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Boundary control of transverse motion of marine risers with actuator dynamics

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Abstract

This paper presents a boundary controller to reduce transverse motion of flexible marine risers driven by a hydraulic system at the top end of the risers under environmental disturbances induced by waves, wind, and ocean currents. The boundary controller is designed based on Lyapunov's direct method and the backstepping technique. Proof of existence and uniqueness of the solutions of the closed-loop control system is carried out by using the Galerkin approximation method. Simulation results illustrate the effectiveness of the proposed boundary controller.

1. Introduction

As exploration and production for natural resources enter deeper ocean waters, control of the dynamics of flexible marine risers connecting a oil and/or gas offshore platform with a well at the sea bed, becomes a vital task for both ocean and control engineers. In general, a riser is subject to nonlinear deformation dependent on hydrodynamic loads induced by waves, ocean currents, tension exerted at the top, distributed/concentrated buoyancy from attached modules, its own weight, inertia forces, and distributed/concentrated torsional couples. Since the riser dynamics is essentially a distributed system and its motion is governed by a set of partial differential equations (PDE) in both time and space variables, modal control and boundary control approaches are often used to control the riser in the literature.

In the modal control approach, see Refs. [1,2], distributed systems are controlled by controlling their modes. As a result, many concepts developed for lumped-parameter systems in Refs. [3,4] can be used for controlling the distributed ones, since both types can be described in terms of modal coordinates. The main difficulty is computation of infinite-dimensional gain matrices. This difficulty can be avoided by using the independent modal-space control method, but this method requires a distributed control force, which can be problematic to implement. One way to overcome this problem is to construct a truncated model consisting a limited number of modes. In order to describe the behavior of a flexible system in a satisfactory fashion, it is necessary to include a large number of modes in the model. Thus, a characteristic of a truncated model is its large

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dimension, i.e. it is impractical to control all modes. Therefore the control of such truncated systems are restricted to a few critical modes. This also means that other modes are not controlled, and could be unstable. In fact, truncation of the infinite-dimensional model divides the system into three groups modes: modeled and controlled, modeled and uncontrolled (residual), and un-modeled. Only the modeled modes are considered in the control design. In addition, observers are needed to provide the system output for these modeled modes from the actual distributed system. The use of these observers in combination with truncated models of distributed system leads to a spill-over phenomenon meaning that the control from actuators not only affects the controlled modes but also influences the residual and un-modeled modes, which can be unstable, [5].

The boundary control approach is more practical and efficient than the modal control approach since it excludes the effect of both observation and control spill-over phenomenon. In the boundary control approach, distributed actuators and sensors are not required. In addition control design based on the original PDE model instead of a truncated model, improves the performance of the control system. In recent years, boundary control has received much attention from the control community. Design of boundary controllers for distributed systems has been usually based on functional analysis and semi-group theory, see Refs. [6,7], and the Lyapunov's direct method, see Refs. [8,9]. The Lyapunov's direct method is widely used since the control Lyapunov functions/functionals directly relate to the kinetic and potential energies of the distributed systems. Using the Lyapunov's direct method, various boundary controllers have been proposed for flexible beam-like systems. In Ref. [10], the boundary stabilization of a beam in free transverse vibration is considered. The control law is a nonlinear function of the slopes and velocity at the boundary of the beam to provide exponential stabilization a free transversely vibrating beam via boundary control without restoring to truncation of the model. The coupling between longitudinal and transversal displacements is also taken into account. Recently, in Ref. [11] an active boundary control is proposed for an Euler–Bernoulli beam, which enables one to generate a desired boundary condition at designated positions of a target beam based on structure transfer matrix and the optimal control methods. It should be noted that the active boundary control in Ref. [11] is implemented at various locations of the beam. Therefore, this method closely relates to the modal control approach although it is called boundary control. In Refs. [12–14], the authors proposed an elegant method, which was developed for stabilizing an unstable heat equation in Ref. [15], to design boundary controllers for strings and beams with pretty simple dynamics. The fundamental idea is to find a coordinate change to transform the string or beam system to a target system, which can be stabilized by a boundary controller. This idea relies on feasibility of finding a kernel, which is a solution of a partial differential equation depending on the system dynamics. The major difference between the controllers proposed in Refs. [12–14], and the damping boundary feedback controllers in Refs. [8,10] is that the controllers in Refs.[12–14] do not rely on a passivity property from the actuator to the sensor. However, the method in Refs. [12–14] is hard to apply to the riser system addressed in this paper due to difficulties in solving a partial differential equation to find a proper kernel. It should be mentioned that in Refs. [16,10,11,8], two-dimensional strings and beams are considered, and distributed forces including the structures' own weight are ignored. Moreover, in Refs. [16,10,11,8] no proof of existence and uniqueness of the solutions of closed-loop systems was given. It is well-known that there are systems governed by initial-boundary PDEs, whose solutions do not exist or are not unique. For any control systems to be useful in practice, existence and uniqueness of the solutions of the closed-loop control systems are as vital as stability. Moreover, there are no actuators that can provide immediate forces/moments for control implementation at the riser boundary. If the actuator dynamics is ignored, the performance of the controlled system can be significantly reduced, and can be unstable in some cases [17]. It is therefore necessary to include the actuator dynamics in the control design.

This paper considers a problem of reducing transverse motion of flexible marine risers driven by a hydraulic system at the top end of the risers under environmental disturbances induced by waves, wind and ocean currents. Based on the energy approach, the equations of motion of the riser-hydraulic system are derived. We show that the Lyapunov direct method and the backstepping technique can be used well to design a controller to drive the hydraulic system at the top end of the riser. Proof of existence and uniqueness of the solutions of the closed-loop control system is carried out by using the Galerkin approximation method. Stability analysis is carefully analyzed. Simulation results illustrate the effectiveness of the proposed boundary controller.

2. Preliminaries and mathematical model

2.1. Preliminaries

This subsection presents two tools that will be used in the control design. The first one is a disturbance observer to estimate un-modeled forces in the dynamics of the hydraulic system. The second tool is a p-times differentiable signum function to approximate the signum function in the dynamics of the hydraulic system.

2.1.1. Disturbance observer

Consider the following system

$$\dot{x} = f(x) + u + d(t,x), \tag{1}$$

where $x \in \mathbb{R}^n$, $f(x)$ is a vector of known functions of x, u the control input vector, and $d(t,x)$ a vector of unknown disturbances. We assume that there exists a nonnegative constant C_d such that $\|\dot{d}(t,x)\| \leqslant C_d$. Now we want to design the control input u to stabilize system (1) at the origin. It is obvious that if we can design a disturbance observer, $\hat{d}(t,x)$, that estimates $d(t,x)$ sufficiently accurately, then the control input u is straightforwardly designed as $u = -\kappa x - f(x) - \hat{d}(t,x)$ with κ a positive-definite matrix. The disturbance observer is given in the following lemma.

Lemma 1. *Consider the following disturbance observer*:

$$\hat{d}(t,x) = \xi + \rho(x),$$

$$\dot{\xi} = -K(x)\xi - K(x)(f(x) + u + \rho(x)), \tag{2}$$

where $K(x) = \partial\rho(x)/\partial x$, $\rho(x)$ is chosen such that the matrix $K(x)$ is positive definite for all $x \in \mathbb{R}^n$. The disturbance observer (2) guarantees that the disturbance observer error $d_e(t,x) = d(t,x) - \hat{d}(t,x)$ exponentially converges to a ball centered at the origin. The radius of this ball can be made arbitrarily small by adjusting the function $\rho(x)$. In the case $C_d = 0$, the disturbance observer error $d_e(t,x)$ exponentially converges to zero.

Proof. (see Do and Pan [18]). The disturbance observer (2) is a dynamical system. The variable ξ is generated by the second equation of Eq. (2), which is an ordinary differential equation, with some initial value $\xi(t_0)$, where t_0 is the initial time. The choice of the function $\rho(x)$, which results in the matrix $K(x)$ directly affects performance of the disturbance observer. The larger eigenvalues of the matrix $K(x)$ are, the faster the response of the disturbance observer is, with a trade-off of a large overshoot of the observer, and vice versa. An application of the disturbance observer (2) to an active heave compensation system is given in Ref. [18]. To illustrate the effectiveness of the disturbance observer (2), we perform some numerical simulations. In the simulations, we consider a scalar system in the form of Eq. (1) with $f(x) = \arctan(x + x^2)$ and $d(t,x) = \sum_{i=1}^{5}(\sin(it) + \sin(x)\sin(it/2))$. The function $\rho(x)$ is taken as$\rho(x) = 20(x + x^3/3)$. This choice gives $K(x) = 20(1 + x^2)$, which is positive for all $x \in \mathbb{R}$. The initial conditions are $x(0) = 1$ and $\xi(0) = 0$. The control law is designed as $u = -\kappa x - f(x) - \hat{d}(t,x)$ with $\kappa = 5$. We run two simulations. In the first one, the disturbance $d(t,x)$ is ignored in the control design, i.e. we set $\hat{d}(t,x) = 0$ in the above control law. Simulation results are presented in the top two sub-figures (A and B) of Fig. 1. In the second simulation, we include $\hat{d}(t,x)$ in the control law. Simulation results are plotted in the bottom two sub-figures (C and D) of Fig. 1. It is seen from the sub-figures (A and C) that in the case where the disturbance $d(t,x)$ is ignored in the control design, the state x converges to a much larger ball than the case where the disturbance observer is used in the control design. In the sub-figures (B and D), the disturbance $d(t,x)$ is plotted in the solid line while the disturbance estimate $\hat{d}(t,x)$ is plotted in the dash-dotted line. From the sub-figure (D), we can see that the disturbance observer provides an excellent estimate of the time and state-dependent disturbance $d(t,x)$. $\square$

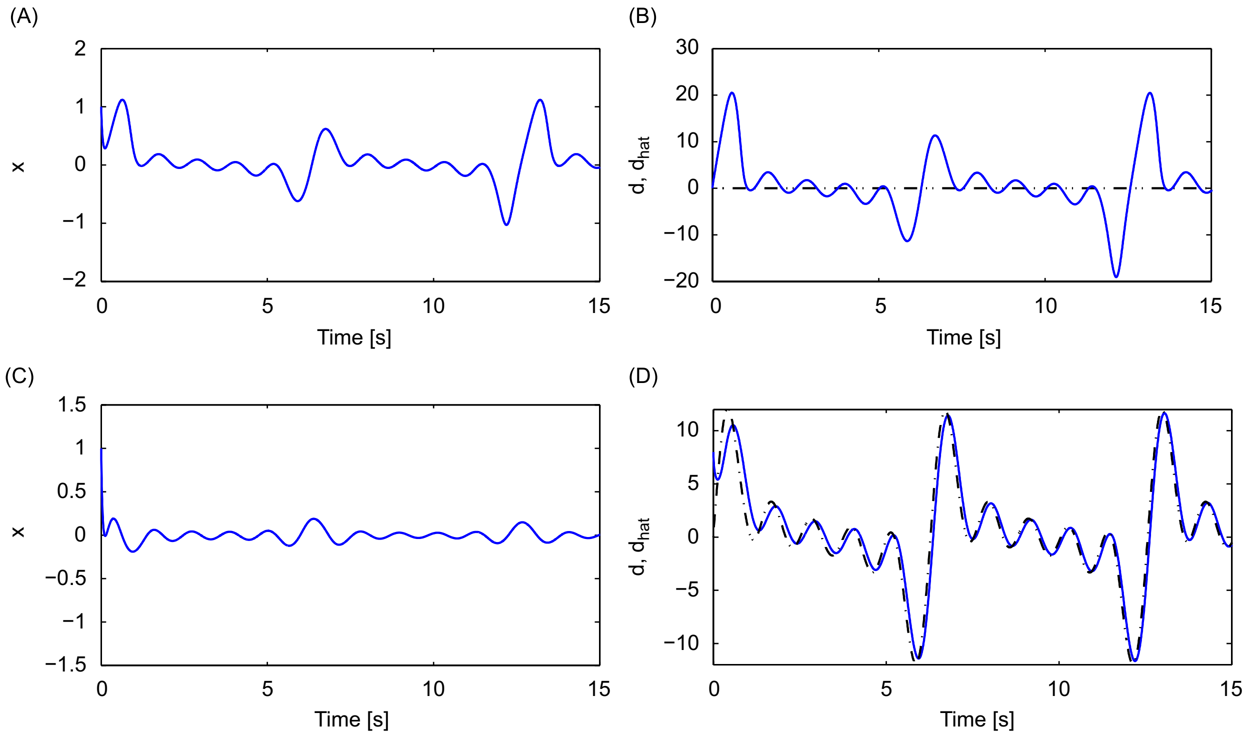

Fig. 1. Effectiveness of the proposed disturbance observer: (A) state x without disturbance observer; (B) actual disturbance $d(t,x)$ plotted by solid line, and estimate of disturbance $\hat{d}(t,x)$ plotted by dash-dot line; (C) state x with disturbance observer; (D) actual disturbance $d(t,x)$ plotted by solid line, and estimate of disturbance $\hat{d}(t,x)$ plotted by dash-dot line.

2.1.2. *p-times differentiable signum function*

Definition 1. A scalar function $h(x,a,b)$ is called a p-times differentiable signum function if it enjoys the following properties:

$$\begin{aligned}&(1)\ h(x,a,b)=-1 \quad \text{if } -\infty<x\leqslant a,\\ &(2)\ h(x,a,b)=1 \quad \text{if } x\geqslant b,\\ &(3)\ -1<h(x,a,b)<1 \quad \text{if } a<x<b,\\ &(4)\ h(x,a,b) \quad \text{is } p \text{ times differentiable with respect to } x\end{aligned} \tag{3}$$

where p is a positive integer, $x\in\mathbb{R}_+$, and a and b are constants such that $a<0<b$. Moreover, if the function $h(x,a,b)$ is infinite-times differentiable with respect to x, then it is called a smooth signum function.

