Available online at www.sciencedirect.com
ScienceDirect
Journal of Sound and Vibration 318 (2008) 757–767
JOURNAL OF SOUND AND VIBRATION
www.elsevier.com/locate/jsvi

Calculation of the rightmost characteristic root of retarded time-delay systems via Lambert W function

Z.H. Wang[a,b,*], H.Y. Hu[b]

[a]Department of Applied Mathematics and Physics, PLA University of Science and Technology, 210007 Nanjing, China
[b]Institute of Vibration Engineering Research, Nanjing University of Aeronautics and Astronautics, 210016 Nanjing, China

Received 2 February 2007; received in revised form 18 February 2008; accepted 27 April 2008
Handling Editor: L.G. Tham
Available online 10 June 2008

Abstract

Generally, it is not easy to analyze the stability of time-delay systems, especially when the systems are of high order or they have multiple delays. For *retarded* time-delay systems, the stability can be determined by the rightmost characteristic root. This paper presents a case study on the calculation of the rightmost root. Three practical time-delay systems are discussed. The first system is an oscillator with delayed state feedback, the second one is a delayed neural network based on the FitzHugh–Nagumo model for neural cells, and the third one is a car model of suspension with a delayed sky-hook damper. By using the Lambert W function, the rightmost root becomes a root of a function associated with the principal branch of the Lambert W function. Then the rightmost root is located by using Newton–Raphson's scheme or Halley's accelerating scheme. Some suggestions for successful application of the proposed method are given.

1. Introduction

With the rapid development of the active control techniques, much attention has been paid to the dynamical systems with time delays over the past decades. It is frequently the time delay, caused usually from controllers, actuators, human–machine interaction, etc. [1–4], that renders the systems unstable, so the stability analysis is essentially important in the dynamics and control of time-delay systems (TDSs for short), and it can be analyzed by means of the Lyapunov's method including the linear matrix inequality (LMI) technique and the root location of the characteristic quasi-polynomials. Each of the two methods has advantages over the other. The Lyapunov's function(al)-based methods work locally around an equilibrium or globally, but they usually yield conservative results. While the method of characteristic function works locally around the equilibrium only, it may give deliberate results about the stability analysis. The books [2–4] present comprehensive discussion on the stability analysis of TDSs.

*Corresponding author at: Institute of Vibration Engineering Research, Nanjing University of Aeronautics and Astronautics, 210016 Nanjing, China.
E-mail address: zhwang@nuaa.edu.cn (Z.H. Wang).

0022-460X/$ - see front matter
doi:10.1016/j.jsv.2008.04.052

The maximal real part of the characteristic roots of a TDS is usually called "abscissa". For TDS with real coefficients, the abscissa corresponds to one real root or one pair of complex conjugate roots of the characteristic quasi-polynomial. Such a root or a pair of conjugate roots will be simply called the rightmost characteristic root (or, the rightmost root). An equilibrium of a TDS is asymptotically stable if and only if the abscissa is negative, namely all the characteristic roots have negative real parts. A great number of criteria are available for checking whether the abscissa is negative or not, including the Pontryakin criterion [4], the Hassard criterion [5] and the Nyquist criterion [6]. Such criteria can be applied in the determination of the admissible feedback gains of PID controllers [7]. In many applications such as in delayed resonator vibration absorbers [8] and vehicle systems [9], the systems are high dimensional and possibly with multiple delays, so that it is usually impossible to get a stability criterion in closed form. On the other hand, a stable system may have very poor performance. This is the case when the abscissa is negative but it is very close to zero. In the design phase of controllers, the admissible feedback gains falling in such a case is useless in practical applications. Thus, it is highly demanded to develop some effective algorithms for the calculation of the abscissa (or, the rightmost root) of a give time-delay system.

For some simple TDSs, the characteristic roots can be expressed explicitly in terms of the Lambert W function (which has infinite many branches) [10–13], so that the abscissa has a closed form in terms of the principal branch only. Unfortunately, it is usually required to find out the characteristic roots numerically such as by using iteration methods, because no closed form of the characteristic roots exists for general TDSs [14]. Though the iteration methods may work effectively in finding a characteristic root, no criteria are available to determine whether the root is the rightmost one or not, because a quasi-polynomial has an infinite number of roots. On the other hand, within a restricted bounded region in the complex plane there are finite many characteristic roots only [4], so in Ref. [15], a computational method was proposed on the basis of the Zakian's iteration [16] and the searching technique. Zakian's algorithm may also work effectively for TDSs if the Nyquist criterion is combined. Another iteration algorithm for finding the abscissa was reported in Ref. [17], where starting from a properly chosen initial guess of the rightmost characteristic root, each iteration needs to fix the exponential factors so that the quasi-polynomial is simplified to a polynomial and then the rightmost root of the simplified polynomial can be obtained to serve as the new guess. The problem is that the iterative sequence in Ref. [17] is frequently not convergent.

The main objective of this paper is to present a case study on the computation of the rightmost characteristic root of time-delay systems, on the basis of the Lambert W function. To this end, some basic facts about the Lambert W function are given firstly in Section 2, then two algorithms are presented for finding the rightmost root. From Sections 3–5, the iteration method is applied to calculate the rightmost root of three practical time-delay systems. The first system is an oscillator with delayed feedback, the second one is a delayed neural network based on the FitzHugh–Nagumo model for neural cells, and the third one is a car model of suspension with a delayed sky-hook damper. Finally, some concluding remarks and suggestions for successful application of the method are given in Section 6.

