Available online at www.sciencedirect.com
ScienceDirect
Journal of Sound and Vibration 318 (2008) 1250–1261
JOURNAL OF SOUND AND VIBRATION
www.elsevier.com/locate/jsvi

On the jump-up and jump-down frequencies of the Duffing oscillator

M.J. Brennan[a,*], I. Kovacic[b], A. Carrella[a], T.P. Waters[a]

[a]*Institute of Sound and Vibration Research, University of Southampton, Southampton, Hampshire SO17 1BJ, UK*
[b]*Faculty of Technical Sciences, Department of Mechanics, University of Novi Sad, Novi Sad 21000, Serbia*

Received 28 November 2007; received in revised form 13 April 2008; accepted 14 April 2008
Handling Editor: M.P. Cartmell
Available online 6 June 2008

Abstract

In this paper, simple approximate non-dimensional expressions, and the corresponding displacement amplitudes for the jump-up and jump-down frequencies of a softening and hardening lightly damped Duffing oscillator with linear viscous damping are presented. Although some of these expressions can be found in the literature, this paper presents a full set of expressions determined using the harmonic balance approach. These analytical expressions are validated for a range of parameters by comparing the predictions with calculations from direct numerical integration of the equation of motion. They are also compared with similar expressions derived using a perturbation method. It is shown that the jump-down frequency is dependent on the degree of nonlinearity *and* the damping in the system, whereas the jump-up frequency is dependent primarily upon the nonlinearity, and is only weakly dependent upon the damping. An expression is also given for the threshold of the excitation force and the nonlinearity that needs to be exceeded for a jump to occur. It is shown that this is only dependent upon the damping in the system.

1. Introduction

The Duffing oscillator has been studied for many years, as it is representative of many nonlinear systems [1]. For this type of system there are frequencies at which the vibration suddenly jumps-up or down, when it is excited harmonically with slowly changing frequency. The frequencies at which these jumps occur depend upon whether the frequency is increasing or decreasing and whether the nonlinearity is hardening or softening. Between these frequencies, multiple solutions exist for a given frequency of excitation, and the initial conditions determine which of these solutions represents the response of the system.

Although the Duffing oscillator has been studied at length, to the authors' knowledge, analytical expressions for the jump frequencies and the amplitudes of vibration at these frequencies for both a softening and hardening system do not appear together in a single publication, which is rather surprising since it has been almost 90 years since Duffing's original publication [2]. There have been two approaches to this problem:

*Corresponding author.
E-mail address: mjbrennan0@btinternet.com (M.J. Brennan).

0022-460X/$ - see front matter
doi:10.1016/j.jsv.2008.04.032

analytical and numerical. The analytical approach has included the use of classical perturbation methods such as the method of multiple scales, which can be applied to weakly nonlinear systems, for which the jump frequencies are close to the undamped natural frequency of the linear system (nonlinear term set to zero). The other analytical method used is harmonic balance, which can be applied to both weakly and strongly nonlinear systems. Nayfeh and Mook [1] and Kevorkian and Cole [3] used the method of multiple scales to determine the jump-down frequency and the peak amplitude of the response, and additionally Kevorkian and Cole have also determined the jump-up frequency and the corresponding amplitude of vibration for a hardening system. Magnus [4] has used the harmonic balance method (HBM) to determine the jump-down frequency and the peak amplitude of a hardening system, but did not consider the jump-up frequency. Unfortunately, all the publications describing these studies have different notation, so that the results cannot be compared in a straightforward manner.

Perturbation methods are quite accurate in predicting the jump-up frequency, because even when there is strong nonlinearity, the jump-up frequency is close to the undamped natural frequency of the linear system. However, this is not the case for the jump-down frequency and it is shown in this paper that the results derived using these techniques are inaccurate for strong nonlinearity. In particular the maximum response is predicted to be the same as the linear system, which is not the case when there is strong nonlinearity.

Since the advent of computers, researchers working on the Duffing oscillator have mainly used a numerical or a hybrid analytical-numerical approach. Peleg [5], studying a hardening isolation system with friction and viscous damping, has proposed *implicit* equations for both the jump-down and jump-up frequencies, and the amplitude of the response at these frequencies by locating the loci of the vertical tangents to the response curve. Murata et al. [6] examined a hardening system described by Duffing's equation. Using catastrophe theory, they determined the jump frequencies numerically and presented their results graphically. Friswell and Penny [7], and Worden [8] computed the jump-up and jump-down frequencies of the Duffing oscillator with linear damping. In both studies, the HBM was used to obtain the frequency response curves. Friswell and Penny [7] used a numerical approach based on Newton's method, including terms up to the ninth harmonic, to compute the jump frequencies, while Worden [8], used a first order expansion and set the discriminant of a cubic polynomial in the square of the amplitude to be equal to zero, and solved the resulting equation numerically. Remarkably the difference between Friswell and Penny's results and Worden's first order approximation never exceeded 0.34% for the parameters chosen in his study. Carrella [9] using the HBM to a first order expansion has found closed form expressions that, with parameters used in [7,8], yield values which differs from those found by Friswell and Worden by less than 1%.