Lemma 2. *Let the scalar function $h(x,a,b)$ be defined by*

$$h(x,a,b)=2\frac{\int_a^x f(\tau-a)f(b-\tau)\,\mathrm{d}\tau}{\int_a^b f(\tau-a)f(b-\tau)\,\mathrm{d}\tau}-1, \tag{4}$$

where the function $f(y)$ is defined as follows:

$$f(y)=0 \text{ if } y\leqslant 0 \text{ and } f(y)=y^p \text{ if } y>0 \tag{5}$$

with p being a positive integer. Then the function $h(x,a,b)$ is a p-times differentiable signum function. Moreover, if the function $f(y)$ is taken as

$$f(y)=0 \text{ if } y\leqslant 0 \text{ and } f(y)=\mathrm{e}^{-1/y} \text{ if } y>0$$

then the function $h(x,a,b)$ is a smooth signum function.

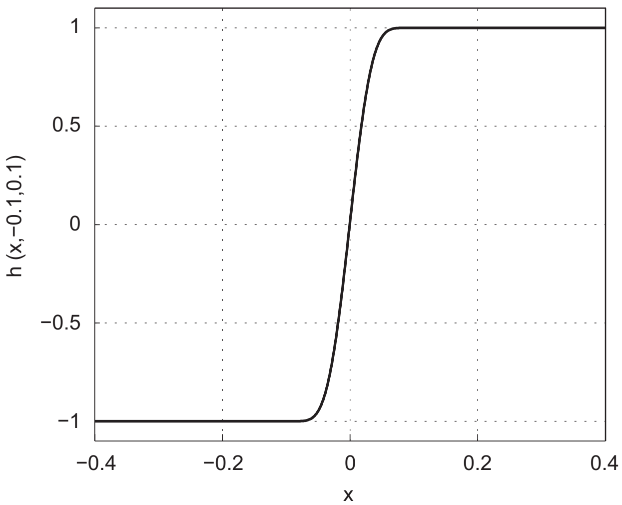

Fig. 2. A twice differentiable signum function.

Proof. (see Do [19]). An illustration of a twice differentiable signum function ($a = -0.1, b = 0.1$) is given in Fig. 2. $\square$

2.2. Mathematical model

In this subsection, we develop equations of the transverse motion of the riser, and of the hydraulic system. These equations will be used for the boundary control design in the next section. In developing these equations, we make the following assumption:

Assumption 1. (1) The riser can be modeled as a beam rather than a shell since the diameter-to-length of the riser is small, i.e. we consider the riser as a slender structure.

(2) Plane sections remain plane after deformation, i.e. warping is neglected.

(3) The riser is locally stiff, i.e. cross sections do not deform and Poisson effect is neglected.

(4) The riser material is homogeneous, isotropic and linearly elastic, i.e. it obeys Hookes's law.

(5) Torsional and distributed moments induced by environmental disturbances are neglected.

(6) The riser deforms in one vertical plane, and its axial motion is ignored.

Remark 1. Items (1–4) mean that the riser will be modeled as a Bernoulli-type beam and not of the Timoshenko type, and that the extension of the riser axis small. Bernoulli–Euler models are satisfactory for modeling low-frequency vibrations of beams. Item (5) implies that we consider fluid/gas transportation risers rather than drilling risers, and that moment induced by the asymmetry of the relative flow due to vortex shedding is ignored. Item (6) means that we consider the transverse motion of the riser. The axial motion of the riser is usually compensated by an active heave compensation system [18].

The riser coordinates and the hydraulic system, which provide the boundary control force in the transverse direction of the riser, are presented in Fig. 3. It is assumed that the riser is subjected to a constant axial force P_0 provided by an active heave compensation system. In Fig. 3, the Earth-fixed coordinate system is denoted by OXZ with O fixed to the sea bed. The riser is connected with the hydraulic system via a ball joint, and is also connected to the sea bed via a ball joint. This configuration results in moment free at both ends of the riser. The Earth-fixed system is ($OXYZ$), where O is the bottom ball-joint of the riser, and the OZ axis is along the initial riser. Let $\eta(z, t)$ be the transverse displacement of the riser. Let $f(z, t, u(z, t), \eta_t(z, t))$ be the transverse distributed damping force and distributed external force induced by waves, wind and ocean currents, where t denotes the time, $u(z, t)$ denotes the component of the water particle velocity in the transverse direction, and $\eta_t(z, t) = (\partial\eta/\partial t)(z, t)$, i.e. the velocity of the riser in the transverse direction at (z, t). The distributed transverse

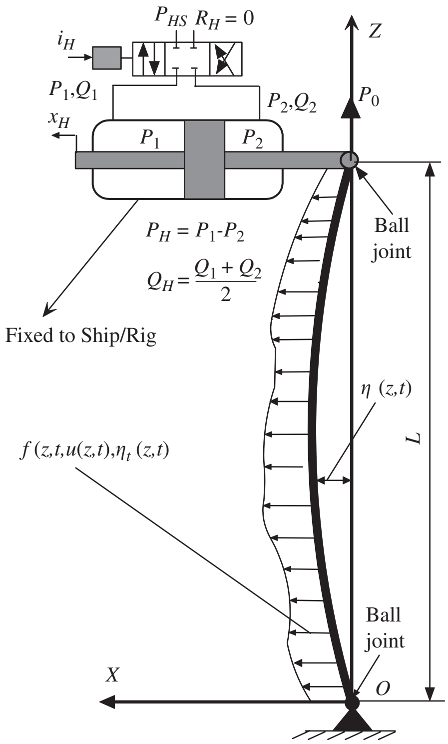

Fig. 3. Riser coordinates and the hydraulic system.

force $f(z, t, u(z, t), \eta_t(z, t))$ can be given as [20,21]:

$$f(z, t, u(z, t), \eta_t(z, t)) = f_D + f_L,$$

$$f_D = -\Omega_D \eta_t(z, t), \quad \Omega_D = \left(c + C_D \frac{\rho_w D}{2} \sqrt{\frac{8}{\pi}} \sigma_u \right),$$

$$f_L = C_M \frac{\rho_w \pi D^2 u_t(z, t)}{4} + C_D \frac{\rho_w D}{2} \sqrt{\frac{8}{\pi}} \sigma_u(z, t) u(z, t), \tag{6}$$

where f_D and f_L are referred to as the distributed damping and external forces, c is the linear viscous damping coefficient, ρ_w the water density, C_D the drag coefficient, D the riser diameter and $\sigma_u(z, t)$ is the root mean square of the water particle velocity, $u(z, t)$. It is noted that in Eq. (6), the quadratic term of the drag force due to the relative water velocity, $\eta_t(z, t) - u(z, t)$ is approximated by a linear expression involving the root mean square of the relative velocity, and moreover, the relative velocity is approximated by the water velocity, $u(z, t)$. We assume that the distributed external force f_L is bounded for all $z \in [0, L]$ and $t \geqslant 0$. To develop equations of motion of the riser-hydraulic system, we first consider the riser and then move to the hydraulic system.

2.2.1. Equations of the transverse motion of the riser

To derive the equations of the transverse motion of the riser, we use the extended Hamilton's principle:

$$\int_{t_1}^{t_2} \delta(T - V + W_c + W_b)\, \mathrm{d}t = 0,$$

$$\delta\eta(z, t_1) = \delta\eta(z, t_2) = 0, \tag{7}$$

where T is the kinetic energy, V is the potential energy, W_c is the virtual work by nonconservative forces, and W_b is the virtual momentum transport at the boundary.

The kinetic energy T consists of the kinetic energy of the riser and the piston of the hydraulic system, and is given by

$$T = \frac{m_o}{2} \int_0^L \eta_t^2(z,t)\,\mathrm{d}z + \frac{m_H}{2}\eta_t^2(L,t), \tag{8}$$

where $m_o = \rho A$ with ρ the mass per unit length and A the cross-section area of the riser, and m_H the mass of the piston of the hydraulic system.

The potential energy V is given by

$$V = \frac{EI}{2}\int_0^L \eta_{zz}^2(z,t)\,\mathrm{d}z + \frac{P_0}{2}\int_0^L \eta_z^2(z,t)\,\mathrm{d}z + \frac{EA}{8}\int_0^L \eta_z^4(z,t)\,\mathrm{d}z, \tag{9}$$

where E is Young's modulus, I the moment of inertia of the riser cross section and P_0 the constant axial force. It is noted that in Eq. (9) the first term is due to the bending moment, the second term is due to the riser tension, and the last term results from the strain energy.

Variation of the virtual work δW_c by nonconservative force $f(z,t,u(z,t),\eta_t(z,t))$ is given by

$$\delta W_c = \int_0^L f(z,t,u(z,t),\eta_t(z,t))\delta\eta(z,t)\,\mathrm{d}z. \tag{10}$$

Variation of the virtual work δW_b by the virtual momentum transport at the boundary is given by

$$\delta W_b = (A_H P_H - \Delta(t,\eta_t(L,t)) - b_H\eta_t(L,t))\delta\eta(L,t), \tag{11}$$

where $P_H = P_1 - P_2$ is the load pressure of the cylinder with P_1 and P_2 being the pressures in the upper and lower compartments of the cylinder, see Fig. 3, A_H is the ram area of the cylinder, b_H represents the combined coefficient of the modeled damping and viscous friction forces on the cylinder rod, and $\Delta(t,\eta_t(L,t))$ is the un-modeled force acting on the cylinder of the hydraulic system. This un-modeled force can include un-modeled friction between the cylinder and the piston of the hydraulic system, and external disturbance acting on the piston of the hydraulic system.

Now substituting Eqs. (8)–(11) into Eq. (7) and integrating by parts result in

$$\begin{aligned}\int_{t_1}^{t_2}\int_0^L \Bigg[&\Bigg(-m_o\eta_{tt}(z,t) - EI\eta_{zzzz}(z,t) + P_0\eta_{zz}(z,t) + \frac{3EA}{2}\eta_z^2(z,t)\eta_{zz}(z,t)\\ &+ f(z,t,u(z,t),\eta_t(z,t))\Bigg)\delta\eta(z,t) - EI\eta_{zz}(z,t)\delta_z(z,t)_0^L\\ &+\left(EI\eta_{zzz}(z,t) - P_0\eta_z(z,t) - \frac{EA}{2}\eta_z^3(z,t)\right)\delta(z,t)\Bigg|_0^L\\ &+ m_H\eta_t(L,t) + (A_H P_H - \Delta(t,\eta_t(L,t)) - b_H\eta_t(L,t))\delta\eta(L,t)\Bigg]\,\mathrm{d}t = 0.\end{aligned} \tag{12}$$

From Eq. (12) and the boundary conditions resulting from the riser configuration (see Fig. 3) we have

$$\begin{gathered}-m_o\eta_{tt}(z,t) - EI\eta_{zzzz}(z,t) + P_0\eta_{zz}(z,t) + \frac{3EA}{2}\eta_z^2(z,t)\eta_{zz}(z,t) + f(z,t,u(z,t),\eta_t(z,t)) = 0,\\ -m_H\eta_{tt}(L,t) + EI\eta_{zzz}(L,t) - P_0\eta_z(L,t) - \frac{EA}{2}\eta_z^3(L,t) + A_H P_H - \Delta(t,\eta_t(L,t)) - b_H\eta_t(L,t) = 0,\\ \eta_{zz}(0,t) = \eta_{zz}(L,t) = 0,\\ \eta(0,t) = 0.\end{gathered} \tag{13}$$

2.2.2. Equations of the hydraulic system

The second equation in Eq. (13) represents the dynamics of the piston of the hydraulic system with

$$\begin{aligned}\eta(L,t) &= x_H,\\ \eta_t(L,t) &= \dot{x}_H.\end{aligned} \tag{14}$$

Neglecting the leakage flows in the cylinder and the servovalve, the actuator or the cylinder dynamics is written as [22]

$$\frac{V_H}{4\beta_{He}}\dot{P}_H = -A_H\dot{x}_H - C_{HT}P_H + Q_H \tag{15}$$

where V_H is the total volume of the cylinder and the hoses between the cylinder and the servovalve, β_{He} the effective bulk modulus, C_{HT} the coefficient of the total internal leakage of the cylinder due to pressure and Q_H the load flow. The load flow Q_H related to the spool displacement of the servovalve, x_{Hv}, by Merritt [22]

$$Q_H = C_{HD}W_H x_{Hv}\sqrt{\frac{P_{HS} - \operatorname{sgn}(x_{Hv})P_H}{\rho_H}} \tag{16}$$

where C_{HD} is the discharge coefficient, W_H the spool valve area gradient, P_{HS} the supply pressure of the fluid, sgn denotes the standard signum function and ρ_H is density of the oil. It is noted that since the supply pressure P_{HS} is always higher than the load pressure P_H, i.e. there exists a strictly positive constant ε_1 such that $P_{HS} - \operatorname{sgn}(x_{Hv})P_H \geqslant \varepsilon_1$. Hence, Eq. (16) is well-defined for all $x_{Hv} \in \mathbb{R}$. The servovalve dynamics can be described by

$$\tau_{Hv}\dot{x}_{Hv} = -x_{Hv} + k_{Hv}i_H \tag{17}$$

where τ_{Hv} and k_{Hv} are the time constant and gain of the servovalve, respectively, i_H is the current input to the hydraulic system. Since P_H is usually very large and τ_{Hv} is usually very small, we scale the pressure P_H and the spool displacement x_{Hv} as $\bar{P}_H = P_H/C_{H3}$ and $\bar{x}_{Hv} = x_{Hv}/C_{H4}$ where C_{H3} and C_{H4} are constants, to avoid numerical error and facilitating the control gain tuning process. With scaling observation in mind, we write the entire system of the riser-hydraulic dynamics in a standard state space form for the purpose of control design in the next section as follows:

$$\begin{aligned}
m_o\eta_{tt}(z,t) &= -EI\eta_{zzzz}(z,t) + P_0\eta_{zz}(z,t) + \frac{3EA}{2}\eta_z^2(z,t)\eta_{zz}(z,t) + f(z,t,u(z,t),\eta_t(z,t))\\
\dot{x}_1 &= x_2,\\
\dot{x}_2 &= -\frac{b_H}{m_H}x_2 - \frac{P_0}{m_H}\eta_z(L,t) - \frac{EA}{2m_H}\eta_z^3(L,t) + \frac{EI}{m_H}\eta_{zzz}(L,t) + \frac{A_HC_{H3}}{m_H}x_3 - \frac{1}{m_H}\Delta(t,\eta_t(L,t)),\\
\dot{x}_3 &= -\frac{4\beta_{He}A_H}{V_HC_{H3}}x_2 - \frac{4\beta_{He}C_{HT}}{V_H}x_3 + \frac{4\beta_{He}C_{HD}C_{H4}W_H}{V_H\sqrt{C_{H3}}}g_3(x_3,x_4),\\
\dot{x}_4 &= -\frac{1}{\tau_{Hv}}x_4 + \frac{k_{Hv}}{\tau_{Hv}C_{H4}}i_H,\\
\eta_{zz}(0,t) &= \eta_{zz}(L,t) = 0,\\
\eta(0,t) &= 0
\end{aligned} \tag{18}$$

where we have defined

$$\begin{aligned}
&x_1 = \eta(L,t), \quad x_2 = \eta_t(L,t), \quad x_3 = \bar{P}_H, \quad x_4 = \bar{x}_{Hv},\\
&g_3(x_3,x_4) = x_4\sqrt{\frac{\bar{P}_{Hs} - h(x_4,a,b)x_3}{\rho_H}}, \quad \bar{P}_{HS} = \frac{P_{HS}}{C_{H3}}
\end{aligned} \tag{19}$$

and the p-times differentiable signum function $h(x_4, a, b)$ has been used to replace the signum function. It is noted that the use of the p-times differentiable signum function $h(x_4, a, b)$ instead of the signum function $\mathrm{sgn}(x_4)$ in Eq. (18) not only makes the function $g_3(x_3, x_4)$ differentiable with respect to x_3 and x_4 but also represents the actual dynamics of the spool dynamics. This is because there is always certain inaccuracy in manufacturing the servovalve, i.e. the flow in the servovalve does not change its direction immediately.

2.3. Control objectives

Under Assumption 1, design the control input i_H for the riser-hydraulic system given by Eq. (18) to stabilize the riser at its vertical position in the sense that all the states of the riser-hydraulic system (18) are bounded and that:

(1) when the external disturbance f_L is ignored, all the terms $|\eta(z,t)|$, $\int_0^L \eta_z^2(z,t)\,\mathrm{d}z$, $\int_0^L \eta_t^2(z,t)\,\mathrm{d}z$ and $\int_0^L \eta_{zz}(z,t)\,\mathrm{d}z$ exponentially converge to zero for all $z \in [0, L]$ and $t \geqslant t_0 \geqslant 0$,
(2) when the external disturbance f_L is present, all the terms $|\eta(z,t)|$, $\int_0^L \eta_z^2(z,t)\,\mathrm{d}z$, $\int_0^L \eta_t^2(z,t)\,\mathrm{d}z$ and $\int_0^L \eta_{zz}(z,t)\,\mathrm{d}z$ exponentially converge to some small positive constant for all $z \in [0, L]$ and $t \geqslant t_0 \geqslant 0$,

It is seen that the control objectives impose on both the displacement and integration of square of the slope, velocity, and curvature of the riser along the riser length.

3. Control design

A close look at the entire system (18) shows that the system is of a strict-feedback form [4]. Therefore, we will use the backstepping technique [4] to design the control input i_H to achieve the control objective stated in the previous section. The control design consists of the following three steps.

3.1. Step 1

At the this step, we consider the scaled pressure $\bar{P}_H$, i.e. x_3, as a control to design a boundary control law (i.e. a control law only uses $\eta(L,t)$ and its spatial and time derivatives) such that it stabilizes the riser at a small neighborhood of its vertical position. Ideally, we want to stabilize the riser at its vertical position but this is impossible due to the distributed external disturbances f_L induced by waves, wind and ocean currents. As such, we define

$$x_{3e} = x_3 - \alpha_1, \tag{20}$$

where α_1 is a virtual control of x_3. To design the virtual boundary control α_1, we use Lyapunov's direct method. Consider the following Lyapunov function candidate:

$$\begin{aligned} W_1 = {} & \frac{m_o}{2}\int_0^L \eta_t^2\,\mathrm{d}z + \frac{P_0}{2}\int_0^L \eta_z^2\,\mathrm{d}z + \frac{EA}{8}\int_0^L \eta_z^4\,\mathrm{d}z + \frac{EI}{2}\int_0^L \eta_{zz}^2\,\mathrm{d}z + \gamma\int_0^L z\eta_t\eta_z\,\mathrm{d}z \\ & + \frac{m_H}{2}\left(\eta_t(L,t) + \frac{\gamma L}{m_o}\eta_z(L,t)\right)^2, \end{aligned} \tag{21}$$

where whenever it is not confusing we drop the arguments z and t of η_t, η_z, and so on; γ is a positive constant to be specified later. Since for all $t \geqslant t_0 \geqslant 0$

$$-\frac{\gamma L}{2}\int_0^L \eta_t^2\,\mathrm{d}z - \frac{\gamma L}{2}\int_0^L \eta_z^2\,\mathrm{d}z \leqslant \int_0^L z\eta_t\eta_z\,\mathrm{d}z \leqslant \frac{\gamma L}{2}\int_0^L \eta_t^2\,\mathrm{d}z + \frac{\gamma L}{2}\int_0^L \eta_z^2\,\mathrm{d}z \tag{22}$$

the function W_1 satisfies

$$W_1 \geqslant \frac{m_o - \gamma L}{2}\int_0^L \eta_t^2\,dz + \frac{P_0 - \gamma L}{2}\int_0^L \eta_z^2\,dz + \frac{EA}{8}\int_0^L \eta_z^4\,dz + \frac{EI}{2}\int_0^L \eta_{zz}^2\,dz + \frac{m_H}{2}\left(\eta_t(L,t) + \frac{\gamma L}{m_o}\eta_z(L,t)\right)^2,$$

$$W_1 \leqslant \frac{m_o + \gamma L}{2}\int_0^L \eta_t^2\,dz + \frac{P_0 + \gamma L}{2}\int_0^L \eta_z^2\,dz + \frac{EA}{8}\int_0^L \eta_z^4\,dz + \frac{EI}{2}\int_0^L \eta_{zz}^2\,dz + \frac{m_H}{2}\left(\eta_t(L,t) + \frac{\gamma L}{m_o}\eta_z(L,t)\right)^2. \tag{23}$$

Hence if we choose γ such that

$$m_o - \gamma L = c_1, \quad P_0 - \gamma L = c_2, \tag{24}$$

where c_1 and c_2 are strictly positive constants, then the function W_1 defined in Eq. (21) is a proper (i.e. positive definite and radially unbounded) function of $\int_0^L \eta_t^2\,dz$, $\int_0^L \eta_{zz}^2\,dz$, $\int_0^L \eta_z^2\,dz$ and $(\eta_t(L,t) + (\gamma L/m_o)\eta_z(L,t))^2$. We do not detail the conditions (24) at the moment, but deal with them after the control design is completed since the constant γ needs to satisfy some more conditions later. It is noted that we do not include the riser transverse displacement η, like $\int_0^L \eta^2\,dz$, in the function W_1 because this term causes difficulties in designing the control α_1 later. As such, after proof of convergence of $\int_0^L \eta_t^2\,dz$, $\int_0^L \eta_{zz}^2\,dz$, and $\int_0^L \eta_z^2\,dz$, convergence of $\int_0^L \eta^2\,dz$ and the riser transverse displacement η will be proved by using Lemmas 3 and 4 in Appendix A. Differentiating both sides of Eq. (21) with respect to time t, along the solutions of the first and the third equations of Eq. (18), and using integration by parts result in

$$\begin{aligned}\dot{W}_1 = {} & \left(P_0\eta_z\eta_t + \frac{EA}{2}\eta_z^3\eta_t - EI\eta_{zzz}\eta_t + EI\eta_{zz}\eta_t + \frac{\gamma P_0}{2m_o}z\eta_z^2 + \frac{3\gamma EA}{8m_o}z\eta_z^4 - \frac{\gamma EI}{m_o}z\eta_{zz}^2\right.\\ & \left.+\gamma EI\eta_z\eta_{zz} + \frac{\gamma}{2}z\eta_t^2\right)\Big|_0^L - \frac{\gamma P_0}{2m_o}\int_0^L \eta_z^2\,dz - \frac{3\gamma EA}{8m_o}\int_0^L \eta_z^4\,dz - \frac{3\gamma EI}{2m_o}\int_0^L \eta_{zz}^2\,dz - \frac{\gamma}{2}\int_0^L \eta_t^2\,dz\\ & + \left(\eta_t(L,t) + \frac{\gamma L}{m_o}\eta_z(L,t)\right)\left(-b_H\eta_t(L,t) - P_0\eta_z(L,t) - \frac{EA}{2}\eta_z(L,t)^3 + EI\eta_{zzz} - \Delta(t,\eta_t(L,t))\right.\\ & \left.+C_{H3}A_H(x_{3e} + \alpha_1) + \frac{m_H\gamma L}{m_o}\eta_{zt}(L,t)\right) + \int_0^L (\eta_t + \gamma z\eta_z)f(\bullet)\,dz,\end{aligned} \tag{25}$$

where we used $f(\bullet)$ to denote $f(z,t,u(z,t),\eta_t(z,t))$ to save some space. Now substituting the boundary conditions given in Eq. (18) into Eq. (25) results in

$$\begin{aligned}\dot{W}_1 = {} & -\frac{\gamma P_0}{2m_o}\int_0^L \eta_z^2\,dz - \frac{3\gamma EA}{8m_o}\int_0^L \eta_z^4\,dz - \frac{3\gamma EI}{2m_o}\int_0^L \eta_{zz}^2\,dz - \frac{\gamma}{2}\int_0^L \eta_t^2\,dz - \frac{\gamma L P_0}{2m_o}\eta_z^2(L,t)\\ & - \frac{\gamma L EA}{8m_o}\eta_z^4(L,t) + \frac{\gamma L}{2}\eta_t^2(L,t) + \left(\eta_t(L,t) + \frac{\gamma L}{m_o}\eta_z(L,t)\right)\left(-b_H\eta_t(L,t) - \Delta(t,\eta_t(L,t))\right.\\ & \left.+C_{H3}A_H(x_{3e} + \alpha_1) + \frac{m_H\gamma L}{m_o}\eta_{zt}(L,t)\right) + \int_0^L (\eta_t + \gamma z\eta_z)f(\bullet)\,dz.\end{aligned} \tag{26}$$

From Eq. (26), we choose the virtual control α_1 as

$$\alpha_1 = \frac{1}{A_H C_{H3}}\left(-\frac{m_H\gamma L}{m_o}\eta_{zt}(L,t) - k_1\left(\eta_t(L,t) + \frac{\gamma L}{m_o}\eta_z(L,t)\right) - b_H\frac{\gamma L}{m_o}\eta_z(L,t) + \hat{\Delta}\right), \tag{27}$$

where k_1 is a positive constant to be selected later, $\hat{\Delta}$ is an estimate of $\Delta(t, \eta_t(L,t))$. The estimate $\hat{\Delta}$ is given by

$$\hat{\Delta} = -(\xi + kx_2),$$

$$\dot{\xi} = -\frac{k}{m_H}\xi - k\left(\Phi + \frac{k}{m_H}x_2\right), \tag{28}$$

where k is a positive constant, and we have defined

$$\Phi = -\frac{b_H}{m_H}x_2 - \frac{P_0}{m_H}\eta_z(L,t) - \frac{EA}{2m_H}\eta_z^3(L,t) + \frac{EI}{m_H}\eta_{zzz}(L,t) + \frac{A_H C_{H3}}{m_H}x_3. \tag{29}$$

It is noted that the disturbance observer (28) is based on Lemma 1 applied to the third equation of Eq. (18) with $\rho(x) = kx$. Define the disturbance observer error as

$$\Delta_e = \Delta - \hat{\Delta}. \tag{30}$$

Differentiating both sides of Eq. (30) along the solutions of Eq. (28) and the third equation of Eq. (18) gives

$$\dot{\Delta}_e = -\frac{k}{m_H}\Delta_e + \dot{\Delta}. \tag{31}$$

This equation will be used in the stability analysis of the closed-loop system after the control design is completed. Now substituting the virtual control α_1 given in Eq. (27) into Eq. (26) results in

$$\begin{aligned}\dot{W}_1 = & -\frac{\gamma P_0}{2m_o}\int_0^L \eta_z^2\,\mathrm{d}z - \frac{3\gamma EA}{8m_o}\int_0^L \eta_z^4\,\mathrm{d}z - \frac{3\gamma EI}{2m_o}\int_0^L \eta_{zz}^2\,\mathrm{d}z - \frac{\gamma}{2}\int_0^L \eta_t^2\,\mathrm{d}z - \frac{\gamma L P_0}{2m_o}\eta_z^2(L,t) \\ & -\frac{\gamma LEA}{8m_o}\eta_z^4(L,t) + \frac{\gamma L}{2}\eta_t^2(L,t) - (k_1 + b_H)\left(\eta_t(L,t) + \frac{\gamma L}{m_o}\eta_z(L,t)\right)^2 \\ & + \left(\eta_t(L,t) + \frac{\gamma L}{m_o}\eta_z(L,t)\right)(\Delta_e + C_{H3}A_H x_{3e}) + \int_0^L (\eta_t + \gamma z\eta_z)f(\bullet)\,\mathrm{d}z.\end{aligned} \tag{32}$$