2. The Iteration Method

In this paper, attention is paid to the stability analysis of retarded-type time–delay systems, whose characteristic function are quasi-polynomials of the form

$$p(\lambda) := \lambda^n + \sum_{i=1}^{n} a_i(e^{-\lambda\tau_1}, e^{-\lambda\tau_2}, \ldots, e^{-\lambda\tau_m})\lambda^{n-i} \tag{1}$$

where $\tau_j > 0$, $(j = 1, 2, \ldots, m)$, are the time delays, and the coefficients $a_i(x_1, x_2, \ldots, x_m)$, $(i = 1, 2, \ldots, n)$, of λ^{n-i} in $p(\lambda)$ are polynomials with respect to $x_1, x_2, \ldots, x_m$. Let α_0 be the abscissa

$$\alpha_0 := \max\{\mathrm{Re}(\lambda) : p(\lambda) = 0\} \tag{2}$$

where $\mathrm{Re}(z)$ stands for the real part of z. Then the equilibrium of the TDS is asymptotically stable if and only if $\alpha_0 < 0$. An iteration method will be presented for the calculation of the rightmost root so that the stability of a given time-delay system is determined.

2.1. The Lambert W function

The proposed iteration method for finding the rightmost characteristic root of TDS is based on the Lambert W function [10], which will be firstly introduced briefly in this subsection. The Lambert W function $w = W(z)$ is defined as the solution of a complex transcendental equation

$$we^{w} = z \quad (z \in \mathbb{C}) \tag{3}$$

The solution $W(z)$ has infinite many branches, denoted by $W_k(z)$, $k = 0, \pm 1, \pm 2, \ldots, \pm\infty$, respectively. The branches can be presented as below [10,11]

$$W_0(z) = \sum_{n=1}^{\infty} \frac{(-n)^{n-1}}{n!} z^n$$

$$W_k(z) = \ln_k(z) - \ln(\ln_k(z)) + \sum_{l=0}^{\infty}\sum_{m=1}^{\infty} C_{lm} \frac{(\ln(\ln_k(z)))^m}{(\ln_k(z))^{l+m}}$$

where $\ln_k(z) = \ln(z) + 2\pi k\mathrm{i}$ indicates the kth logarithm branch, and the coefficients C_{lm} can be expressed in terms of Stirling cycle numbers $C_{lm} = \frac{(-1)^l}{m!}\left[\begin{smallmatrix} l+m \\ l+1 \end{smallmatrix}\right]$. $W_0(z)$ is the unique branch that is analytic at the origin $z = 0$ and is called the principal branch. For more details about the Lambert W function, it is referred to Refs. [10–12]. In particular, it has been proved in Ref. [13] that for given $z \in \mathbb{C}$, $\mathrm{Re}[W_k(z)]$ is decreasing with respect to k if the integer $k > 0$, and it is increasing in k if $k < 0$. In addition, one has [13]

$$\mathrm{Re}[W_0(z)] = \max_{k=0,\pm 1,\pm 2,\ldots,\pm\infty} \mathrm{Re}[W_k(z)] \quad (\forall z \in \mathbb{C}) \tag{4}$$

Thus, the zeros of the following quasi-polynomial:

$$p(\lambda) = \lambda - a - b\mathrm{e}^{-\lambda\tau} \quad (a, b \in \mathbb{C}) \tag{5}$$

have a closed form in terms of the Lambert W function [11]. In fact, $p(\lambda) = 0$ implies $(\lambda - a)\tau\mathrm{e}^{(\lambda-a)\tau} = \tau b\mathrm{e}^{-a\tau}$, thus, the infinite many roots of $p(\lambda) = 0$ can be expressed by

$$\lambda = a + \frac{1}{\tau} W_k(\tau b\mathrm{e}^{-a\tau}), \quad (k = 0, \pm 1, \pm 2, \ldots) \tag{6}$$

Thus, all the roots of $p(\lambda)$ have negative real part if and only if

$$\alpha_0 = \mathrm{Re}(a) + \frac{1}{\tau}\mathrm{Re}(W_0(\tau b\mathrm{e}^{-a\tau})) < 0 \tag{7}$$

An extension is also made in Ref. [11] for some TDSs of high order. It is worthy to note that Maple, Matlab and Mathematica, the three well-known computer algebras, provide a calculator of the Lambert W function. So with the help of Maple, the computation related to the Lambert W function is easy tractable.

For demonstration, let us study the stability of a time-delay system arising in visually guided movement [18]

$$\dot{\phi}(t) = -\alpha \sin[\phi(t) - \omega\tau] - \beta \sin[\phi(t - \tau)] \tag{8}$$

where ϕ stands for the phase difference between the target signal and the tracking signal, α, β and ω are positive real constants. Consider the equilibrium of the form [18]

$$\phi_0 = \frac{\omega\tau}{2} + \arctan\left(\frac{\alpha - \beta}{\alpha + \beta}\tan\frac{\omega\tau}{2}\right)$$

Let $y(t) = \phi(t) - \phi_0$, then the characteristic quasi-polynomial of the linearized equation $\dot{y}(t) = a(\tau)y(t) + b(\tau)y(t - \tau)$ is $p(\lambda) = \lambda - a(\tau) - b(\tau)\mathrm{e}^{-\lambda\tau}$ with $a(\tau)$ and $b(\tau)$ given by

$$a(\tau) = -\alpha\cos\left[\frac{\omega\tau}{2} - \arctan\left(\frac{\alpha - \beta}{\alpha + \beta}\tan\frac{\omega\tau}{2}\right)\right], \quad b(\tau) = -\beta\cos\phi_0$$

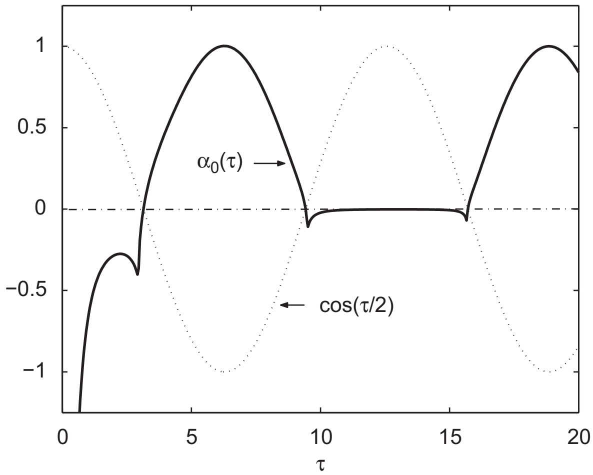

Fig. 1. The plot of the abscissa α_0 vs τ of Eq. (8) for $\omega = 1$, $\alpha = 1$ and $\beta = 1$. Stability switches [1] occur at $\tau = \hat{\tau}_k = (2k+1)\pi$, $(k = 0, 1, 2, \ldots)$. In $\tau \in (\hat{\tau}_1, \hat{\tau}_2)$, the stability is very "poor".