The primary aim of this paper is to re-visit the analytical approach in the determination of the jump frequencies and the corresponding vibration amplitudes of the Duffing oscillator. Using the HBM, in which the response is approximated by a single harmonic, a consistent approach is used to determine these for both softening and hardening systems with linear viscous damping. The results are presented in tabular form for ease of reference. A comparison is made between these results and those in the literature determined using the method of multiple scales. The limitation in the multiple scales method in determining the jump-down frequency is also discussed. The secondary aim of the paper is to provide a link, in terms of a reference trail, between the earlier analytical work and the later numerical studies.

In some recent work, Malatkar and Nayfeh [10] determined the minimum excitation force required for the jump phenomenon to appear. Their key results are also presented in this paper, but in a modified and simplified form, which is consistent with the analysis for the jump frequencies.

2. Calculation of the jump frequencies

Duffing's equation is given by [1,2]

$$m\ddot{x} + c\dot{x} + k_1 x + k_3 x^3 = F\cos(\omega t) \tag{1}$$

which is the equation of motion of a single degree-of-freedom system of mass m, suspended on a parallel combination of a dashpot with damping coefficient c, and a spring with nonlinear restoring force, $k_1x + k_3x^3$, excited by a harmonic force $F\cos(\omega t)$. If $k_3 > 0$ the system is hardening and if $k_3 < 0$ the system is softening.

It is convenient to write Eq. (1) in non-dimensional form as

$$y'' + 2\zeta y' + y + \alpha y^3 = \cos(\Omega\tau) \tag{2}$$

where

$$y = \frac{x}{x_0}, \quad \alpha = \frac{k_3 x_0^2}{k_1}, \quad \zeta = \frac{c}{2m\omega_0}, \quad \omega_0^2 = \frac{k_1}{m}, \quad \tau = \omega_0 t, \quad \Omega = \frac{\omega}{\omega_0}$$

with x_0 defined as $x_0 = F/k_1|_{k_3=0,\omega=0}$ and $(\bullet)' = \mathrm{d}/\mathrm{d}\tau(\bullet)$. Note that the term α is the product of two factors, one related to the degree of nonlinearity, k_3/k_1, and the other related to the square of the amplitude of the applied force. It can also be thought of as the ratio of the force due to the nonlinear stiffness at displacement x_0 to the force due to the linear stiffness at displacement x_0. It is important to note that as the amplitude of the force increases (decreases) then α increases (decreases) and the non-dimensional displacement of the system y decreases (increases).

In this article, the relationship between the jump frequencies and α is determined. The amplitude of vibration at the jump-up and jump-down frequencies is also given. It should be noted that a change in α can be interpreted as either a change in the degree of nonlinearity or a change in the amplitude of excitation.

Applying the HBM and assuming a solution of the form $y = Y\cos(\Omega\tau + \phi)$, with the term containing $\cos 3\Omega\tau$ neglected, leads to the frequency–amplitude relationship

$$\left(\frac{3}{4}\alpha Y^3 + (1-\Omega^2)Y\right)^2 + 4\zeta^2\Omega^2 Y^2 = 1 \tag{3}$$

This can be expanded and arranged as

$$\frac{9}{16}\alpha^2 Y^6 + \frac{3}{2}\alpha(1-\Omega^2)Y^4 + (4\zeta^2\Omega^2 + (1-\Omega^2)^2)Y^2 - 1 = 0 \tag{4a}$$

or alternatively as

$$Y^2\Omega^4 - Y^2\left(\frac{3}{2}\alpha Y^2 + 2(1-2\zeta^2)\right)\Omega^2 + Y^2\left(\frac{3}{4}\alpha Y^2 + 1\right)^2 - 1 = 0 \tag{4b}$$

Worden [8] considered a hardening system only. He set the discriminant of Eq. (4a), which is a cubic polynomial in the square of the amplitude Y, equal to zero and found the bifurcation points numerically; these are the jump frequencies. In this paper, the approach taken by Magnus [4] and Hagedorn [11] is followed, and Eq. (4b), which is quadratic in Ω^2, is considered instead. Solving for Ω, the two positive solutions are

$$\Omega_{1,2} = \left(\frac{3}{4}\alpha Y^2 + (1-2\zeta^2) \pm \frac{(1 - 4\,Y^2\zeta^2(1-\zeta^2) - 3\alpha Y^4\zeta^2)^{1/2}}{Y}\right)^{1/2} \tag{5a,b}$$

In this paper *it is assumed that damping is small* such that $\zeta^2 \ll 1$ so Eqs. (5a,b) reduce to

$$\Omega_{1,2} \approx \left(1 + \frac{3}{4}\alpha Y^2 \pm \left(\frac{1}{Y^2} - 4\zeta^2\left(1 + \frac{3}{4}\alpha Y^2\right)\right)^{1/2}\right)^{1/2} \tag{6a,b}$$

These equations yield real values of Ω provided that

$$1 - 4Y^2\zeta^2\left(1 + \frac{3}{4}\alpha Y^2\right) \geqslant 0 \tag{7a}$$

and also

$$1 + \frac{3}{4}\alpha Y^2 \pm \left(\frac{1}{Y^2} - 4\zeta^2\left(1 + \frac{3}{4}\alpha Y^2\right)\right)^{1/2} \geqslant 0 \tag{7b}$$

Eqs. (6a,b) are used to plot the frequency response curve for the range of Y that satisfies the inequalities of Eqs. (7a) and (7b). This is shown in Fig. 1 for a softening system (two cases), a linear system and a hardening system, with the damping ratio set to $\zeta = 0.02$. The values of α used are discussed below. The jump-down