On the other hand, substituting the virtual control α_1 into the third equation of Eq. (18) gives

$$\begin{aligned}\dot{x}_2 = & -\frac{b_H}{m_H}x_2 - \frac{P_0}{m_H}\eta_z(L,t) - \frac{EA}{2m_H}\eta_z^3(L,t) + \frac{EI}{m_H}\eta_{zzz}(L,t) + \left(-\frac{m_H\gamma L}{m_o}\eta_{zt}(L,t)\right. \\ & \left. -k_1\left(\eta_t(L,t) + \frac{\gamma L}{m_o}\eta_z(L,t)\right) - b_H\frac{\gamma L}{m_o}\eta_z(L,t) + \hat{\Delta}\right) + \frac{A_H C_{H3}}{m_H}x_{3e} - \frac{1}{m_H}\Delta_e.\end{aligned} \tag{33}$$

3.2. Step 2

Our goal at this step is to regulate x_{3e} to a small neighborhood of the origin by considering the fourth equation of the entire system (18) where for simplicity of the design process, we consider $g_3(x_3, x_4)$ as a control instead of x_4. As such, we define

$$x_{4e} = g_3(x_3, x_4) - \alpha_2, \tag{34}$$

where α_2 is a virtual control of $g_3(x_3, x_4)$. To design α_2, we first calculate $\dot{x}_{3e}$. Differentiating both sides of Eq. (20) along the solutions of Eq. (27), Eq. (34) and the third equation of Eq. (18) gives

$$\begin{aligned}\dot{x}_{3e} = & -\frac{4\beta_{He}A_H}{V_H C_{H3}}x_2 - \frac{4\beta_{He}C_{HT}}{V_H}x_3 + \frac{4\beta_{He}C_{HD}C_{H4}W_H}{V_H\sqrt{C_{H3}}}(x_{4e} + \alpha_2) + \frac{m_H\gamma L}{m_o}\eta_{ztt}(L,t) \\ & + \frac{(k_1 + b_H)\gamma L}{m_o}\eta_{zt}(L,t) - \frac{k}{m_H}\xi - \frac{k^2}{m_H}x_2 + k_1\Phi - \frac{k + k_1}{m_H}\Delta(t, \eta_t(L,t)).\end{aligned} \tag{35}$$

To design the virtual control α_2, we consider the following Lyapunov function candidate:

$$W_2 = W_1 + \tfrac{1}{2}x_{3e}^2 \tag{36}$$

whose derivative along the solutions of Eqs. (32) and (35) is

$$\begin{aligned}\dot{W}_2 = & -\frac{\gamma P_0}{2m_o}\int_0^L \eta_z^2\,\mathrm{d}z - \frac{3\gamma EA}{8m_o}\int_0^L \eta_z^4\,\mathrm{d}z - \frac{3\gamma EI}{2m_o}\int_0^L \eta_{zz}^2\,\mathrm{d}z - \frac{\gamma}{2}\int_0^L \eta_t^2\,\mathrm{d}z - \frac{\gamma L P_0}{2m_o}\eta_z^2(L,t) \\ & - \frac{\gamma LEA}{8m_o}\eta_z^4(L,t) + \frac{\gamma L}{2}\eta_t^2(L,t) - (k_1+b_H)\left(\eta_t(L,t) + \frac{\gamma L}{m_o}\eta_z(L,t)\right)^2 \\ & + x_{3e}\left[\left(\eta_t(L,t) + \frac{\gamma L}{m_o}\eta_z(L,t)\right)C_{H3}A_H - \frac{4\beta_{He}A_H}{V_H C_{H3}}x_2 - \frac{4\beta_{He}C_{HT}}{V_H}x_3 + \frac{4\beta_{He}C_{HD}C_{H4}W_H}{V_H\sqrt{C_{H3}}}\alpha_2\right. \\ & \left. + \frac{m_H\gamma L}{m_o}\eta_{ztt}(L,t) + \frac{(k_1+b_H)\gamma L}{m_o}\eta_{zt}(L,t) - \frac{k}{m_H}\xi - \frac{k^2}{m_H}x_2 + k_1\Phi - \frac{k+k_1}{m_H}\Delta(t,\eta_t(L,t))\right] \\ & + \left(\eta_t(L,t) + \frac{\gamma L}{m_o}\eta_z(L,t)\right)\Delta_e + \frac{4\beta_{He}C_{HD}C_{H4}W_H}{V_H\sqrt{C_{H3}}}x_{3e}x_{4e} + \int_0^L (\eta_t + \gamma z\eta_z)f(\bullet)\,\mathrm{d}z.\end{aligned} \tag{37}$$

Eq. (37) suggests that we choose the virtual control α_2 as follows:

$$\begin{aligned}\alpha_2 = & \frac{V_H\sqrt{C_{H3}}}{4\beta_{He}C_{HD}C_{H4}W_H}\left[-k_2x_{3e} - \left(\eta_t(L,t) + \frac{\gamma L}{m_o}\eta_z(L,t)\right)C_{H3}A_H + \frac{4\beta_{He}A_H}{V_H C_{H3}}x_2 + \frac{4\beta_{He}C_{HT}}{V_H}\alpha_1\right. \\ & \left. - \frac{m_H\gamma L}{m_o}\eta_{ztt}(L,t) - \frac{(k_1+b_H)\gamma L}{m_o}\eta_{zt}(L,t) + \frac{k}{m_H}\xi + \frac{k^2}{m_H}x_2 - k_1\Phi + \frac{k+k_1}{m_H}\hat{\Delta}\right],\end{aligned} \tag{38}$$

where k_2 is a positive constant, and we did not cancel the useful damping term $-(4\beta_{He}C_{HT}/V_H)x_{3e}$. It is seen from Eq. (38) that α_2 is a smooth function of x_2, x_3, $\eta_z(L,t)$, $\eta_{zt}(L,t)$, $\eta_{ztt}(L,t)$, $\eta_{zzz}(L,t)$ and ξ. Now substituting Eq. (38) into Eq. (37) gives

$$\begin{aligned}\dot{W}_2 = & -\frac{\gamma P_0}{2m_o}\int_0^L \eta_z^2\,\mathrm{d}z - \frac{3\gamma EA}{8m_o}\int_0^L \eta_z^4\,\mathrm{d}z - \frac{3\gamma EI}{2m_o}\int_0^L \eta_{zz}^2\,\mathrm{d}z - \frac{\gamma}{2}\int_0^L \eta_t^2\,\mathrm{d}z - \frac{\gamma L P_0}{2m_o}\eta_z^2(L,t) \\ & - \frac{\gamma LEA}{8m_o}\eta_z^4(L,t) + \frac{\gamma L}{2}\eta_t^2(L,t) - (k_1+b_H)\left(\eta_t(L,t) + \frac{\gamma L}{m_o}\eta_z(L,t)\right)^2 - \left(k_2 + \frac{4\beta_{He}C_{HT}}{V_H}\right)x_{3e}^2 \\ & - \frac{k+k_1}{m_H}x_{3e}\Delta_e + \left(\eta_t(L,t) + \frac{\gamma L}{m_o}\eta_z(L,t)\right)\Delta_e + \frac{4\beta_{He}C_{HD}C_{H4}W_H}{V_H\sqrt{C_{H3}}}x_{3e}x_{4e} + \int_0^L (\eta_t + \gamma z\eta_z)f(\bullet)\,\mathrm{d}z.\end{aligned} \tag{39}$$

Moreover, substituting Eq. (38) into Eq. (35) results in

$$\begin{aligned}\dot{x}_{3e} = & -\left(k_2 + \frac{4\beta_{He}C_{HT}}{V_H}\right)x_{3e} - \left(\eta_t(L,t) + \frac{\gamma L}{m_o}\eta_z(L,t)\right)C_{H3}A_H \\ & + \frac{4\beta_{He}C_{HD}C_{H4}W_H}{V_H\sqrt{C_{H3}}}x_{4e} - \frac{k+k_1}{m_H}\Delta_e.\end{aligned} \tag{40}$$

3.3. Step 3

This is the final step. The actual control input i_H will be designed to regulate x_{4e} to a small neighborhood of the origin. Since α_2 is a smooth function of x_2, x_3, $\eta_z(L,t)$, $\eta_{zt}(L,t)$, $\eta_{ztt}(L,t)$, $\eta_{zzz}(L,t)$ and ξ, differentiating both sides of Eq. (34) along the solutions of Eq. (38) and the entire system (18) gives

$$\begin{aligned}\dot{x}_{4e} = &\left(\frac{\partial g_3(x_3,x_4)}{\partial x_3} - \frac{\partial \alpha_2}{\partial x_3}\right)\left(-\frac{4\beta_{He}A_H}{V_H C_{H3}}x_2 - \frac{4\beta_{He}C_{HT}}{V_H}x_3 + \frac{4\beta_{He}C_{HD}C_{H4}W_H}{V_H\sqrt{C_{H3}}}g_3(x_3,x_4)\right)\\ &+ \frac{\partial g_3(x_3,x_4)}{\partial x_4}\left(-\frac{1}{\tau_{Hv}}x_4 + \frac{k_{Hv}}{\tau_{Hv}C_{H4}}i_H\right) - \frac{\partial \alpha_2}{\partial x_2}\left(\Phi - \frac{1}{m_H}\Delta(t,\eta_t(L,t))\right) - \frac{\partial \alpha_2}{\partial \eta_z(L,t)}\eta_{zt}(L,t)\\ &- \frac{\partial \alpha_2}{\partial \eta_{zt}(L,t)}\eta_{ztt}(L,t) - \frac{\partial \alpha_2}{\partial \eta_{ztt}(L,t)}\eta_{zttt}(L,t) - \frac{\partial \alpha_2}{\partial \eta_{zzz}(L,t)}\eta_{zzzt}(L,t)\\ &- \frac{\partial \alpha_2}{\partial \xi}\left(-\frac{k}{m_H}\xi - k\left(\Phi + \frac{k}{m_H}x_2\right)\right). \end{aligned} \tag{41}$$

To design the actual control i_H, we consider the following Lyapunov function candidate:

$$W_3 = W_2 + \tfrac{1}{2}x_{4e}^2 \tag{42}$$

whose derivative along the solutions of Eqs. (41) and (39) is

$$\begin{aligned}\dot{W}_3 = &-\frac{\gamma P_0}{2m_o}\int_0^L \eta_z^2\,\mathrm{d}z - \frac{3\gamma EA}{8m_o}\int_0^L \eta_z^4\,\mathrm{d}z - \frac{3\gamma EI}{2m_o}\int_0^L \eta_{zz}^2\,\mathrm{d}z - \frac{\gamma}{2}\int_0^L \eta_t^2\,\mathrm{d}z - \frac{\gamma L P_0}{2m_o}\eta_z^2(L,t)\\ &- \frac{\gamma LEA}{8m_o}\eta_z^4(L,t) + \frac{\gamma L}{2}\eta_t^2(L,t) - (k_1 + b_H)\left(\eta_t(L,t) + \frac{\gamma L}{m_o}\eta_z(L,t)\right)^2 - \left(k_2 + \frac{4\beta_{He}C_{HT}}{V_H}\right)x_{3e}^2\\ &+ x_{4e}\left[\frac{4\beta_{He}C_{HD}C_{H4}W_H}{V_H\sqrt{C_{H3}}}x_{3e} + \left(\frac{\partial g_3(x_3,x_4)}{\partial x_3} - \frac{\partial \alpha_2}{\partial x_3}\right)\left(-\frac{4\beta_{He}A_H}{V_H C_{H3}}x_2 - \frac{4\beta_{He}C_{HT}}{V_H}x_3\right.\right.\\ &\left.+ \frac{4\beta_{He}C_{HD}C_{H4}W_H}{V_H\sqrt{C_{H3}}}g_3(x_3,x_4)\right) + \frac{\partial g_3(x_3,x_4)}{\partial x_4}\left(-\frac{1}{\tau_{Hv}}x_4 + \frac{k_{Hv}}{\tau_{Hv}C_{H4}}i_H\right)\\ &- \frac{\partial \alpha_2}{\partial x_2}\left(\Phi - \frac{1}{m_H}\Delta(t,\eta_t(L,t))\right) - \frac{\partial \alpha_2}{\partial \eta_z(L,t)}\eta_{zt}(L,t) - \frac{\partial \alpha_2}{\partial \eta_{zt}(L,t)}\eta_{ztt}(L,t)\\ &\left.- \frac{\partial \alpha_2}{\partial \eta_{ztt}(L,t)}\eta_{zttt}(L,t) - \frac{\partial \alpha_2}{\partial \eta_{zzz}(L,t)}\eta_{zzzt}(L,t) - \frac{\partial \alpha_2}{\partial \xi}\left(-\frac{k}{m_H}\xi - k\left(\Phi + \frac{k}{m_H}x_2\right)\right)\right]\\ &- \frac{k + k_1}{m_H}x_{3e}\Delta_e + \left(\eta_t(L,t) + \frac{\gamma L}{m_o}\eta_z(L,t)\right)\Delta_e + \int_0^L (\eta_t + \gamma z\eta_z)f(\bullet)\,\mathrm{d}z. \end{aligned} \tag{43}$$