The equilibrium ϕ_0 is asymptotically stable if and only if

$$\alpha_0 := a(\tau) + \frac{1}{\tau}\mathrm{Re}[W_0(\tau b(\tau)\mathrm{e}^{-a(\tau)\tau})] < 0 \tag{9}$$

Fig. 1 is the plot of the abscissa α_0 vs τ for $\alpha = \beta = \omega = 1$. The equilibrium ϕ_0 is asymptotically stable for $\tau \in [0, \pi) \cup (3\pi, 5\pi) \cup \cdots$, but the performance of the stable system with $\tau \in (3\pi, 5\pi) \cup \cdots$ is very *poor*.

2.2. Algorithms

Assume that λ is a root of the characteristic equation $p(\lambda) = 0$, then

$$(a\lambda + b)\mathrm{e}^{a\lambda+b} = (a\lambda + b - p(\lambda))\mathrm{e}^{a\lambda+b}$$

for some properly chosen constants $a > 0$ and b, thus, there is a $i \geqslant 0$ such that λ is a root of

$$a\lambda + b = W_i((a\lambda + b - p(\lambda))\mathrm{e}^{a\lambda+b})$$

Due to Eq. (4), it is expected to find out the rightmost root through the principal branch W_0 of the Lambert W function, hence, an iteration sequence can be constructed intuitively from

$$a\lambda_{k+1} + b = W_0((a\lambda_k + b - p(\lambda_k))\mathrm{e}^{a\lambda_k+b}) \quad (k = 0, 1, 2, \ldots) \tag{10}$$

However, such a sequence is frequently not convergent. This is the common problem that encountered in solving nonlinear equation by using the iteration method. That is, one usually cannot find out a root of $f(x) = 0$ simply from the iteration sequence $x_{k+1} = x_k - f(x_k)$. Instead, the algorithms based on the Newton–Raphson scheme should be employed. Let

$$F(\lambda) := a\lambda + b - W_0((a\lambda + b - p(\lambda))\mathrm{e}^{a\lambda+b}) \tag{11}$$

and λ_0 be an initial guess, then the rightmost root of $p(\lambda)$ can be found from one of the following two iteration schemes:

- *Newton–Raphson's scheme* [19]

$$\lambda_{i+1} = \lambda_i - \frac{F(\lambda_i)}{F'(\lambda_i)} \quad (i = 0, 1, 2, \ldots) \tag{12}$$

- *Halley's accelerating scheme* [19]

$$\lambda_{i+1} = \lambda_i - \frac{F(\lambda_i)}{F'(\lambda_i)}\left(1 - \frac{F(\lambda_i)F''(\lambda_i)}{2(F'(\lambda_i))^2}\right)^{-1} \quad (i = 0, 1, 2, \ldots) \tag{13}$$

where the derivative can be computed by using the property

$$W_0'(z) = \frac{W_0(z)}{z + zW_0(z)}$$

For a given tolerance ε, the iteration is stopped if

$$|\lambda_{i+1} - \lambda_i| < \varepsilon \tag{14}$$

Halley's scheme is the standard algorithm in solving nonlinear equations related to the Lambert W function in the software Maple.

In general, an iteration method works only if the initial guess is close to the root of concern. Thus, a successful application of the two iteration schemes depends on a proper chosen λ_0. There are many ways to choose an initial guess. For example, for a freely given $\hat{\lambda}_0$, we firstly solve the polynomial equation

$$\lambda^n + \sum_{i=1}^{n} a_i(\mathrm{e}^{-\hat{\lambda}_0\tau_1}, \mathrm{e}^{-\hat{\lambda}_0\tau_2}, \ldots, \mathrm{e}^{-\hat{\lambda}_0\tau_m})\lambda^{n-i} = 0 \tag{15}$$

then the initial guess λ_0 can be taken as the rightmost root of Eq. (15), by following the idea of Krodkiewski and Jintanawan [17]. Eq. (15) can also be replaced with $\lambda^n + \sum_{i=1}^{n} a_i(1, 1, \ldots, 1)\lambda^{n-i} = 0$, the case of $\hat{\lambda}_0 = 0$ for Eq. (15).

Finally it remains to confirm the iteration result. The Nyquist plot is preferred for this purpose. Let

$$R(\omega) + \mathrm{i}, S(\omega) := \frac{p(\mathrm{i}\omega)}{(1 + \mathrm{i}\omega)^n} \tag{16}$$