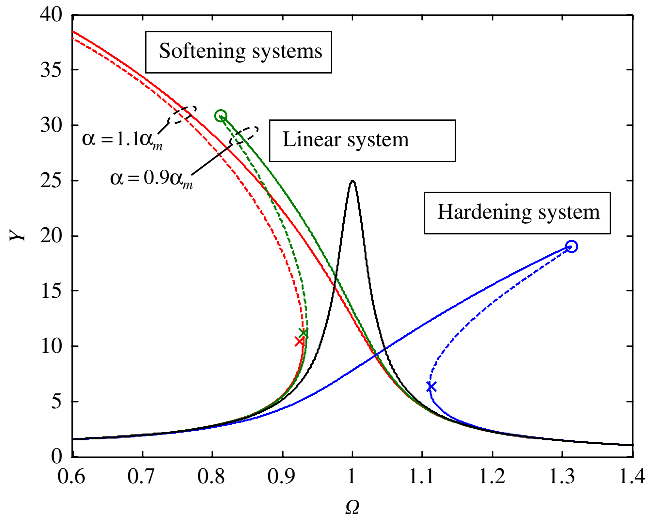

Fig. 1. Frequency response curves for the Duffing oscillator. The dashed lines denote unstable solutions. The crosses denote the responses at the jump-up frequencies and the circles denote the maximum values of the response, which occur approximately at the jump-down frequencies. Expressions for the jump-up and jump-down frequencies, and the response amplitudes at these frequencies are given in Table 1. For all the simulations $\zeta = 0.02$; for the softening systems $\alpha = 0.9\alpha_m$ and $1.1\alpha_m$ where $\alpha_m = -4/3\zeta^2$, and for the hardening system $\alpha = -5\alpha_m$. Note that there is no jump-down frequency, when $\alpha = 1.1\alpha_m$.

frequency is approximately when the curves described by Eqs. (6a,b) are equal, which occurs when the peak displacement is a maximum; this coincides with the so called backbone curve given by $\Omega^2_{\text{backbone}} = 1 + 3/4\alpha Y^2$. The amplitude of vibration Y_d, at the jump-down frequency can be determined by setting the inequality in Eq. (7a) to be an equality, to give

$$Y_d \approx \left(\frac{2}{3\alpha}\left(\left(1+\frac{3\alpha}{4\zeta^2}\right)^{1/2} - 1\right)\right)^{1/2} \tag{8}$$

Eq. (8) can be substituted into either Eq. (6a) or (6b), or the expression for the backbone curve, to give the equation for the jump-down frequency

$$\Omega_d \approx \frac{1}{2^{1/2}}\left(1 + \left(1+\frac{3\alpha}{4\zeta^2}\right)^{1/2}\right)^{1/2} \tag{9}$$

Eqs. (8) and (9) are similar to those given by Magnus in his book [4], but with different notation, and with higher powers of ζ neglected compared to unity. They are different to the expressions given by Nayfeh and Mook [1] and Kevorkian and Cole [3], who used the perturbation method (PM) to determine the jump-down frequency and the amplitude of vibration at this frequency. The expressions given by Nayfeh and Mook, and Kervorkian and Cole are the same, but with different notation. These are given in Appendix A for convenience.

The expression for the jump-down frequency and for the amplitude of vibration at this frequency given in Eqs. (8) and (9) are more general than those determined using the method of multiple scales given in Appendix A, because they allow for a stronger nonlinearity. To make the comparison between the two approaches it must be assumed that $3\alpha/4\zeta^2 \ll 1$, which is quite restrictive if the damping is light. However, if this assumption is made, then the jump-down frequency is given by $\Omega_d \approx 1 + 3\alpha/32\zeta^2$ and the amplitude at the jump-down frequency is given by $Y_d \approx 1/2\zeta$, which are the same as the expressions derived using the perturbation method. This maximum amplitude is the same as that for a linear system. It can be seen that, for this to be true, then the degree of nonlinearity alone is not important, but it is how it compares with the damping in the system, i.e., it is the ratio of $3\alpha/4\zeta^2$ and how this ratio compares with unity which is important. This fact is not clear when the system is considered using the PM alone; it only becomes clear when the system is analysed using the HBM, and the results compared with those from the perturbation method, as outlined above.

Eq. (8) is valid for both a softening and hardening system. However, for the softening system, α is negative so the term in the right inner brackets in Eq. (9) can become negative resulting in Ω_d becoming complex. For Ω_d to be real for a softening system, the following condition is necessary:

$$\alpha \geqslant \alpha_m \tag{10}$$

where $\alpha_m = -4/3\zeta^2$. If this condition is not met then a jump-down does not occur because the two frequency response curves do not meet.

