van der Pol method allows analytical periodic solutions to be obtained and their stability and bifurcations to be analysed. This is the advantage of the method. However, it is not yet clear how to apply this method to systems with several degrees of freedom and this will be the topic of future work.

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