As proved in Ref. [6], all the roots of $p(\lambda)$ have negative real parts if and only if the Nyquist plot of $p(\mathrm{i}\omega)/(1 + \mathrm{i}\omega)^n$, namely the plot of

$$\{(R(\omega), S(\omega)) : \omega \in (-\infty, +\infty)\} \tag{17}$$

does not encircle the origin of the complex plane, and it has at least one root with positive real part if the Nyquist plot encircles the origin. Let $\lambda = s + \alpha$, then $\mathrm{Re}(\lambda) \leqslant \alpha$ if and only if $\mathrm{Re}(s) \leqslant 0$. Hence, $\lambda = \lambda^*$ is the rightmost root if and only if (i). the Nyquist plot of $p(\mathrm{i}\omega + \mathrm{Re}(\lambda^*))/(1 + \mathrm{i}\omega)^n$ *passes through* the origin of the complex plane, and (ii). the Nyquist plot of $p(\mathrm{i}\omega + \mathrm{Re}(\lambda^*) + \eta)/(1 + \mathrm{i}\omega)^n$ *does not encircle* the origin, for *any small* $\eta > 0$. The first condition indicates that $p(s + \mathrm{Re}(\lambda^*))$ has a root with zero real part, namely $p(\lambda)$ has a root with real part $\mathrm{Re}(\lambda^*)$. The second condition implies that all the roots of $p(s + \mathrm{Re}(\lambda^*) + \eta)$ have negative real parts, namely $p(\lambda)$ have roots with real parts *less than* $\mathrm{Re}(\lambda^*) + \eta$ only, and consequently, $p(\lambda)$ have roots with $\mathrm{Re}(\lambda) \leqslant \mathrm{Re}(\lambda^*)$ only, because η is an arbitrary small number.

3. The rightmost characteristic root of an oscillator with delayed state feedback

Now, we give three practical examples to demonstrate the proposed iteration method for finding the rightmost root of the characteristic equations. Firstly, let us study the stability of an oscillator with delayed state feedback

$$\ddot{x}(t) + \xi\dot{x}(t) + kx(t) = ux(t - \tau_1) + v\dot{x}(t - \tau_2) \tag{18}$$

whose characteristic equation reads

$$p(\lambda) := \lambda^2 + \xi\lambda + k - u\mathrm{e}^{-\lambda\tau_1} - v\lambda\mathrm{e}^{-\lambda\tau_2} = 0 \tag{19}$$

where $\tau_1 > 0$, $\tau_2 > 0$ are the time delays. Without loss of generality, assume that $\xi \neq 0$. The function $F(\lambda)$ in Eq. (11) reads

$$F(\lambda) = \xi\lambda + k - W_0(-(\lambda^2 - u\mathrm{e}^{-\lambda\tau_1} - v\lambda\mathrm{e}^{-\lambda\tau_2})\mathrm{e}^{\xi\lambda+k}) \tag{20}$$

In order to avoid numerical problems, the values of ξ and k should be assumed *not large*. Otherwise, a scaling factor β is usually required so that the constants $\beta\xi$ and βk in the function

$$F(\lambda) = \beta(\xi\lambda + k) - W_0(-\beta(\lambda^2 - u\mathrm{e}^{-\lambda\tau_1} - v\lambda\mathrm{e}^{-\lambda\tau_2})\mathrm{e}^{\beta(\xi\lambda+k)}) \tag{21}$$

are not large. Then, the rightmost characteristic root can be found by using Newton–Raphson's scheme (12) or Halley's accelerating scheme (13). To show the validity of this method, let us check some special cases (randomly chosen), where the tolerance $\varepsilon = 10^{-4}$ is fixed.

Example 1. $\xi = 1$, $k = 4$, $u = -2$, $v = 0$ and $\tau_1 = 0.5$. In this case, a good initial guess is $\lambda_0 = -0.5 + 2.0\mathrm{i}$, which is close to one of the solutions of the simplified polynomial equation $\lambda^2 + \lambda + 4 - (-2)\mathrm{e}^{0\cdot\tau_1} = 0$. Then with the tolerance $\varepsilon = 10^{-4}$, the third Newton–Raphson's iteration and the second Halley's iteration give the same result $\lambda = -0.7906 \times 10^{-1} + 0.2206 \times 10^1\mathrm{i}$. The same result can be obtained after 23 iterations (Newton–Raphson's scheme) and 7 iterations (Halley's scheme), respectively, if $\lambda_0 = 15 + 1.1\mathrm{i}$. Moreover, one can confirm that $\lambda = -0.7906 \times 10^{-1} \pm 0.2206 \times 10^1\mathrm{i}$ is the rightmost characteristic root by using the Nyquist plot in Fig. 2.

Example 2. $\xi = 1$, $k = 4$, $u = -2$, $v = 0$ and $\tau_1 = 2.5$. In this case, $\lambda_0 = 15 + 1.1\mathrm{i}$ is a *bad choice* of initial guess for Newton–Raphson's scheme which results in an un-convergent sequence, but it gives a convergent result $\lambda = -0.1264 + 0.1131 \times 10^1\mathrm{i}$ after 7 iterations of Halley's scheme. Again, $\lambda_0 = -0.5 + 2.0\mathrm{i}$ is a good choice of the initial guess. Together with the Nyquist test shown in Fig. 2, the iteration method computes the rightmost root to be $\lambda = -0.1264 \pm 0.1131 \times 10^1\mathrm{i}$ after 5 iterations (Newton–Raphson's scheme) and 4 iterations (Halley's scheme), respectively.

Example 3. $\xi = -1$, $k = 4$, $u = -2$, $v = -0.1$, $\tau_1 = 2.5$ and $\tau_2 = 1.5$. In this case, starting from the initial $\lambda_0 = 0.5 + 2.0\mathrm{i}$, the rightmost root can be found out and confirmed by means of the Nyquist plot to be $\lambda = 0.3209 \pm 0.2017 \times 10^1\mathrm{i}$ after 3 iterations (Newton–Raphson's scheme) and 2 iterations (Halley's scheme), respectively.

Example 4. $\xi = -1$, $k = 1.2$, $u = -2$, $v = -1.2$, $\tau_1 = 1.2$ and $\tau_2 = 1.8$. In this case, let $\lambda_0 = 0.5 + 2.0\mathrm{i}$ be the initial guess, then the rightmost root can be found out and confirmed by means of the Nyquist plot to be $\lambda = 0.3209 \pm 0.2017 \times 10^1\mathrm{i}$ after 6 iterations (Newton–Raphson's scheme) and 3 iterations (Halley's scheme), respectively.