The differences between the predicted jump-down frequency and the amplitude of vibration at this frequency when using the HBM and the PM can be seen in Fig. 2a and b. In Fig. 2a the independent variable is α/ζ^2 and varies from $-4/3$, which is the minimum value for a jump to occur for a softening system, to $5 \times \frac{4}{3}$. It

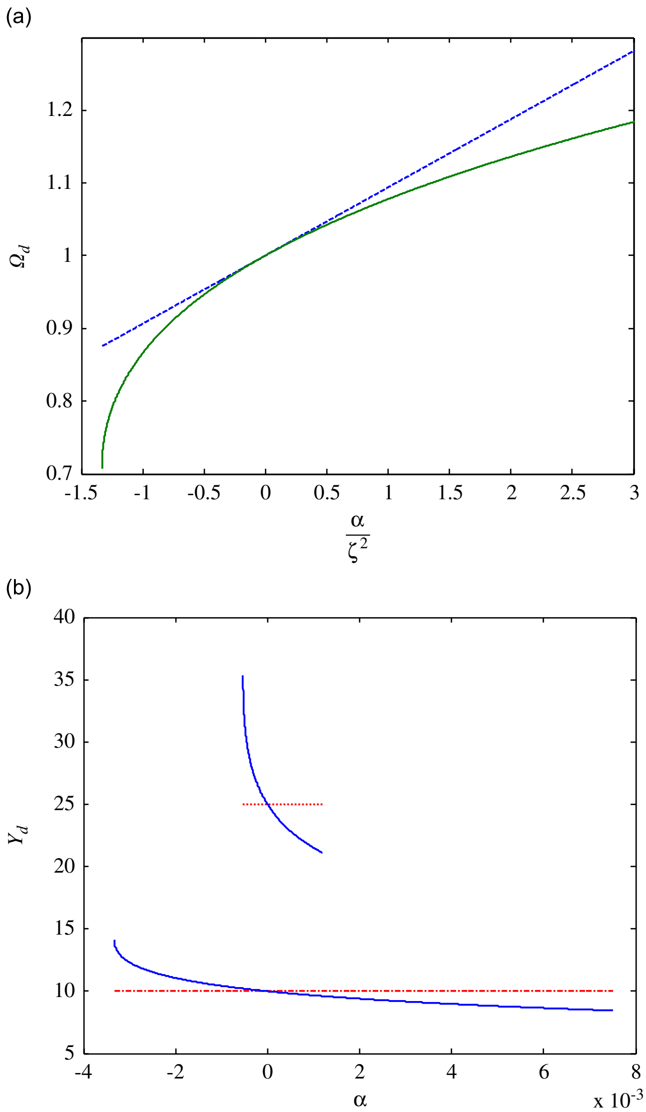

Fig. 2. Comparison between the harmonic balance method (HBM) and perturbation method (PM) and in calculating the jump-down frequency and the amplitude of vibration at this frequency: (a) jump-down frequency; solid line—HBM, dashed line—PM; and (b) non-dimensional amplitude of vibration, solid line—HBM, $\zeta = 0.02$, dotted line…PM, $\zeta = 0.02$, dashed-dotted line—HBM, $\zeta = 0.05$, dashed-dotted line—PM, $\zeta = 0.05$.

can be seen that as the ratio of the nonlinear parameter to the square of the damping ratio increases, the PM becomes more inaccurate. In Fig. 2b it can be seen that the PM predicts that the peak amplitude is independent of nonlinearity, which is clearly not the case for strong nonlinearity, which can be seen in Fig. 1. The accuracy of the HBM in the determination of the jump frequencies is discussed later.

To determine the frequency at which the jump-up occurs, it is noted that this frequency is only weakly dependent upon damping [4,10]. Thus the amplitude of vibration at the jump-up frequency Y_u can be determined by setting $\zeta = 0$ in Eqs. (6a,b) and solving $\mathrm{d}\Omega_1/\mathrm{d}Y = 0$ for Y, to give

$$Y_u \approx \left(\frac{2}{3}\right)^{1/3} \frac{1}{|\alpha|^{1/3}} \tag{11}$$

Eq. (11) can be combined with Eqs. (5a,b) to give the jump-up frequency

$$\Omega_u \approx \left(1 \pm \left(\frac{3}{2}\right)^{4/3} |\alpha|^{1/3}\right)^{1/2} \tag{12}$$

In general $|\alpha| \ll 1$ so that Eq. (12) simplifies to

$$\Omega_u \approx 1 \pm \frac{1}{2}\left(\frac{3}{2}\right)^{4/3} |\alpha|^{1/3} \tag{13}$$

where the $-$ sign is for the softening system when $\alpha < 0$, and the $+$ sign is for the hardening system when $\alpha > 0$. Eq. (13) is the same as the result given by Kevorkian and Cole [3], which is discussed in Appendix A. The jump-up and jump-down frequencies, and the corresponding responses at these frequencies are summarised in Table 1. It should be noted that the jump-up frequency is dependent only upon the nonlinear parameter α, but the jump-down frequency is dependent upon the ratio α/ζ^2.