Eq. (43) suggests that we choose the actual control i_H as follows:

$$\begin{aligned} i_H = &\frac{\tau_{Hv}C_{H4}}{k_{Hv}\dfrac{\partial g_3(x_3,x_4)}{\partial x_4}}\left[-k_3 x_{4e} - \frac{4\beta_{He}C_{HD}C_{H4}W_H}{V_H\sqrt{C_{H3}}}x_{3e} - \left(\frac{\partial g_3(x_3,x_4)}{\partial x_3} - \frac{\partial \alpha_2}{\partial x_3}\right)\left(-\frac{4\beta_{He}A_H}{V_H C_{H3}}x_2\right.\right.\\ &\left.- \frac{4\beta_{He}C_{HT}}{V_H}x_3 + \frac{4\beta_{He}C_{HD}C_{H4}W_H}{V_H\sqrt{C_{H3}}}g_3(x_3,x_4)\right) + \frac{\partial g_3(x_3,x_4)}{\partial x_4}\frac{1}{\tau_{Hv}}x_4 + \frac{\partial \alpha_2}{\partial x_2}\left(\Phi - \frac{1}{m_H}\hat{\Delta}\right)\\ &+ \frac{\partial \alpha_2}{\partial \eta_z(L,t)}\eta_{zt}(L,t) + \frac{\partial \alpha_2}{\partial \eta_{zt}(L,t)}\eta_{ztt}(L,t) + \frac{\partial \alpha_2}{\partial \eta_{ztt}(L,t)}\eta_{zttt}(L,t) + \frac{\partial \alpha_2}{\partial \eta_{zzz}(L,t)}\eta_{zzzt}(L,t)\\ &\left.+ \frac{\partial \alpha_2}{\partial \xi}\left(-\frac{k}{m_H}\xi - k\left(\Phi + \frac{k}{m_H}x_2\right)\right)\right], \end{aligned} \tag{44}$$

where k_3 is a positive constant. It is seen from Eq. (44) that the signals $\eta(L,t)$, $\eta_z(L,t)$, $\eta_{zt}(L,t)$, $\eta_{ztt}(L,t)$, $\eta_{zttt}(L,t)$, $\eta_{zzz}(L,t)$, $\eta_{zzzt}(L,t)$, x_3 and x_4, which are measurable or numerically calculated from measurable signals, are required for implementation. It is noted that differentiating twice and three times the slope $\eta_z(L,t)$ with respect to time to get $\eta_{ztt}(L,t)$ and $\eta_{zttt}(L,t)$, respectively, is undesirable in practice due to noise amplification. Therefore, it is suggested to use the boundary condition, the second equation of Eq. (13) to estimate $\eta_{ztt}(L,t)$ and $\eta_{zttt}(L,t)$. We define an estimate of $\eta_{tt}(L,t)$ by $\hat{\eta}_{tt}(L,t)$ when the disturbance $\Delta(t,\eta_t(L,t))$ in the second equation of Eq. (13) is replaced by its estimate $\hat{\Delta}$ given in Eq. (28). With this notation and the second equation of Eq. (13), we have

$$\hat{\eta}_{tt}(L,t) = -\frac{b_H}{m_H}\eta_t(L,t) - \frac{P_0}{m_H}\eta_z(L,t) - \frac{EA}{2m_H}\eta_z^3(L,t) + \frac{EI}{m_H}\eta_{zzz}(L,t) + \frac{A_H}{m_H}P_H - \frac{1}{m_H}\hat{\Delta}. \tag{45}$$

Now, numerical differentiation of both sides of Eq. (45) with respect to the spatial variable z gives an estimate $\hat{\eta}_{ztt}(L,t)$ of $\eta_{ztt}(L,t)$. On the other hand, numerical differentiation of both sides of Eq. (45) with respect to time t, then with respect to the spatial variable z, gives an estimate $\hat{\eta}_{zttt}(L,t)$ of $\eta_{zttt}(L,t)$. The estimates $\hat{\eta}_{ztt}(L,t)$ and $\hat{\eta}_{zttt}(L,t)$ can be used in the control expression (44) instead of $\eta_{ztt}(L,t)$ and $\eta_{zttt}(L,t)$, respectively.

Substituting Eq. (44) into Eq. (43) results in

$$\begin{aligned}\dot{W}_3 = &-\frac{\gamma P_0}{2m_o}\int_0^L \eta_z^2\,\mathrm{d}z - \frac{3\gamma EA}{8m_o}\int_0^L \eta_z^4\,\mathrm{d}z - \frac{3\gamma EI}{2m_o}\int_0^L \eta_{zz}^2\,\mathrm{d}z - \frac{\gamma}{2}\int_0^L \eta_t^2\,\mathrm{d}z - \frac{\gamma L P_0}{2m_o}\eta_z^2(L,t)\\ &-\frac{\gamma LEA}{8m_o}\eta_z^4(L,t) + \frac{\gamma L}{2}\eta_t^2(L,t) - (k_1+b_H)\left(\eta_t(L,t)+\frac{\gamma L}{m_o}\eta_z(L,t)\right)^2 - \left(k_2+\frac{4\beta_{He}C_{HT}}{V_H}\right)x_{3e}^2\\ &-k_3x_{4e}^2 + \frac{\partial\alpha_2}{\partial x_2}\frac{1}{m_H}x_{4e}\Delta_e - \frac{k+k_1}{m_H}x_{3e}\Delta_e + \left(\eta_t(L,t)+\frac{\gamma L}{m_o}\eta_z(L,t)\right)\Delta_e + \int_0^L(\eta_t+\gamma z\eta_z)f(\bullet)\,\mathrm{d}z.\end{aligned} \tag{46}$$

Before going further, let us consider the following:

$$Q_{W31} = -\frac{\gamma L P_0}{2m_o}\eta_z^2(L,t) + \frac{\gamma L}{2}\eta_t^2(L,t) - (k_1+b_H)\left(\eta_t(L,t)+\frac{\gamma L}{m_o}\eta_z(L,t)\right)^2,$$

$$Q_{W32} = \int_0^L(\eta_t+\gamma z\eta_z)f(\bullet)\,\mathrm{d}z. \tag{47}$$

Using Eq. (6), a simple calculation shows that

$$Q_{W31} \leqslant -(k_1+b_H-\gamma L)\left(\eta_t(L,t)+\frac{\gamma L}{m_o}\eta_z(L,t)\right)^2 - \frac{\gamma L}{m_o}\left(\frac{P_0}{2}-\frac{(\gamma L)^2}{m_o}\right)\eta_z^2(L,t),$$

$$Q_{W32} \leqslant -\left(\Omega_D - \frac{\gamma L\Omega_D}{4\varepsilon_1} - \varepsilon\right)\int_0^L \eta_t^2\,\mathrm{d}z + (\gamma L\Omega_D\varepsilon_1 + \gamma L\varepsilon)\int_0^L \eta_z^2\,\mathrm{d}z + \frac{1+\gamma L}{4\varepsilon}\int_0^L f_L^2\,\mathrm{d}z, \tag{48}$$

where ε and ε_1 are arbitrarily positive constants. Now substituting Eq. (48) into Eq. (46) gives

$$\begin{aligned}\dot{W}_3 \leqslant &-\left(\frac{\gamma P_0}{2m_o} - \gamma L\Omega_D\varepsilon_1 - \gamma L\varepsilon\right)\int_0^L \eta_z^2\,\mathrm{d}z - \frac{3\gamma EA}{8m_o}\int_0^L \eta_z^4\,\mathrm{d}z - \frac{3\gamma EI}{2m_o}\int_0^L \eta_{zz}^2\,\mathrm{d}z\\ &-\left(\frac{\gamma}{2}+\Omega_D-\frac{\gamma L\Omega_D}{4\varepsilon_1}-\varepsilon\right)\int_0^L \eta_t^2\,\mathrm{d}z - \frac{\gamma L}{m_o}\left(\frac{P_0}{2}-\frac{(\gamma L)^2}{m_o}\right)\eta_z^2(L,t) - \frac{\gamma LEA}{8m_o}\eta_z^4(L,t)\\ &-(k_1+b_H-\gamma L)\left(\eta_t(L,t)+\frac{\gamma L}{m_o}\eta_z(L,t)\right)^2 - \left(k_2+\frac{4\beta_{He}C_{HT}}{V_H}\right)x_{3e}^2 - k_3x_{4e}^2\\ &+\frac{\partial\alpha_2}{\partial x_2}\frac{1}{m_H}x_{4e}\Delta_e - \frac{k+k_1}{m_H}x_{3e}\Delta_e + \left(\eta_t(L,t)+\frac{\gamma L}{m_o}\eta_z(L,t)\right)\Delta_e + \frac{1+\gamma L}{4\varepsilon}\int_0^L f_L^2\,\mathrm{d}z.\end{aligned} \tag{49}$$

Therefore, it is sufficient to choose the constant k_1 and γ such that

$$\frac{\gamma P_0}{2m_o} - \gamma L\Omega_D\varepsilon_1 - \gamma L\varepsilon = c_3,$$

$$\frac{\gamma}{2} + \Omega_D - \frac{\gamma L\Omega_D}{4\varepsilon_1} - \varepsilon = c_4,$$

$$k_1 + b_H - \gamma L = c_5,$$

$$\frac{P_0}{2}-\frac{(\gamma L)^2}{m_o}=c_6, \tag{50}$$

where c_3, c_4, c_5, and c_6 are strictly positive constants. It is recalled that the constant γ also needs to satisfy condition (24). A straightforward verification shows that there always exist design constants γ and k_1 simultaneously satisfying conditions (24) and (50). On the other hand, substituting Eq. (44) into Eq. (41) gives

$$\dot{x}_{4e}=-k_3x_{4e}-\frac{4\beta_{He}C_{HD}C_{H4}W_H}{V_H\sqrt{C_{H3}}}x_{3e}+\frac{\partial\alpha_2}{\partial x_2}\frac{1}{m_H}\Delta_e. \tag{51}$$

The control design has been completed. For convenience of stability analysis later, we rewrite the closed-loop system consisting of the first and the last two equations of Eqs. (18), (33), (40), (51), and (31) as follows:

$$\begin{aligned}
m_o\eta_{tt}(z,t)&=-EI\eta_{zzzz}(z,t)+P_0\eta_{zz}(z,t)+\frac{3EA}{2}\eta_z^2(z,t)\eta_{zz}(z,t)+f(z,t,u(z,t),\eta_t(z,t)),\\
\dot{x}_1&=x_2,\\
\dot{x}_2&=-\frac{b_H}{m_H}x_2-\frac{P_0}{m_H}\eta_z(L,t)-\frac{EA}{2m_H}\eta_z^3(L,t)+\frac{EI}{m_H}\eta_{zzz}(L,t)+\Bigg(-\frac{m_H\gamma L}{m_o}\eta_{zt}(L,t)\\
&\quad-k_1\left(\eta_t(L,t)+\frac{\gamma L}{m_o}\eta_z(L,t)\right)-b_H\frac{\gamma L}{m_o}\eta_z(L,t)+\hat{\Delta}\Bigg)+\frac{A_HC_{H3}}{m_H}x_{3e}-\frac{1}{m_H}\Delta_e,\\
\dot{x}_{3e}&=-\left(k_2+\frac{4\beta_{He}C_{HT}}{V_H}\right)x_{3e}-\left(\eta_t(L,t)+\frac{\gamma L}{m_o}\eta_z(L,t)\right)C_{H3}A_H\\
&\quad+\frac{4\beta_{He}C_{HD}C_{H4}W_H}{V_H\sqrt{C_{H3}}}x_{4e}-\frac{k+k_1}{m_H}\Delta_e,\\
\dot{x}_{4e}&=-k_3x_{4e}-\frac{4\beta_{He}C_{HD}C_{H4}W_H}{V_H\sqrt{C_{H3}}}x_{3e}+\frac{\partial\alpha_2}{\partial x_2}\frac{1}{m_H}\Delta_e,\\
\eta_{zz}(0,t)&=\eta_{zz}(L,t)=0,\\
\eta(0,t)&=0,\\
\dot{\Delta}_e&=-\frac{k}{m_H}\Delta_e+\dot{\Delta}.
\end{aligned} \tag{52}$$

We are ready to state the main result of our paper in the following theorem.

Theorem 1. *Under Assumption* 1, *the control input i_H given in Eq.* (44) *solves the control objective provided that the initial tension P_0 is strictly positive, and the design constants γ and k_1 are chosen such that conditions* (24) *and* (50) *hold. In particular, the solutions of the closed-loop system* (52) *exist and are unique. Moreover, when the external distributed disturbance f_L is zero, and the disturbance $\Delta(t,\eta_t(L,t))$ is constant, all the terms $|\eta(z,t)|$, $\int_0^L\eta_z^2(z,t)\,dz$, $\int_0^L\eta_t^2(z,t)\,dz$, and $\int_0^L\eta_{zz}^2(z,t)\,dz$ exponentially converge to zero, and when the external distributed disturbance f_L is different from zero but bounded, and the disturbance $\Delta(t,\eta_t(L,t))$ is time varying with bounded derivatives, all the terms $|\eta(z,t)|$, $\int_0^L\eta_z^2(z,t)\,dz$, $\int_0^L\eta_t^2(z,t)\,dz$ and $\int_0^L\eta_{zz}^2(z,t)\,dz$ exponentially converge to some small positive constants.*

Proof. See Appendix B. □

4. Simulations

To illustrate the effectiveness of the controller proposed in the previous section, we carry out some simulations in this section. The parameters of the hydraulic system are taken based on [23] as follows: $m_H=1000\,\mathrm{kg}$, $A_H=0.65\,\mathrm{m}^2$, $b_H=40\,\mathrm{N/(m/s)}$, $4\beta_{He}/V_H=4.53\times10^8\,\mathrm{N/m^5}$, $C_{HD}=2.21\times10^{-14}\,\mathrm{m^5/N\,s}$,

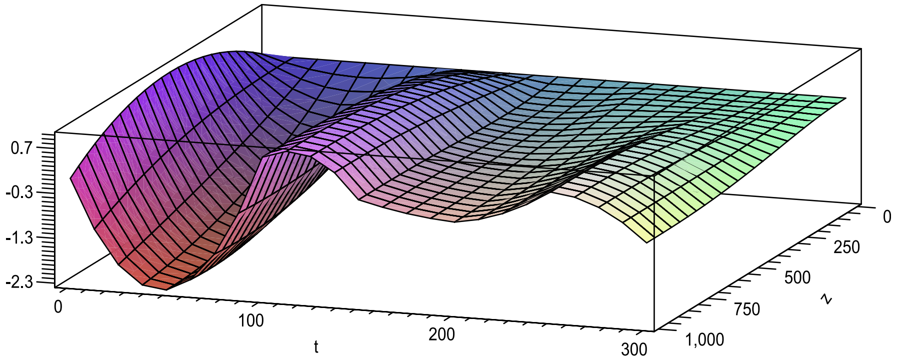

Fig. 4. Simulation result without the proposed controller: transverse displacement $\eta(z, t)$.