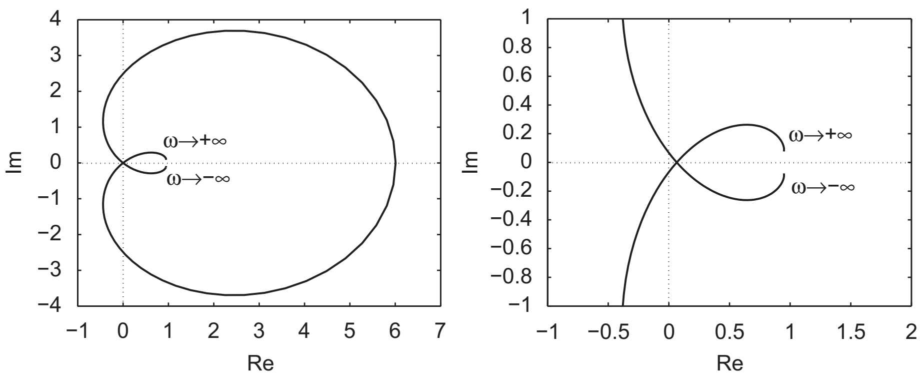

Fig. 2. For $\xi = 1$, $k = 4$, $u = -2$, $v = 0$, $\tau_1 = 0.5$ and $\tau_2 = 0$, the rightmost root of $p(\lambda)$ for Eq. (18) is $\lambda = -0.7906 \times 10^{-1} \pm 0.2206 \times 10^1\mathrm{i}$. (Left) The Nyquist plot of $p(\mathrm{i}\omega - 0.7906 \times 10^{-1})/(1 + \mathrm{i}\omega)^2$ *passes through* the origin of the complex plane. (Right) Zoom around the origin of the Nyquist plot of $p(\mathrm{i}\omega - 0.7906 \times 10^{-1} + 0.05)/(1 + \mathrm{i}\omega)^2$, which *does not encircle* the origin of the complex plane.

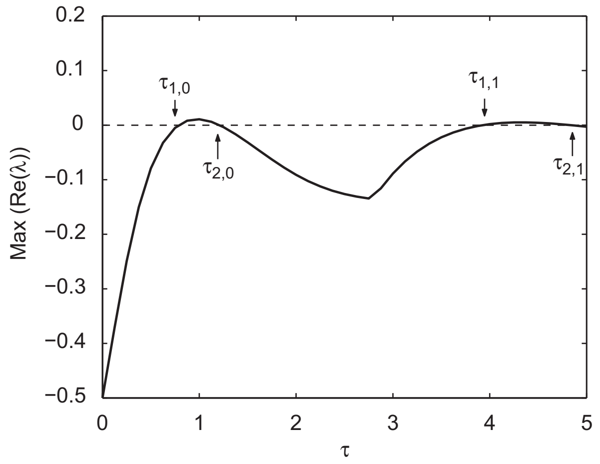

Fig. 3. The plot of the abscissa α_0 vs τ for Eq. (18) with $\xi = 1$, $k = 4$, $u = -2$ and $v = 0$, computed by using Halley's scheme (13) with $\lambda_0 = 0.3 + 2.0\mathrm{i}$ and $\varepsilon = 10^{-8}$.

The iteration method can produce the curve of the abscissa vs a parameter, similar to Fig. 1. For example, let $\tau_1 = \tau_2 = \tau$ be the control parameter. At $\tau = 0$, the trivial solution is asymptotically stable because $p(\lambda)$ is Hurwitz. As τ increases, the trivial solution keeps to be asymptotically stable until $\tau = \tau_j$ for which the characteristic equation has a pair of conjugate pure imaginary roots $\lambda = \pm\mathrm{i}\omega$. More precisely, let $\xi = 1$, $k = 4$, $u = -2$, $v = 0$, as discussed in Example 1, and $\tau_1 = \tau$ be the control parameter, then $p(\pm\mathrm{i}\omega) = 0$ yields $|(\mathrm{i}\omega)^2 + \mathrm{i}\omega + 4|^2 - (-2)^2 = 0$, namely $\omega^4 - 7\omega^2 + 12 = 0$. This equation has four roots $\omega_{1,2} = \pm 2$, $\omega_{3,4} = \pm\sqrt{3}$. The corresponding critical delay values are

$$\tau_{1,j} = j\pi + \frac{\pi}{4}, \quad \tau_{2,j} = [(2j+1)\pi - \frac{\pi}{3}]/\sqrt{3} \quad (j = 0, 1, 2, \ldots)$$

Since $\tau_{2,0} = 1.209 > 0.7854 = \tau_{1,0}$ and

$$\mathrm{Re}[\lambda'(\tau_{1,j})] > 0, \quad \mathrm{Re}[\lambda'(\tau_{2,j})] < 0 \quad (j = 0, 1, 2, \ldots)$$

the trivial solution $x = 0$ is asymptotically stable for $\tau \in [0, \tau_{1,0}) \cup (\tau_{2,0}, \tau_{1,1}) \cup (\tau_{2,1}, \tau_{1,2}) \cdots \cup (\tau_{2,5}, \tau_{1,6})$, and it is unstable for $\tau \in (\tau_{1,0}, \tau_{2,0}) \cup (\tau_{1,1}, \tau_{2,1}) \cup \cdots (\tau_{1,5}, \tau_{2,5}) \cup (\tau_{1,6}, +\infty)$ [1]. Fig. 3 shows that the stability analysis given above is the same with the plot of the abscissa by using the iteration method.