As mentioned previously, Fig. 1 shows the frequency response curves of the Duffing oscillator calculated using Eqs. (6a,b). For the softening system α was set to $0.9\alpha_m$ and $1.1\alpha_m$ to illustrate the fact that if $\alpha \geqslant \alpha_m$ then

Table 1
Normalised jump-up and jump-down frequencies and the normalised response amplitudes at these frequencies calculated using the harmonic balance method

	Softening system	Hardening system
Jump-up frequency	$\Omega_u = 1 - \frac{1}{2}\left(\frac{3}{2}\right)^{4/3}\|\alpha\|^{1/3}$	$\Omega_u = 1 + \frac{1}{2}\left(\frac{3}{2}\right)^{4/3}\|\alpha\|^{1/3}$
Amplitude at the jump-up frequency	$Y_u = \left(\frac{2}{3}\right)^{1/3}\frac{1}{\|\alpha\|^{1/3}}$	
Jump-down frequency	$\Omega_d \approx \frac{1}{2^{1/2}}\left(1 + \left(1 + \frac{3\alpha}{4\zeta^2}\right)^{1/2}\right)^{1/2}$	
Maximum amplitude	$Y_d \approx \left(\frac{2}{3\alpha}\left(\left(1 + \frac{3\alpha}{4\zeta^2}\right)^{1/2} - 1\right)\right)^{1/2}$	
Threshold value of $\|\alpha\|$ for a jump to occur	$\|\alpha\| \geqslant \frac{2^8}{3^{5/2}}\zeta^3$	
Frequency at which the jump occurs for the threshold value of $\|\alpha\|$	$\Omega = 1 \pm 3^{1/2}\zeta$	

The jump-up frequency and corresponding amplitude are the same as those determined using the perturbation method. The jump-down frequency and amplitude at this frequency calculated using perturbation methods are given by $\Omega_d \approx 1 + 3\alpha/32\zeta^2$ and $Y_d \approx 1/2\zeta$, respectively.

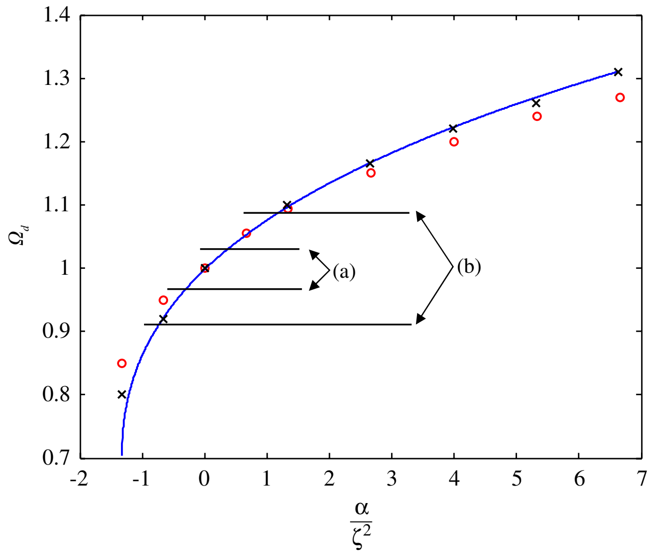

Fig. 3. Jump-down frequencies of the Duffing oscillator as a function of the ratio of non-dimensional parameters α and ζ^2, which governs the degree of nonlinearity and damping, respectively. A negative value of α/ζ^2 denotes a softening system and a positive value of α/ζ^2 denotes a hardening system; solid line—Eq. (9) ○, numerical integration $\zeta = 0.02$; ×, numerical integration $\zeta = 0.05$; (a) and (b) denote the frequencies at which a jump will occur for the threshold value of $|\alpha|$ for $\zeta = 0.02$ and 0.05, respectively.

there is a jump-down, but for $\alpha < \alpha_m$ then a jump-down does not occur. For the hardening system the parameter was set so that $\alpha = -5\alpha_m$, and for the linear system α was set to zero.

The dashed lines in Fig. 1, which are evident between the jump-up and jump-down frequencies, correspond to unstable solutions, which cannot occur in practice [1]. The jump-up and jump-down frequencies calculated using the expressions in Table 1 are marked by '×' and '○', respectively.

The jump-down frequencies as a function of α/ζ^2, calculated using Eq. (9), and the corresponding numerical results calculated for $\zeta = 0.02$ and 0.05 are plotted in Fig. 4. It can be seen that, in this case, the jump frequencies are dependent upon α/ζ^2 for $\zeta \ll 1$. It can also be seen that for the softening system as α approaches α_m, i.e. as $\alpha/\zeta^2 \to -4/3$, the value of the jump-down frequency asymptotes to $1/\sqrt{2}$.

Another check on the validity of the HBM in the determination of the jump frequencies, assuming the response is given approximately by the first harmonic, is to calculate the ratio of the third harmonic to the first harmonic of the response at these frequencies. This was done by calculating the spectrum of the respective time series and computing the ratio numerically. The ratio did not exceed 5% for the parameters used to generate the graphs in Figs. 2 and 3.

There are two sets of frequencies marked (a) and (b), in Fig. 3, and these are discussed in the following section.

The jump-up frequencies as a function of α are plotted in Fig. 4 for $\zeta = 0.02$ and 0.05. The minimum value of α is α_m and maximum is $-5\alpha_m$. Also plotted in Fig. 3 are the responses calculated using numerical integration for some values of α. The weak dependence on damping can be seen in the numerical results, whereas the analytical result of Eq. (12) is, of course, independent of damping. The frequencies marked (a) and (b) are discussed in Section 3.