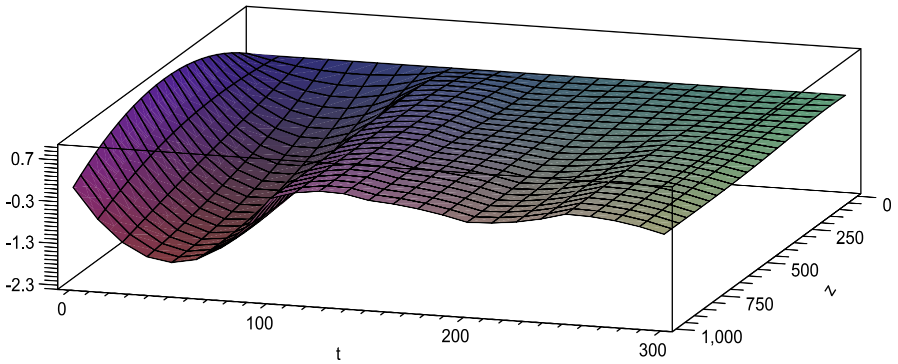

Fig. 5. Simulation result with the proposed controller: transverse displacement $\eta(z, t)$.

$C_{HD}W_H/\sqrt{\rho} = 3.42 \times 10^{-5}\,\mathrm{m}^3\sqrt{N}\,\mathrm{s}$, $P_{HS} = 10,342,500\,\mathrm{Pa}$, $k_{Hv} = 0.0324$, $\tau_{Hv} = 0.00636$. The scale factors are taken as $C_{H3} = 6 \times 10^5$, $C_{H4} = 5 \times 10^{-7}$ to scale P_{HS} down and τ_{Hv} up as discussed in Section 2.2.2. The riser parameters assumed are: length $L = 1000\,\mathrm{m}$, diameter $D = 0.14\,\mathrm{m}$, density $\rho = 8200\,\mathrm{kg/m}^3$, Young's modulus $E = 2 \times 10^8\,\mathrm{kg/m}^2$, initial tension $P_0 = 60 \times 10^3\,\mathrm{N}$. The parameters of the distributed damping and external forces are taken as follows: $c = 120\,\mathrm{N\,s/m}$, $\rho_w = 1024\,\mathrm{kg/m}^3$, and $C_D = 1.2$. Using the linear wave theory, the horizontal water particle velocity and acceleration are expressed respectively as [20]

$$u(z,t) = \sum_{i=1}^{N_w}\left(A_{wi}w_{wi}\frac{\cosh(k_{wi}z)}{\sinh(k_{wi}L)}\sin(w_{wi}t + \varphi_{wi})\right), \tag{53}$$

where the amplitude A_{wi}, wave number k_{wi}, frequency w_{wi}, phase φ_{wi} of the wave ith are given by

$$w_{wi} = w_m + \frac{w_m - w_M}{N_w}i, \quad S_{wi} = \frac{1.25}{4}\frac{w_o^4}{w_{wi}^5}H_{sw}^2\mathrm{e}^{-1.25(w_o/w_{wi})^4},$$

$$A_{wi} = \sqrt{2S_{wi}\frac{w_{mi} - w_{Mi}}{N_w}}, \quad 9.8k_{wi}\tanh(k_{wi}L) = w_{wi}^2, \quad \varphi_{wi} = 2\pi\,\mathrm{rand}(). \tag{54}$$

In Eq. (54), minimum and maximum wave frequencies are $w_m = 0.2\,\mathrm{rand/s}$, $w_M = 2.5\,\mathrm{rand/s}$; the two-parameter Bretschneider spectrum S_{wi} is used with the significant wave height $H_{sw} = 4\,\mathrm{m}$; the modal frequency is $w_o = 2\pi/T_w$ with the period $T_w = 7.8$; $N_w = 10$; and rand() is a random number between 0 and 1. The observer and control gains are chosen as: $k = 2$, $\gamma = 10^{-3}$, $k_1 = k_2 = k_3 = 2$. It is directly verified that conditions (24) and (50) are well satisfied with $\varepsilon = \varepsilon_1 = 10^{-3}$. We assume that the disturbance $\Delta(t, \eta_t(L,t)) = 0.5m_H\sin(0.5t + 2\pi\,\mathrm{rand}())$. The initial conditions are taken as $t_0 = 0$, $\eta(z, t_0) = 4z(z-L)/L^2$,

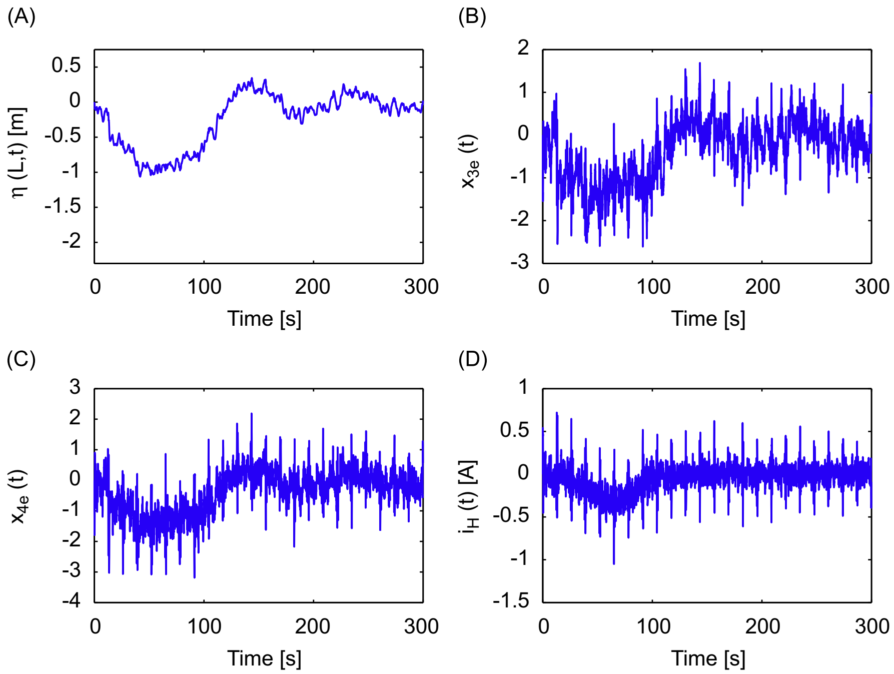

Fig. 6. Simulation result with the proposed controller: (A) transverse displacement at the top end $\eta(L,t)$; (B) virtual error $x_{3e}(t)$; (C) virtual error $x_{4e}(t)$; (D) control input $i_H(t)$.

$\eta_t(z,t_0)=0$ and $\xi(t_0)=0$. We use a finite difference scheme [24] to approximate the uncontrolled/controlled (continuous) system for the simulation purposes.

We run simulations without the proposed boundary controller, i.e. we set all the observer and control gains to zero ($k=k_1=k_2=k_3=0$), and with the proposed boundary controller, i.e. $k=2$, $\gamma=10^{-3}$, $k_1=k_2=k_3=2$. The length of simulation time for both cases is 300 s. The transverse displacement $\eta(z,t)$ for the uncontrolled and controlled cases are displayed in Figs. 4 and 5, respectively. It is seen from these figures that the proposed boundary controller can reduce deflections of the riser transverse motion significantly, i.e. the displacement magnitudes are significantly reduced. This illustrates the effectiveness of the proposed boundary controller in the sense that it is able to drive the riser to the small neighborhood of its equilibrium position (Fig. 6).

5. Conclusions

Based on the energy approach, the equations of motion of a marine riser-hydraulic system were presented. These equations were then used for the design of the boundary controller at the top end of the riser based on Lyapunov's direct method. The proposed controller robustly stabilized the riser at its equilibrium vertical position. Proof of existence and uniqueness of the solutions of the closed-loop system was given. Future work is to carry out experiments to validate the proposed controller.

Acknowledgments

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Appendix A. Useful lemmas

This appendix provides two useful lemmas that will be used in the proof of Theorem 1.

Lemma 3. *For any $y = [y_1, \ldots, y_i, \ldots, y_n]^{\mathrm{T}}$ with $y_i \in C^1[0, L]$, $i = 1, \ldots, n$, the following inequalities hold*:

$$\int_0^L y(s).y(s)\,\mathrm{d}s \leqslant 2Ly(0).y(0) + 4L^2 \int_0^L y_s(s).y_s(s)\,\mathrm{d}s, \tag{55}$$

$$\int_0^L y(s).y(s)\,\mathrm{d}s \leqslant 2Ly(L).y(L) + 4L^2 \int_0^L y_s(s).y_s(s)\,\mathrm{d}s. \tag{56}$$

Proof. We prove Eq. (56). The proof of Eq. (55) is similar by using a change of coordinate $\xi = L - s$. Using integration by parts, we have

$$\begin{aligned}\int_0^L y(s).y(s)\,\mathrm{d}s &= y(s).y(s)s|_0^L - 2\int_0^L sy(s).y_s(s)\,\mathrm{d}s \\ &\leqslant Ly(L).y(L) + \frac{1}{2}\int_0^L y(s).y(s)\,\mathrm{d}s + 2\int_0^L s^2 y_s(s).y_s(s)\,\mathrm{d}s \\ &\leqslant Ly(L).y(L) + \frac{1}{2}\int_0^L y(s).y(s)\,\mathrm{d}s + 2L^2\int_0^L y_s(s).y_s(s)\,\mathrm{d}s\end{aligned} \tag{57}$$

which gives Eq. (56). □

Lemma 4. *For any $y = [y_1, \ldots, y_i, \ldots, y_n]^{\mathrm{T}}$ with $y_i \in C^1[0, L]$, $i = 1, \ldots, n$, the following inequalities hold*:

$$\max_{s\in[0,L]}(y(s).y(s)) \leqslant y(0).y(0) + 2\sqrt{\int_0^L y(s).y(s)\,\mathrm{d}s}\sqrt{\int_0^L y_s(s).y_s(s)\,\mathrm{d}s}, \tag{58}$$

$$\max_{s\in[0,L]}(y(s).y(s)) \leqslant y(L).y(L) + 2\sqrt{\int_0^L y(s).y(s)\,\mathrm{d}s}\sqrt{\int_0^L y_s(s).y_s(s)\,\mathrm{d}s}. \tag{59}$$

Proof. We prove Eq. (58). The proof of Eq. (59) is similar by using a change of coordinate $\xi = L - s$. From fundamental of calculus, we have

$$\begin{aligned}y(s).y(s) &= y(0).y(0) + 2\int_0^s y(\zeta).y_\zeta(\zeta)\,\mathrm{d}\zeta \\ &\leqslant y(0).y(0) + 2\sqrt{\int_0^s y(\zeta).y(\zeta)\,\mathrm{d}\zeta}\sqrt{\int_0^s y_\zeta(\zeta).y_\zeta(\zeta)\,\mathrm{d}\zeta},\end{aligned} \tag{60}$$

where we have used the Cauchy–Schwartz inequality. □

Appendix B. Proof of Theorem 1

We first prove existence and uniqueness of the solutions of the closed-loop system (52) then move to prove convergence of the solutions.

B.1. Existence and uniqueness

Let $H^2(0, L)$ be the usual Hilbert space [25]. Our analysis is based on the Sobolev spaces:

$$V_S = \eta \in H^2(0, L)|\eta(0, t) = 0 \tag{61}$$

equipped with the norm $\|\eta\|_{V_S} = \|\eta_{zz}\|_2$, and

$$W_S = \eta \in V_S \cap H^4(0, L)|\eta_{zz}(0, t) = 0, \eta_{zz}(L, t) = 0 \tag{62}$$

equipped with the norm $\|\eta\|_{W_S} = \|\eta_{zz}\|_2 + \|\eta_{zzzz}\|_2$ where $\|.\|_p$ denotes the L^p norms. From the Poincare inequality, it follows that $\|.\|_{V_S}$ and $\|.\|_{W_S}$ are equivalent to the standard norms of $H^2(0, L)$ and $H^4(0, L)$, respectively. Next, we consider $\phi \in V_S$. Now inner producting both sides of the first equation of Eq. (52) by ϕ

then integrating from 0 to L results in

$$m_o\int_0^L \eta_{tt}\phi\,\mathrm{d}z = -EI\int_0^L \eta_{zzzz}\phi\,\mathrm{d}z + P_0\int_0^L \eta_{zz}\phi\,\mathrm{d}z + \frac{3EA}{2}\int_0^L \eta_z^2\eta_{zz}\phi\,\mathrm{d}z + \int_0^L f(\bullet)\phi\,\mathrm{d}z. \tag{63}$$