4. The rightmost characteristic root of a third-order time-delay system arising from neuron networks

Next, let us consider the following time-delay system based on the FitzHugh–Nagumo model for neural cells [20]

$$\begin{cases} \tau\dot{u}(t) = -u(t) + qg(v(t-T)) + e \\ \dot{v}(t) = c(w(t) + v(t) - \frac{1}{3}v^3(t)) + u(t) \\ \dot{w}(t) = (a - v(t) - bw(t))/c \end{cases} \tag{22}$$

where u denotes the total postsynaptic potential, v is the membrane potential, w is an auxiliary variable, and the time delay T is the propagation time of neural signals between the axon and dendrites. When $g(x) = 1/(1 + \mathrm{e}^{-4x})$, the characteristic function corresponding to the stationary solution $(\bar{u}, \bar{v}, \bar{w})$ reads

$$p(\lambda) = \lambda^3 + k_{20}\lambda^2 + (k_{11}\mathrm{e}^{-\lambda T} + k_{10})\lambda + k_{01}\mathrm{e}^{-\lambda T} + k_{00}$$

or in short

$$p(\lambda) = k_{10}\lambda + k_{00} + q(\lambda, \mathrm{e}^{-\lambda T})$$

where

$$k_{20} = (\tau c^2 \bar{v}^2 + c + \tau b - \tau c^2)/(\tau c)$$

$$k_{11} = (-4q\mathrm{e}^{-4\bar{v}})/((1 + \mathrm{e}^{-4\bar{v}})^2 \tau)$$

$$k_{10} = (-\tau cb + \tau c\bar{v}^2 b + \tau c + b + c^2\bar{v}^2 - c^2)/(\tau c)$$

$$k_{01} = bk_{11}/c$$

$$k_{00} = (c + c\bar{v}^2 b - cb)/(\tau c)$$

By using the Lambert W function, it is easy to see that each root of $p(\lambda)$ is a root of one of the following functions:

$$F(j,\lambda) := k_{10}\lambda + k_{00} - W_j(-q(\lambda, \mathrm{e}^{-\lambda T})\mathrm{e}^{k_{10}\lambda + k_{00}}), \quad j = 0, \pm 1, \pm 2, \ldots$$

In what follows, Halley's scheme will be used to find the rightmost root of Eq. (22) through the function

$$F(\lambda) := k_{10}\lambda + k_{00} - W_0(-q(\lambda, \mathrm{e}^{-\lambda T})\mathrm{e}^{k_{10}\lambda + k_{00}}) \tag{23}$$

When $a = 0.9$, $b = 0.9, c = 2.0$, $q = -1.0$, $e = -2.5$, $\tau = 40.0$, $T = 30.0$ [20], the system has a stationary solution $(\bar{u}, \bar{v}, \bar{w}) = (-2.5374, -0.8120, 1.9022)$. In this case, the simplified polynomial (the transcendental

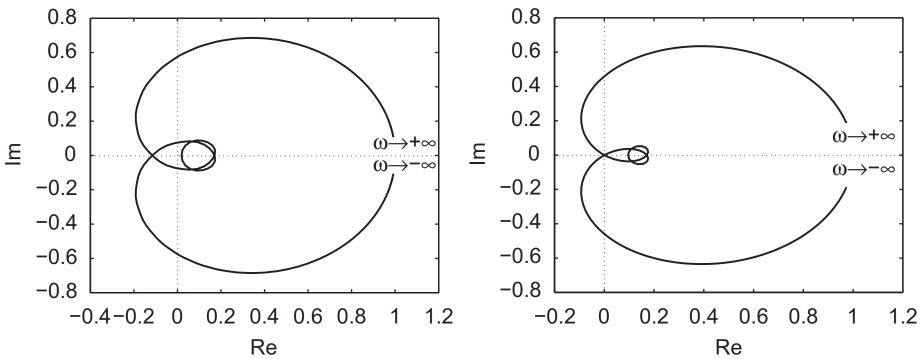

Fig. 4. When $a = 0.9$, $b = 0.9$, $c = 2.0$, $q = -1.0$, $e = -2.5$, $\tau = 40.0$, $T = 30.0$, Eq. (22) has a stationary solution $(\bar{u}, \bar{v}, \bar{w}) = (-2.5374, -0.8120, 1.9022)$. (Left) The Nyquist plot of $p(\mathrm{i}\omega)/(1 + \mathrm{i}\omega)^3$ *encircles* the origin of the complex plane, where the curve ends at the limit point $(1, 0)$. (Right). The Nyquist plot of $p(\mathrm{i}\omega + 0.1156)/(1 + \mathrm{i}\omega)^3$ *passes through* the origin of the complex plane.

Table 1
The rightmost characteristic root for Eq. (22) with $a = 0.9$, $b = 0.9$, $c = 2.0$, $q = -1.0$, $\tau = 40.0$ and $T = 30.0$, starting from $0.1156 \pm 0.8247\mathrm{i}$

Parameter e	The rightmost root λ
$e = -2.55$	$0.6795 \times 10^{-1} + 0.8557\mathrm{i}$
$e = -2.6$	$0.2137 \times 10^{-1} + 0.8823\mathrm{i}$
$e = -2.62$	$0.3073 \times 10^{-2} + 0.8915\mathrm{i}$
$e = -2.6234$	$0.8929\mathrm{i}$
$e = -2.625$	$-0.1473 \times 10^{-2} + 0.8936\mathrm{i}$
$e = -2.63$	$-0.6008 \times 10^{-2} + 0.8960\mathrm{i}$

terms involving $e^{-T\lambda}$ in $p(\lambda)$ are replaced with 0) is

$$\lambda^3 - 0.2062551500\lambda^2 + 0.6876538038\lambda + 0.01733587956 = 0$$

which has 3 roots: $0.1156 \pm 0.8247i, -0.2500 \times 10^{-1}$. Thus, $\lambda_0 = 0.1156 \pm 0.8247i$ is a proper initial guess for the iteration scheme. The first Halley's iteration is the same as λ_0. It means that the rightmost characteristic root $0.1156 \pm 0.8247i$, which is the same as that obtained in Ref. [20] by using DDE BIFTOOL. In addition, the Nyquist plots in Fig. 4 confirms that the rightmost characteristic root is $0.1156 \pm 0.8247i$.