3. Effect of nonlinearity and damping on the occurrence of the jump phenomena

If the nonlinear parameter $|\alpha|$ is increased from zero, the frequency response curve bends to the left or the right depending on whether the system is softening or hardening, respectively. At a threshold value of $|\alpha|$, the frequency response curve becomes multi-valued above a certain frequency for a hardening system and below a certain frequency for a softening system. This frequency and threshold value has been determined by Malatkar and Nayfeh [10], who formulated the problem in a slightly different way to that presented here. They also used

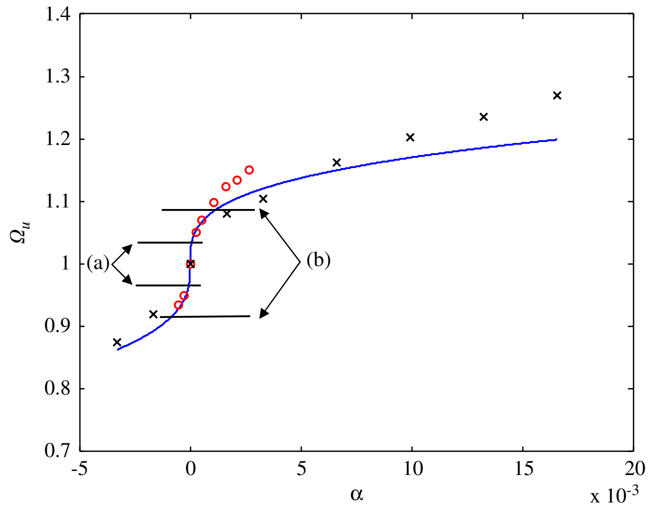

Fig. 4. Jump-up frequencies of the Duffing oscillator as a function of the non-dimensional parameter α, which governs the degree of nonlinearity. A negative value of α denotes a softening system and a positive value of α denotes a hardening system; solid line—Eq. (12); ○, numerical integration $\zeta = 0.02$; ×, numerical integration $\zeta = 0.05$; (a) and (b) denote the frequencies at which a jump will occur for the threshold value of $|\alpha|$ for $\zeta = 0.02$ and 0.05, respectively.

different notation. Thus, the relevant results from their paper and how they relate to those presented here are given in Appendix B. Malatkar and Nayfeh calculated the threshold of the applied force amplitude required for a multi-valued frequency response curve, and hence jump-up and jump-down frequencies; they also calculated the frequency at which this would occur. Using the PM they determined the conditions for an inflexion to occur in the frequency response curve and found that the frequency at which this occurs $\Omega_{\text{inflexion}}$, is given by

$$\Omega_{\text{inflexion}} = 1 \pm 3^{1/2}\zeta \tag{14}$$

which shows that this is dependent upon the damping. The lines marked (a) and (b) in Figs. 3 and 4 indicate the frequencies at which the inflexion given by Eq. (14) occurs, and hence the frequency at which jumps can occur for the threshold value of α. By manipulating the expressions given in Ref. [10], which are repeated in Appendix B, and writing them in terms of the variables used in this article, the criterion for a jump *not* to occur is given by

$$|\alpha| < \frac{2^8}{3^{5/2}}\zeta^3 \tag{15}$$

from which it is very clear that the threshold value of $|\alpha|$ is proportional to the cube of the damping ratio. The value of Y at the frequency given by Eq. (14) and the threshold value of α (when Eq. (15) is an equality) is given by

$$Y = \frac{2^2}{3^{3/4}}\left(\frac{\zeta}{|\alpha|}\right)^{1/2} \tag{16}$$

The maximum displacement response of the Duffing oscillator at the frequency of excitation occurs approximately at the jump-down frequency and is given by Eq. (8). This is plotted together with the numerical simulations as a function of α in Fig. 5 for $\zeta = 0.02$ and 0.05. The threshold values of α, below which a jump will not occur are calculated using Eq. (15) and are also marked on Fig. 5 as (a) and (b) for $\zeta = 0.02$ and 0.05, respectively.

It is interesting to note that a similar dependence of the threshold value of α on the damping ratio, as well as the corresponding frequency at which the inflexion in the frequency response curve, can be derived by equating the jump-down and jump-up frequencies, given by Eqs. (9) and (12), respectively. The resulting value of α is given by

$$\alpha = \pm 3^{1/2} 2^4 \zeta^3 \tag{17}$$

It can be seen, however, by comparing Eq. (15) with (17) that this second approach quantitatively overestimates the threshold value of α determined by Malatakar and Nayfeh by $3^3/2^4$. The corresponding value of frequency associated with the threshold value of α can be determined by substituting for $|\alpha|$ from

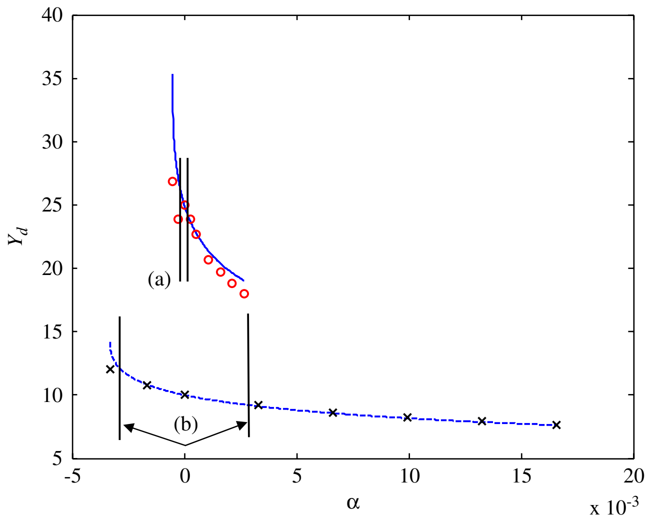

Fig. 5. Non-dimensional maximum amplitude of the Duffing oscillator, which occurs approximately at the jump-down frequency as a function of the non-dimensional parameter α, which governs the degree of nonlinearity. A negative value of α denotes a softening system and a positive value of α denotes a hardening system; solid line—Eq. (8) with $\zeta = 0.02$; dashed line- - -Eq. (8) with $\zeta = 0.05$; ○, numerical integration $\zeta = 0.02$; ×, numerical integration $\zeta = 0.05$; (a) and (b) denote the threshold value of $|\alpha|$ for $\zeta = 0.02$ and 0.05, respectively.