We will use the Galerkin approximation to show that for all $\phi \in V_S$ there exists $\eta \in W_S$ such that Eq. (63) holds. Let ϕ^j be a component of a complete orthogonal system of W_S for which $\{\eta(z,t_0), \eta_t(z,t_0)\} \in \mathrm{Span}\{\phi^1, \phi^2\}$. For each $n \in N$, let $W_{Sn} = \mathrm{Span}\{\phi^1, \phi^2, \ldots, \phi^n\}$. We search for a function $\eta^n(z,t) = \sum_{j=1}^n k^j(t)\phi^j$ such that for any $\phi \in W_{Sn}$, it satisfies the approximate closed-loop system

$$m_o\int_0^L \eta^n_{tt}\phi\,\mathrm{d}z = -EI\int_0^L \eta^n_{zzzz}\phi\,\mathrm{d}z + P_0\int_0^L \eta^n_{zz}\phi\,\mathrm{d}z + \frac{3EA}{2}\int_0^L \eta_z^{n2}\eta^n_{zz}\phi\,\mathrm{d}z + \int_0^L f^n(\bullet)\phi\,\mathrm{d}z,$$

$$\dot{x}_1^n = x_2^n,$$

$$\begin{aligned}\dot{x}_2^n = &-\frac{b_H}{m_H}x_2^n - \frac{P_0}{m_H}\eta_z^n(L,t) - \frac{EA}{2m_H}\eta_z^{n3}(L,t) + \frac{EI}{m_H}\eta^n_{zzz}(L,t) + \left(-\frac{m_H\gamma L}{m_o}\eta^n_{zt}(L,t)\right.\\ &\left.-k_1\left(\eta_t^n(L,t) + \frac{\gamma L}{m_o}\eta_z^n(L,t)\right) - b_H\frac{\gamma L}{m_o}\eta_z^n(L,t) + \hat{\Delta}^n\right) + \frac{A_H C_{H3}}{m_H}x_{3e}^n - \frac{1}{m_H}\Delta_\mathrm{e}^n,\end{aligned}$$

$$\begin{aligned}\dot{x}_{3e}^n = &-\left(k_2 + \frac{4\beta_{He}C_{HT}}{V_H}\right)x_{3e}^n - \left(\eta_t^n(L,t) + \frac{\gamma L}{m_o}\eta_z^n(L,t)\right)C_{H3}A_H\\ &+\frac{4\beta_{He}C_{HD}C_{H4}W_H}{V_H\sqrt{C_{H3}}}x_{4e}^n - \frac{k+k_1}{m_H}\Delta_\mathrm{e}^n,\end{aligned}$$

$$\dot{x}_{4e}^n = -k_3x_{4e}^n - \frac{4\beta_{He}C_{HD}C_{H4}W_H}{V_H\sqrt{C_{H3}}}x_{3e}^n + \frac{\partial\alpha_2}{\partial x_2}\frac{1}{m_H}\Delta_\mathrm{e}^n,$$

$$\eta^n_{zz}(0,t) = \eta^n_{zz}(L,t) = 0,$$

$$\eta^n(0,t) = 0,$$

$$\dot{\Delta}_\mathrm{e}^n = -\frac{k}{m_H}\Delta_\mathrm{e}^n + \dot{\Delta}^n. \tag{64}$$

with the initial conditions

$$\eta^n(z,t_0) = \eta(z,t_0), \quad \eta_t^n(z,t_0) = \eta_t(z,t_0) \tag{65}$$

which are possible since $(\eta(z,t_0), \eta_t(z,t_0))$ belongs to W_{Sn} for $n \geqslant 2$. Note that Eqs. (64) and (65) are in fact a system of ordinary differential equations in the variable t, which has a local solution in $[0, t_n)$. After the estimates below, the approximate solution will be extended to the interval $[0, T]$ for any given $T > 0$.

Estimate I: *Upper bound of* $\int_0^L \eta_t^{n2}\,\mathrm{d}z + \int_0^L \eta_{zz}^{n2}\,\mathrm{d}z$. In Eq. (64), we take $\phi = \eta_t^n$ and consider the following Lyapunov function candidate:

$$\begin{aligned}W_n = &\frac{m_o}{2}\int_0^L \eta_t^{n2}\,\mathrm{d}z + \frac{P_0}{2}\int_0^L \eta_z^{n2}\,\mathrm{d}z + \frac{EA}{8}\int_0^L \eta_z^{n4}\,\mathrm{d}z + \frac{EI}{2}\int_0^L \eta_{zz}^{n2}\,\mathrm{d}z + \gamma\int_0^L z\eta_t^n\eta_z^n\,\mathrm{d}z\\ &+\frac{m_H}{2}\left(\eta_t^n(L,t) + \frac{\gamma L}{m_o}\eta_z^n(L,t)\right)^2 + \frac{1}{2}x_{3e}^{n2} + \frac{1}{2}x_{4e}^{n2} + \frac{\lambda}{2}\Delta_e^{n2},\end{aligned} \tag{66}$$

where γ is the positive constant specified in Section 3, and λ is a positive constant to be specified later. Indeed, as in Section 3, the function W_n is proper (i.e. positive definite and radially unbounded). We use the same technique in Section 3 to calculate the time derivative of the function W_n along the solutions of Eq. (64):

$$\begin{aligned}\dot{W}_n \leqslant & -c_3\int_0^L \eta_z^{n2}\,\mathrm{d}z - \frac{3\gamma EA}{8m_o}\int_0^L \eta_z^{n4}\,\mathrm{d}z - \frac{3\gamma EI}{2m_o}\int_0^L \eta_{zz}^{n2}\,\mathrm{d}z - c_4\int_0^L \eta_t^{n2}\,\mathrm{d}z - \frac{\gamma L}{m_o}c_6\eta_z^{n2}(L,t)\\ & -\frac{\gamma LEA}{8m_o}\eta_z^{n4}(L,t) - (c_5-\sigma)\left(\eta_t^n(L,t)+\frac{\gamma L}{m_o}\eta_z^n(L,t)\right)^2 - \left(k_2+\frac{4\beta_{He}C_{HT}}{V_H}-\sigma\right)x_{3e}^{n2} - (k_3-\sigma)x_{4e}^{n2}\\ & -\left(\frac{\lambda k}{m_H}-\frac{1}{4\sigma}\left(\frac{\partial\alpha_2^n}{\partial x_2^n}\frac{1}{m_H}\right)^2-\frac{1}{4\sigma}\left(\frac{k+k_1}{m_H}\right)^2-\frac{1}{4\sigma}-\sigma\right)\Delta_{\mathrm{e}}^{n2}+\frac{\lambda}{4\sigma}\dot{\Delta}^{n2}+\frac{1+\gamma L}{4\varepsilon}\int_0^L f_L^{n2}\,\mathrm{d}z,\end{aligned} \tag{67}$$

where the positive constants c_3, c_4, and c_5 are specified in Section 3, see Eqs. (24) and (50), and σ is an arbitrarily positive constant. From Eqs. (66) and (67), there exist sufficiently small σ and sufficiently large λ such that

$$\dot{W}_n \leqslant -c_nW_n + Q_n, \tag{68}$$

where c_n is a strictly positive constant, and Q_n is the maximum value of $(\lambda/4\sigma)\dot{\Delta}^{n2}+((1+\gamma L)/4\varepsilon)\int_0^L f_L^{n2}\,\mathrm{d}z$, which is bounded by some nonnegative constant. The differential inequality (68) implies that $W_n(t)\leqslant(W_n(t_0)-(Q_n/c_n))\mathrm{e}^{(t-t_0)}+(Q_n/c_n)$, which in turn means there exists a nonnegative constant M_1 such that

$$\int_0^L \eta_t^{n2}\,\mathrm{d}z+\int_0^L \eta_z^{n2}\,\mathrm{d}z+\int_0^L \eta_{zz}^{n2}\,\mathrm{d}z\leqslant M_1 \quad \forall t\in[0,T],\ \ n\in N. \tag{69}$$

Estimate II: *Upper bound of* $\eta_{tt}(z,t_0)$ in L^2-*norm*. In the first equation of Eq. (64), taking $\phi=\eta_{tt}^n(z,t_0)$ and $t=t_0$ gives

$$\begin{aligned}m_o\int_0^L \eta_{tt}^{n2}(z,t_0)\,\mathrm{d}z = & -EI\int_0^L \eta_{zzzz}^n(z,t_0)\eta_{tt}^n(z,t_0)\,\mathrm{d}z + P_0\int_0^L \eta_{zz}^n(z,t_0)\eta_{tt}^n(z,t_0)\,\mathrm{d}z\\ & +\frac{3EA}{2}\int_0^L \eta_z^{n2}(z,t_0)\eta_{zz}^n(z,t_0)\eta_{tt}^n(z,t_0)\,\mathrm{d}z+\int_0^L f^n(\bullet)|_{t=t_0}\eta_{tt}^n(z,t_0)\,\mathrm{d}z.\end{aligned} \tag{70}$$

A simple calculation shows that

$$\begin{aligned}(m_o-4\mu_1)\int_0^L \eta_{tt}^{n2}(z,t_0)\,\mathrm{d}z\leqslant & \frac{(EI)^2}{4\mu_1}\int_0^L \eta_{zzzz}^{n2}(z,t_0)\,\mathrm{d}z+\frac{P_0^2}{4\mu_1}\int_0^L \eta_{zz}^{n2}(z,t_0)\,\mathrm{d}z\\ & +\left(\frac{3EA}{2}\right)^2\frac{1}{4\mu_1}\int_0^L \eta_z^{n4}(z,t_0)\eta_{zz}^{n2}(z,t_0)\,\mathrm{d}z+\frac{1}{4\mu_1}\int_0^L f^{n2}(\bullet)\Big|_{t=t_0}\,\mathrm{d}z,\end{aligned} \tag{71}$$

where μ_1 is an arbitrarily positive constant. Since the initial values $\eta(z,t_0)$ and $\eta_t(z,t_0)$ are sufficiently smooth, and we have already proved that $\int_0^L \eta_t^{n2}(z,t)\,\mathrm{d}z$, $\int_0^L \eta_z^{n2}(z,t)\,\mathrm{d}z$, $\int_0^L \eta_{zz}^{n2}(z,t)\,\mathrm{d}z$ are bounded, see Estimate I section, from Eq. (71) picking μ_1 strictly less than $m_o/4$ shows that there exists a nonnegative constant M_2 such that

$$\int_0^L \eta_{tt}^{n2}(z,t_0)\,\mathrm{d}z\leqslant M_2,\quad \forall t\in[0,T],\ \ n\in N. \tag{72}$$

Estimate III: *Upper bound of* $\eta_{tt}(z,t)$ *and* $\eta_{zzt}(z,t)$ *in* L^2-*norm*. To estimate the upper bound of these terms, we use a difference approach. Let us fix t and ξ such that $\xi<T-t$. Now taking the difference of the first equation of Eq. (64) with $t=t+\xi$ and $t=t$, and then letting $\phi=\eta_t^n(t+\xi)-\eta_t^n(t)$ result in

$$\frac{m_o}{2}\int_0^L \frac{\mathrm{d}}{\mathrm{d}t}(\eta_t^n(z,t+\xi)-\eta_t^n(z,t))^2\,\mathrm{d}z+\frac{EI}{2}\int_0^L \frac{\mathrm{d}}{\mathrm{d}t}(\eta_{zz}^n(z,t+\xi)-\eta_{zz}^n(z,t))^2\,\mathrm{d}z+\Omega_1+\Omega_2=0, \tag{73}$$

where

$$\begin{aligned}\Omega_1 &= \frac{P_0}{2}\int_0^L \frac{\mathrm{d}}{\mathrm{d}t}(\eta_z^n(z,t+\xi)-\eta_z^n(z,t))^2\,\mathrm{d}z\\&\quad+\frac{EA}{4}\int_0^L (\eta_z^{n2}(z,t+\xi)+\eta_z(z,t+\xi)\eta_z^n(z,t)+\eta_z^{n2}(z,t)\frac{\mathrm{d}}{\mathrm{d}t}(\eta_z^n(z,t+\xi)-\eta_z^n(z,t))^2\,\mathrm{d}z,\\ \Omega_2 &= (m_H(\eta_{tt}^n(L,t+\xi)-\eta_{tt}^n(L,t))+b_H(\eta_t^n(L,t+\xi)-\eta_t^n(L,t))-A_H(P_H(t+\xi)-P_H(t)))\\&\quad\times(\eta_t^n(L,t+\xi)-\eta_t(L,t))-\int_0^L (f^n(\bullet)|_{t=t+\xi}-f^n(\bullet)|_{t=t})(\eta_t^n(L,t+\xi)-\eta_t^n(L,t)).\end{aligned} \tag{74}$$

Since the initial values $\eta(z,t_0)$ and $\eta_t(z,t_0)$ are sufficiently smooth, $\eta(0,t)=0$, $\eta_{zz}(0,t)=0$, $\eta_{zz}(L,t)=0$ for all $\eta\in W_S$ and all the terms $\int_0^L \eta_t^{n2}\,\mathrm{d}z$, $\int_0^L \eta_z^{n2}(z,t)\,\mathrm{d}z$, $\int_0^L \eta_{zz}^{n2}(z,t)\,\mathrm{d}z$ are bounded, see Estimate I section, using the Mean Value Theorem and Lemmas 3 and 4 shows that there exist nonnegative constants M_{31} and M_{32} such that

$$|\Omega_1|+|\Omega_2|\leqslant M_{31}\int_0^L (\eta_t^n(z,t+\xi)-\eta_t^n(z,t))^2\,\mathrm{d}z+M_{32}\int_0^L \left(\eta_{zz}^n(z,t+\xi)-\eta_{zz}^n(z,t)\right)^2\mathrm{d}z. \tag{75}$$

Using Eq. (75), we can write Eq. (73) as

$$\frac{\mathrm{d}\Phi^n}{\mathrm{d}t}(t,\xi)\leqslant M_{33}\Phi^n(t,\xi)\Rightarrow \Phi(t,\xi)\leqslant\Phi(t_0,\xi)\mathrm{e}^{M_{33}(t-t_0)}, \tag{76}$$

where M_{33} is a nonnegative constant, and

$$\Phi^n(t,\xi)=m_o\int_0^L (\eta_t^n(z,t+\xi)-\eta_t^n(z,t))^2\,\mathrm{d}z+EI\int_0^L (\eta_{zz}^n(z,t+\xi)-\eta_{zz}^n(z,t))^2\,\mathrm{d}z. \tag{77}$$

Dividing both sides of the last inequality in Eq. (76) by ξ^2 then taking the limit $\xi\to 0$ gives

$$m_o\int_0^L \eta_{tt}^{n2}(z,t)\,\mathrm{d}z+EI\int_0^L \eta_{zzt}^{n2}(z,t)\,\mathrm{d}z\leqslant\left[m_o\int_0^L \eta_{tt}^{n2}(z,t_0)\,\mathrm{d}z+EI\int_0^L \eta_{zzt}^{n2}(z,t_0)\,\mathrm{d}z\right]\mathrm{e}^{M_{33}(t-t_0)} \tag{78}$$

for all $0\leqslant t_0\leqslant t\leqslant T$. Now from the estimates given in Eqs. (69) and (71), we can deduce from Eq. (78) that there exists $M_3\geqslant 0$ depending on T such that

$$m_o\int_0^L \eta_{tt}^{n2}(z,t)\,\mathrm{d}z+EI\int_0^L \eta_{zzt}^{n2}(z,t)\,\mathrm{d}z\leqslant M_3. \tag{79}$$

From the estimates given in Eqs. (69), (72) and (79), we can use the Lions–Aubin theorem to get the necessary compactness to pass the nonlinear system (64) to the limit. Then it is a matter of routine to conclude the existence of global solutions in $[0,T]$.