The roots of $p(\lambda)$ depend continuously on the parameter e, so a complex number near the rightmost root $0.1156 \pm 0.8247i$ for $e = -2.5$ is a suitable initial estimation for e near -2.5. Table 1 shows the variation of the rightmost root of the characteristic quasi-polynomial of Eq. (22) with respect to e, and it tells that a Hopf bifurcation occurs at $e = -2.6234$. The corresponding Nyquist plots confirm all the numerical rightmost roots given in Table 1.

5. The rightmost characteristic root of a quarter car model of suspension with a delayed sky-hook damper

Finally let us consider a time-delay system of two degrees of freedom arising from vehicle dynamics in dimensionless form [9,1]

$$\begin{cases} \ddot{x}(t) + c[\dot{x}(t) - \dot{y}(t)] + [x(t) - y(t)] + v\dot{x}(t-\tau) = 0 \\ \ddot{y}(t) - \beta c[\dot{x}(t) - \dot{y}(t)] - \beta[x(t) - y(t)] + \beta k y(t) - \beta v \dot{x}(t-\tau) = 0 \end{cases} \tag{24}$$

where x denotes the vertical displacement of the vehicle body, y denotes the vertical displacement of the unsprung mass, and τ is the time delay in the feedback control of the sky-hook damper, all are in dimensionless form. The characteristic quasi-polynomial is

$$p(\lambda) = \lambda^4 + c(1+\beta)\lambda^3 + (1 + \beta + \beta k)\lambda^2 + \beta c k \lambda + \beta k + v\lambda(\lambda^2 + \beta k)e^{-\lambda\tau} \tag{25}$$

In what follows, $\beta = 4.9153$, $k = 11.301$ and $c = 0.4$ are fixed. The function $F(\lambda)$ for finding out the rightmost characteristic root for given τ is chosen in the form:

$$F(\lambda) := c\lambda + 1 - W_0\left(\frac{(\beta c k \lambda + \beta k - p(\lambda))}{\beta k} e^{c\lambda+1}\right) \tag{26}$$

where the factor $e^{c\lambda+1}$, rather than $e^{\beta c k\lambda + \beta k}$, is applied, since the latter one is a *large* factor that should be avoided in numerical computation. We will show that the iteration method works effectively in finding out the rightmost root for any given parameters.

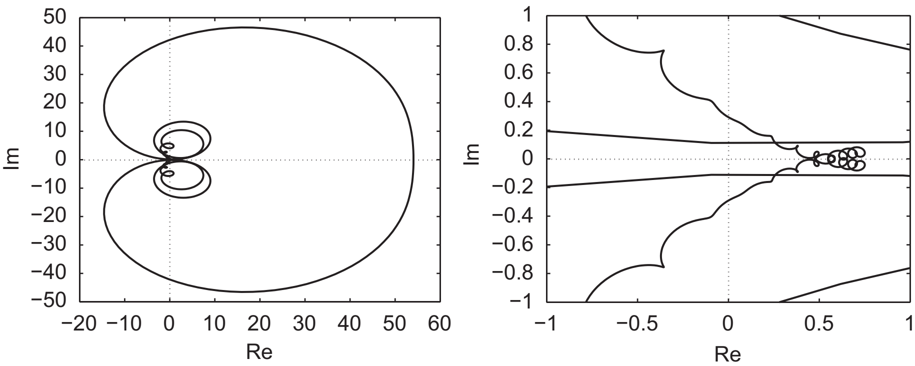

Fig. 5. For $\beta = 4.9153$, $k = 11.301$, $c = 0.4$, $v = 0.6$ and $\tau = 6.0$, the fifth iteration of Newton–Raphson's scheme finds the rightmost root of $p(\lambda)$ for Eq. (25) to be $\lambda^* = -0.2582 \times 10^{-1} \pm 0.7109i$. (Left) The Nyquist plot of $p(i\omega - 0.2582 \times 10^{-1} + 0.01)/(1 + i\omega)^4$ does not encircle the origin of the complex plane. (Right) Zoom around the origin.

Example 1. $v = 0.6$ and $\tau = 0.5$. We chose $\hat{\lambda}_0 = 3 + 8.9\mathrm{i}$ (randomly chosen), then the simplified polynomial

$$\lambda^4 + c(1+\beta)\lambda^3 + (1+\beta+\beta k)\lambda^2 + \beta ck\lambda + \beta k + v\lambda(\lambda^2 + \beta k)\mathrm{e}^{-0.5\hat{\lambda}_0}$$

has 4 roots $-0.9954 + 0.7670 \times 10\mathrm{i}$, $-0.1457 + 0.8921\mathrm{i}$, $-0.1611 - 0.1013 \times 10\mathrm{i}$, and $-0.1029 \times 10 - 0.7678 \times 10\mathrm{i}$. Hence a proper choice of the initial guess is $\lambda_0 = -0.1457 + 0.8921\mathrm{i}$. The third iteration of the Newton–Raphson's scheme (or Halley's scheme) gives the rightmost root $-0.6149 \pm 0.1072 \times 10\mathrm{i}$.

Example 2. $v = 0.6$ and $\tau = 6$. Begin with the initial guess $\lambda_0 = -0.1457 + 0.8921\mathrm{i}$, the iteration method finds the rightmost root $-0.2582 \times 10^{-1} \pm 0.7109\mathrm{i}$ after 5 iterations (Newton–Raphson's scheme) or 3 iterations (Halley's scheme) respectively, which can be confirmed by the Nyquist plot given in Fig. 5.