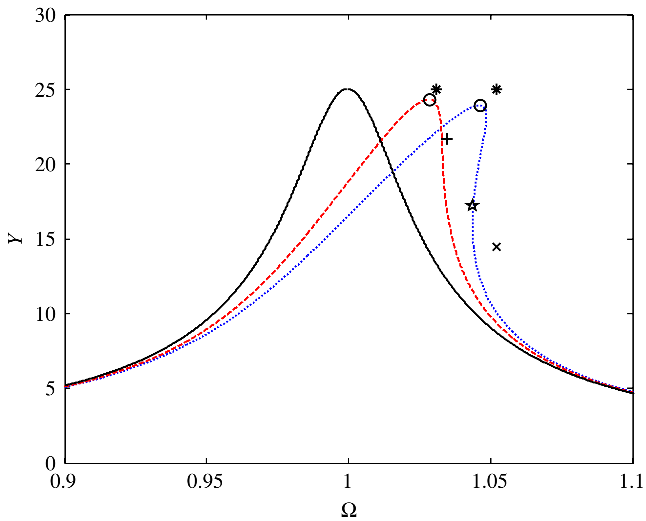

Fig. 6. Non-dimensional displacement as a function of non-dimensional frequency for various values of α and with $\zeta = 0.02$; solid line—linear system with $\alpha = 0$; dashed line- - -α is given by Eq. (15) (as an equality); +, denotes the value of Y given by Eq. (16) at the frequency given by Eq. (14); dotted line…α is given by Eq. (17) (+ve); ○, maximum amplitude given by Eq. (8) at the jump-down frequency given by Eq. (9); *, maximum amplitude given by Eq. (8) at $\Omega_d \approx 1 + 3\alpha/32\zeta^2$; × and ☆, denote the jump-up frequencies calculated using Eq. (13) for threshold values of α given by Eqs. (15) and (17), respectively.

Eq. (17) into (13) to give the linear dependence of frequency on the damping ratio

$$\Omega_{\text{inflexion}} = 1 \pm \frac{3^{3/2}}{2}\zeta \tag{18}$$

which is an overestimate of the critical frequency. The reason why this, and the threshold value of α are overestimated, is because of the approximations made in the estimates of the jump-down and jump-up frequencies given by Eqs. (9) and (12), respectively. The greatest error is in the estimate of the jump-up frequency, because in the derivation it was assumed that damping has little effect on this, which is clearly not the case when α is close to the threshold value. The estimates of the jump-down frequency are also inaccurate when α is small because this frequency is assumed to occur at the maximum value of the frequency response curve, which is also a relatively poor assumption in his case. These points are demonstrated in Fig. 6, which shows three frequency response curves; one is with the threshold value of α given by Eq. (15), the other is with the threshold value of α given by Eq. (17), and the final one is the linear system, plotted for comparison. In all cases damping is set, so that $\zeta = 0.02$. Also marked on the figure are the approximate jump-up and jump-down frequencies and the displacement amplitudes at these frequencies calculated using the expressions derived using the harmonic balance approach, and those calculated using the perturbation method. The quantitative errors in these approximations are evident.

4. Conclusions

In this paper approximate expressions for the jump frequencies and the corresponding displacement amplitudes for the softening and hardening lightly damped Duffing oscillator have been presented. The expressions derived using the HBM have been compared with those derived using a perturbation method, which are valid at frequencies close to the undamped natural frequency of the corresponding linear system. The fundamental assumption in the derivations is that the dominant response of the oscillator to harmonic excitation is at the excitation frequency. In contrast to previous work the results have been presented in terms of two parameters; α, which encompasses the degree of nonlinearity and the magnitude of the excitation force, and ζ, the damping ratio of the system. It has been shown that for a lightly damped system, the jump-up frequency is a function of α, but the jump-down frequency is a function of α/ζ^2. By re-writing results from Ref. [10] in terms of the notation used in this paper, it was shown that for a jump to occur it is necessary for $|\alpha|$ to reach a threshold value, and this is proportional to ζ^3. In addition to determining the jump frequencies, the values of the non-dimensional displacements of the system at these frequencies have also been determined. The analytical results, which have been validated with some numerical simulations for a range of parameters, have been presented in tabular form for ease of reference.

Appendix A. Selected results from Kevorkian and Cole [3]

In Appendix A the results given by Kevorkian and Cole which are pertinent to this paper are given. Where a simple one-to-one mapping of variables exists between analyses, the notation adopted is as used in this work rather than in Ref. [3].