Uniqueness: Let u and v be two solutions of the closed-loop system (52). Letting $\varpi=u-v$, we have $\varpi(z,t_0)=0$ and $\varpi_t(z,t_0)=0$ and from Eq. (63) we have

$$\begin{aligned}m_o\int_0^L \varpi_{tt}\phi\,\mathrm{d}z &= -EI\int_0^L \varpi_{zzzz}\phi\,\mathrm{d}z+P_0\int_0^L \varpi_{zz}\phi\,\mathrm{d}z+\frac{3EA}{2}\int_0^L (u_z^2u_{zz}-v_z^2v_{zz})\phi\,\mathrm{d}z\\&\quad+\int_0^L (f(\bullet)|_{\eta=u}-f(\bullet)|_{\eta=v})\phi\,\mathrm{d}z.\end{aligned} \tag{80}$$

By taking $\phi=\varpi_t(z,t)$ in Eq. (80) and using the Mean Value Theorem and passing of the limit of all the estimates given in Eqs. (69), (71) and (79) previously, we readily have

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_0^L \varpi_t^2\,\mathrm{d}z+\int_0^L \varpi_{ss}^2\right)\leqslant M_4\left(\int_0^L \varpi_t^2\,\mathrm{d}z+\int_0^L \varpi_{ss}^2\,\mathrm{d}z\right), \tag{81}$$

where M_4 is a positive constant. Since $\varpi(z,t_0)=0$ and $\varpi_t(z,t_0)=0$, using Gronwall's Lemma shows that $\varpi=0$, i.e. $u=v$ for all $t\geqslant t_0\geqslant 0$ and $z\in[0,L]$.

B.2. Proof of convergence

First, we consider the case where $f_L = 0$ and $\Delta(t, \eta_t(z,t)$ is constant, i.e. $\dot{\Delta} = 0$ then move to the case where $f_L \neq 0$ and $\dot{\Delta} \neq 0$.

(1) *Case $f_L = 0$ and $\dot{\Delta} = 0$*: In this case, we consider the following Lyapunov function candidate:

$$U_1 = W_3 + \frac{\lambda}{2}\Delta_e^2, \tag{82}$$

where λ is a positive constant to be specified. Differentiating both sides of Eq. (82) along the solutions of Eq. (49) and the last equation of Eq. (52) results in

$$\begin{aligned}\dot{U}_1 \leqslant & -c_3\int_0^L \eta_z^2\,\mathrm{d}z - \frac{3\gamma EA}{8m_o}\int_0^L \eta_z^4\,\mathrm{d}z - \frac{3\gamma EI}{2m_o}\int_0^L \eta_{zz}^2\,\mathrm{d}z - c_4\int_0^L \eta_t^2\,\mathrm{d}z - \frac{\gamma L}{m_o}c_6\eta_z^2(L,t) \\ & - \frac{\gamma LEA}{8m_o}\eta_z^4(L,t) - (c_5-\sigma)\left(\eta_t(L,t) + \frac{\gamma L}{m_o}\eta_z(L,t)\right)^2 - \left(k_2 + \frac{4\beta_{He}C_{HT}}{V_H} - \sigma\right)x_{3e}^2 - (k_3-\sigma)x_{4e}^2 \\ & - \left(\frac{\lambda k}{m_H} - \frac{1}{4\sigma}\left(\frac{\partial\alpha_2}{\partial x_2}\frac{1}{m_H}\right)^2 - \frac{1}{4\sigma}\left(\frac{k+k_1}{m_H}\right)^2 - \frac{1}{4\sigma}\right)\Delta_e^2,\end{aligned} \tag{83}$$

where the positive constants c_3, c_4, and c_5 are specified in Section 3 (see Eqs. (24) and (50)), and σ is an arbitrarily positive constant, and we have used $f_L = 0$ and $\dot{\Delta} = 0$ in this case. Since c_5 and k are strictly positive constants, and $\partial\alpha_2/\partial x_2$ is equal to a constant, we can always pick sufficiently small σ and sufficiently large λ such that $c_5 - \sigma$ and

$$\left(\frac{\lambda k}{m_H} - \frac{1}{4\sigma}\left(\frac{\partial\alpha_2}{\partial x_2}\frac{1}{m_H}\right)^2 - \frac{1}{4\sigma}\left(\frac{k+k_1}{m_H}\right)^2 - \frac{1}{4\sigma}\right)$$

are strictly positive constants. Using Eq. (23), it is straightforwardly deduced that

$$\dot{U}_1 \leqslant -\kappa U_1, \tag{84}$$

where

$$\kappa = \frac{\min\left(c_3, \frac{3\gamma EA}{8m_o}, \frac{3\gamma EI}{2m_o}, c_4, c_5-\sigma, \left(\frac{\lambda k}{m_H} - \frac{1}{4\sigma}\left(\frac{\partial\alpha_2}{\partial x_2}\frac{1}{m_H}\right)^2 - \frac{1}{4\sigma}\left(\frac{k+k_1}{m_H}\right)^2 - \frac{1}{4\sigma}\right)\right)}{\max\left(\frac{m_o+\gamma L}{2}, \frac{P_0+\gamma L}{2}, \frac{EA}{8}, \frac{EI}{2}, \frac{m_H}{2}, \lambda\right)}. \tag{85}$$

Therefore

$$U_1(t) \leqslant U_1(t_0)\mathrm{e}^{-\kappa(t-t_0)}, \quad \forall t \geqslant t_0 \geqslant 0 \tag{86}$$

which combines with the low and upper bounds of $W_1(t)$ given in Eq. (23), we have

$$\begin{aligned}&\frac{m_o-\gamma L}{2}\int_0^L \eta_t^2(z,t)\,\mathrm{d}z + \frac{P_0-\gamma L}{2}\int_0^L \eta_z^2(z,t)\,\mathrm{d}z + \frac{EA}{8}\int_0^L \eta_z^4(z,t)\,\mathrm{d}z + \frac{EI}{2}\int_0^L \eta_{zz}^2(z,t)\,\mathrm{d}z \\ &+ \frac{m_H}{2}\left(\eta_t(L,t) + \frac{\gamma L}{m_o}\eta_z(L,t)\right)^2 + \frac{1}{2}x_{3e}^2(t) + \frac{1}{2}x_{4e}^2(t) + \frac{1}{2}\Delta_e^2(t) \leqslant \left[\frac{m_o+\gamma L}{2}\int_0^L \eta_t^2(z,t_0)\,\mathrm{d}z\right. \\ &+ \frac{P_0+\gamma L}{2}\int_0^L \eta_z^2(z,t_0)\,\mathrm{d}z + \frac{EA}{8}\int_0^L \eta_z^4(z,t_0)\,\mathrm{d}z + \frac{EI}{2}\int_0^L \eta_{zz}^2(z,t_0)\,\mathrm{d}z + \frac{m_H}{2}\Big(\eta_t(L,t_0) \\ &\left.+ \frac{\gamma L}{m_o}\eta_z(L,t_0)\Big)^2 + \frac{1}{2}x_{3e}^2(t_0) + \frac{1}{2}x_{4e}^2(t_0) + \frac{1}{2}\Delta_e^2(t_0)\right]\mathrm{e}^{-\kappa(t-t_0)}, \quad \forall t \geqslant t_0 \geqslant 0.\end{aligned} \tag{87}$$

Since the initial values of $\eta_t(z,t_0), \eta_z(z,t_0), \eta_{zz}(z,t_0)$ for all $z \in [0,L]$, $x_3(t_0)$ and $x_4(t_0)$ are bounded and sufficiently smooth, all the terms in side the square bracket in the right-hand side of Eq. (87) are bounded.

Hence, the right-hand side of Eq. (87) is bounded and converges exponentially to zero. Boundedness and exponential convergence of the right-hand side of Eq. (87) to zero imply that the left-hand side of Eq. (87) must be bounded and exponentially converges to zero. This in turn implies that all the terms $\int_0^L \eta_t^2(z,t)\,\mathrm{d}z$, $\int_0^L \eta_{zz}^2(z,t)\,\mathrm{d}z$, $\int_0^L \eta_z^2(z,t)\,\mathrm{d}z$, $x_{3e}(t)$, $x_{4e}(t)$, and $\Delta_e(t)$ are bounded and exponentially converge to zero. Next, we use Lemmas 3 and 4 to show that $\int_0^L \eta^2(z,t)\,\mathrm{d}z$ and $|\eta(z,t)|$ are bounded and exponentially converge to zero. An application of Lemma 3 gives

$$\int_0^L \eta^2(z,t)\,\mathrm{d}z \leqslant 2\eta^2(0,t) + 4L^2 \int_0^L \eta_z^2(z,t)\,\mathrm{d}z. \tag{88}$$

Since $\eta(0,t)=0$ and we have already proved that $\int_0^L \eta_z^2(z,t)\,\mathrm{d}z$ is bounded and exponentially converges to zero, Eq. (88) implies that $\int_0^L \eta^2(z,t)\,\mathrm{d}z$ must be bounded and exponentially converges to zero. On the other hand, an application of Lemma 4 shows that

$$\max_{s\in[0,L]}(\eta^2(z,t)) \leqslant \eta^2(0,t) + 2\sqrt{\int_0^L \eta^2(z,t)\,\mathrm{d}z}\sqrt{\int_0^L \eta_z^2(z,t)\,\mathrm{d}z}. \tag{89}$$

Since $\eta(0,t)=0$ and we have already proved that $\int_0^L \eta_z^2(z,t)\,\mathrm{d}z$ and $\int_0^L \eta^2(z,t)\,\mathrm{d}z$ are bounded and exponentially converge to zero, Eq. (89) implies that $|\eta(z,t)|$ must be bounded and exponentially converges to zero.

(2) *Case* $f_L \neq 0$ *and* $\dot{\Delta} \neq 0$: Similarly to the previous case, we consider the following Lyapunov function candidate:

$$U_2 = W_3 + \frac{\lambda}{2}\Delta_\mathrm{e}^2, \tag{90}$$

where λ is a positive constant to be specified. Differentiating both sides of Eq. (82) along the solutions of Eq. (49) and the last equation of Eq. (52) results in

$$\begin{aligned}
\dot{U}_2 \leqslant & -c_3\int_0^L \eta_z^2\,\mathrm{d}z - \frac{3\gamma EA}{8m_o}\int_0^L \eta_z^4\,\mathrm{d}z - \frac{3\gamma EI}{2m_o}\int_0^L \eta_{zz}^2\,\mathrm{d}z - c_4\int_0^L \eta_t^2\,\mathrm{d}z - \frac{\gamma L}{m_o}c_6\eta_z^2(L,t) \\
& - \frac{\gamma LEA}{8m_o}\eta_z^4(L,t) - (c_5-\sigma)\left(\eta_t(L,t) + \frac{\gamma L}{m_o}\eta_z(L,t)\right)^2 - \left(k_2 + \frac{4\beta_{He}C_{HT}}{V_H} - \sigma\right)x_{3e}^2 - (k_3-\sigma)x_{4e}^2 \\
& - \left(\frac{\lambda k}{m_H} - \frac{1}{4\sigma}\left(\frac{\partial \alpha_2}{\partial x_2}\frac{1}{m_H}\right)^2 - \frac{1}{4\sigma}\left(\frac{k+k_1}{m_H}\right)^2 - \frac{1}{4\sigma}\right)\Delta_\mathrm{e}^2 + \frac{\lambda}{4\sigma}\dot{\Delta}^2 + \frac{1+\gamma L}{4\varepsilon}\int_0^L f_L^2\,\mathrm{d}z,
\end{aligned} \tag{91}$$

where all the positive constants c_3, c_4, c_5, σ, and λ are specified in the previous case. Processing the same as the previous case, we have

$$\dot{U}_2 \leqslant -\kappa U_2 + C_{U2}, \tag{92}$$

where κ is given in Eq. (85), and $C_{U2} = \max|\lambda/4\sigma\dot{\Delta}^2 + (1+\gamma L)/4\varepsilon \int_0^L f_L^2\,\mathrm{d}z|$. The differential inequality (92) implies that

$$U_2(t) \leqslant \left(U_2(t_0) - \frac{C_{U2}}{\kappa}\right)\mathrm{e}^{-\kappa(t-t_0)} + \frac{C_{U2}}{\kappa}, \quad \forall t \geqslant t_0 \geqslant 0. \tag{93}$$

Hence $U_2(t)$ exponentially converges to the nonnegative constant C_{U2}/κ. This in turn implies that all the terms $\int_0^L \eta_t^2(z,t)\,\mathrm{d}z$, $\int_0^L \eta_{zz}^2(z,t)\,\mathrm{d}z$, $\int_0^L \eta_z^2(z,t)\,\mathrm{d}z$, $x_{3e}(t)$, $x_{4e}(t)$ and $\Delta_e(t)$ exponentially converge to some nonnegative constant less than

$$\frac{C_{U2}}{\kappa\min\left(\frac{m_o+\gamma L}{2}, \frac{P_0+\gamma L}{2}, \frac{EA}{8}, \frac{EI}{2}, \frac{m_H}{2}, \lambda\right)}$$

due to Eq. (23). Proof of boundedness (not exponential convergence to zero) of $\int_0^L \eta^2(z,t)\,\mathrm{d}z$ and $|\eta(z,t)|$ can be carried out in the same lines as in the previous case.

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