Example 3. $v = -0.1$ and $\tau = 2.5$. Let $\hat{\lambda}_0 = 10 + 3.5\mathrm{i}$ (randomly chosen), then the rightmost root of the simplified polynomial

$$\lambda^4 + c(1+\beta)\lambda^3 + (1+\beta+\beta k)\lambda^2 + \beta ck\lambda + \beta k + v\lambda(\lambda^2 + \beta k)\mathrm{e}^{-2.5\hat{\lambda}_0}$$

is $-0.1699 \pm 0.9483\mathrm{i}$. Starting from the initial guess $\lambda_0 = -0.1699 \pm 0.9483\mathrm{i}$, the third iteration of Newton–Raphson's scheme (or the second iteration of Halley's scheme) gives the rightmost root $-0.2032 \pm 0.8793\mathrm{i}$.

6. Conclusions

With the help of the Lambert W function, the rightmost characteristic root (or the abscissa) of a retarded time-delay system can be found simply by using the famous Newton–Raphson's scheme or Halley's scheme. The idea of using an iteration method to find out a root of a nonlinear equation is not new, the present work makes it possible to find out the rightmost root by using iteration methods. Three points are important for a successful application of the proposed method. (i) The freely chosen constants a and b in the function $F(\lambda)$ should be not large so as to avoid numerical problems. (ii) The initial guess can be chosen in different ways, say, taken as the rightmost root of the simplified polynomial defined by Eq. (15). (iii) The computation result should be confirmed by using the Nyquist graphical test. As shown in the 3 illustrative examples, the rightmost root of TDSs can be located successfully by a few number of iterations of Newton–Raphson's scheme or Halley's scheme.

Though the proposed iteration methods work effectively for TDSs of retarded type as shown in this paper, a rigorous mathematical proof to the main results is still not available. In addition, the method may fail in finding the rightmost root of TDSs of neutral type or that of the degenerated case: polynomials, for which more tricks are usually required. This problem is left for future consideration.

Acknowledgments

This work was supported by FANEDD of China under Grant 200430 and by the NSF of China under Grant 10532050.

References

[1] H.Y. Hu, Z.H. Wang, *Dynamics of Controlled Mechanical Systems with Delayed Feedback*, Springer, Heidelberg, Verlag, 2002.
[2] S.-I. Niculescu, *Delay Effects on Stability: A Robust Control Approach*, Springer, London, Verlag, 2001.
[3] G. Stépán, *Retarded Dynamical Systems: Stability and Characteristic Functions*, Longman Scientific and Technical, Essex, 1989.
[4] Y.X. Qin, Y.Q. Liu, L. Wang, Z.X. Zhen, *Stability of Dynamical Systems with Time Lag*, second ed., Science Press, Beijing, 1989.
[5] B. Hassard, Counting roots of the characteristic equation for linear delay–differential systems, *Journal of Differential Equations* 136 (1997) 222–235.
[6] M.Y. Fu, A.W. Olbrot, M.P. Polis, Robust stability for time-delay systems: the edge theorem and graphical tests, *IEEE Transactions on Automatic Control* 34 (1989) 813–820.
[7] A. Datta, M.-T. Ho, S.P. Bhattacharyya, *Structure and Synthesis of PID Controllers*, Springer, London, 2000.
[8] N. Jalili, N. Olgac, Multiple delayed resonator vibration absorbers for multi-degree-of-freedom mechanical structures, *Journal of Sound and Vibration* 223 (1999) 567–585.

[9] L. Palkovics, P.J.Th. Venhovens, Investigation on stability and possible chaotic motions in the controlled wheel suspension system, *Vehicle System Dynamics* 21 (1992) 269–296.
[10] R.M. Corless, G.H. Gonnet, D.E.G. Hare, D.J. Jeffrey, D.E. Knuth, On the Lambert W function, *Advances in Computational Mathematics* 5 (1996) 329–359.
[11] F.M. Asl, A.G. Ulsoy, Analysis of a system of linear delay differential equations, *ASME Journal of Dynamic Systems, Measurement, and Control* 125 (2003) 215–223.
[12] C. Hwang, Y.-C. Cheng, A note on the use of the Lambert W function in the stability analysis of time-delay systems, *Automatica* 41 (2005) 1979–1985.
[13] H. Shinozaki, T. Mori, Robust stability analysis of linear time-delay systems by Lambert W function: some extreme point results, *Automatica* 42 (2006) 1791–1799.
[14] Y.Q. Chen, D.Y. Xue, J. Gu, Analytical and numerical computation of stability bound for a class of linear delay differential equations using Lambert function, *Dynamics of Continuous, Discrete and Impulsive Systems, Series B* (Suppl. S) (2003) 489–494.
[15] S. Arunsawtwong, Stability of retarded differential equations, *International Journal of Control* 65 (2) (1996) 347–364.
[16] A. Zakian, Computation of the Abscissa of stability by repeated use of the Routh test, *IEEE Transactions on Automatic Control* 24 (1979) 604–607.
[17] J.M. Krodkiewski, T. Jintanawan, Stability improvement of periodic vibration of multi-degree-of-freedom systems by means of time-delay control, *Proceedings of the International Conference on Vibration*, *Noise and Structural Dynamics'99*, Vol. 1, Venice, Italy, 28–30 April, 1999, pp. 340–351.
[18] P. Tass, J. Kurths, M.G. Rosenblum, G. Guasti, H. Hefter, Delay-induced transitions in visually guided movements, *Physical Review E* 54 (1996) R2224–R2227.
[19] J.H. Mathews, K.D. Fink, *Numerical Methods Using Matlab*, third ed., Prentice Hall & Publishing House of Electronic Industry, Beijing, 2002.
[20] A. Gail, *Bursting in a Model with Delay for Networks of Neurons*, Dissertation, University of Cologne, Germany, 2004.