The equation of motion for the Duffing oscillator given by Kevorkian and Cole (Eq. 3.2.142), who use the method of multiple scales, has the form

$$y'' + \alpha\beta y' + y + \alpha y^3 = \alpha f\ \cos((1+\alpha\omega)\tau) \tag{A.1}$$

where the relationship between these variables and those used in this paper are given by

$$\beta = \frac{2\zeta}{\alpha}, \quad f = \frac{1}{\alpha}, \quad 1 + \alpha\omega = \Omega \tag{A.2a−c}$$

It should be noted that because of the term on the right-hand side of Eq. (A.1), the solution is only valid at frequencies close to the resonance frequency of the linear system. Kevorkian and Cole give the jump-up frequency and corresponding amplitude for the hardening system to be (Eq. 3.2.178)

$$\omega = \frac{3^{4/3}}{2^{7/3}} f^{2/3} \tag{A.3}$$

and

$$Y = \left(\frac{2}{3} f\right)^{1/3} \tag{A.4}$$

By using Eqs. (A.2b,c), (A.3) and (A.4) transform to those for the jump-up frequency given by Eq. (13) and the corresponding amplitude at this frequency given by Eq. (11).

Kevorkian and Cole [3] and Nayfeh and Mook [1] both give the same expressions for the peak amplitude and the frequency at which it occurs. They are (Eq. 3.2.180) in [3]

$$Y = \frac{f}{\beta} \tag{A.5}$$

and

$$\omega = \frac{3}{8} Y^2 \tag{A.6}$$

If the appropriate substitutions are made using Eqs. (A.2a–c), then the results $\Omega_d \approx 1 + 3\alpha/32\zeta^2$ and $Y_d \approx 1/2\zeta$ are found.

Appendix B. Selected results from Malatkar and Nayfeh [10]

In Appendix B the results given by Malatkar and Nayfeh [10] which are pertinent to this paper are given. Again, if there is a simple one-to-one mapping of variables between analyses, the notation adopted is as used in this work rather than in Ref. [10].

The critical force required for a jump to occur is given by

$$f = 8\mu\omega_0 \left(\frac{2\mu\omega_0}{3^{3/2}|\gamma|}\right)^{1/2} \tag{B.1}$$

and the corresponding value of the displacement at the inflexion point in the frequency response curve is given by

$$a = \left(\frac{8\mu\omega_0}{3^{1/2}|\gamma|}\right)^{1/2} \tag{B.2}$$

where

$$f = \frac{F}{m}, \quad \mu = \zeta\omega_0, \quad \gamma = \frac{3\omega_0^2}{2x_0^2}\alpha, \quad a = Yx_0 \tag{B.3a−d}$$

Substituting for the variables given in Eqs. (B.3a–c), as well as those for x_0 and ω_0^2 given after Eqs. (2) into (B.1) gives the threshold value for $|\alpha|$ as

$$|\alpha| = \frac{2^8}{3^{5/2}} \zeta^3 \tag{B.4}$$

and substituting for the variables given in Eqs. (B.3b–d) into (B.2) gives the corresponding value of the displacement at the inflexion point

$$Y = \frac{2^2}{3^{3/4}} \left(\frac{\zeta}{|\alpha|}\right)^{1/2} \tag{B.5}$$

Finally, the frequency at which a jump occurs for the threshold value of $|\alpha|$ is

$$\sigma = \pm 3^{1/2}\mu \tag{B.6}$$

where

$$\sigma = \omega_0(\Omega - 1) \tag{B.7}$$

Combining Eqs. (B.3b), (B.6) and (B.7) gives the corresponding non-dimensional frequency to be

$$\Omega_{\text{inflexion}} = 1 \pm 3^{1/2}\zeta \tag{B.8}$$

References

[1] A.H. Nayfeh, D.T. Mook, *Nonlinear Oscillations*, Wiley, New York, 1979.
[2] G. Duffing, *Erzwungene Schwingungen bei veränderlicher Eigenfrequenz und ihre technische Bedeutung*, Vieweg Braunschweig, 1918.
[3] J. Kevorkian, J.D. Cole, *Perturbation Methods in Applied Mathematics*, Springer, New York, 1981.
[4] K. Magnus, *Vibrations*, Blackie and Son, London, 1965.
[5] K. Peleg, Frequency response of non-linear single degree-of-freedom systems, *International Journal of Mechanical Science* 21 (1979) 75–84.
[6] A. Murata, Y. Kume, F. Hashimoto, Application of catastrophe theory to the forced vibration of a diaphragm air spring, *Journal of Sound and Vibration* 112 (1) (1987) 31–44.
[7] M.I. Friswell, J.E.T. Penny, The accuracy of jump frequencies in series solutions of the response of a Duffing oscillator, *Journal of Sound and Vibration* 169 (2) (1994) 261–269.
[8] K. Worden, On jump frequencies in the response of a Duffing oscillator, *Journal of Sound and Vibration* 198 (4) (1996) 522–525.
[9] A. Carrella, *Passive Vibration Isolators with High-Static-Low-Dynamic-Stiffness*, PhD Thesis, University of Southampton, 2008.
[10] P. Malatkar, A.H. Nayfeh, Calculation of the jump frequencies in the response of a s.d.o.f. non-linear systems, *Journal of Sound and Vibration* 254 (5) (2002) 1005–1011.
[11] P. Hagedorn, *Non-linear Oscillations (Translated and edited by Wolfram Stadler)*, Clarendon Press, Oxford, 1981.